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4. Exercise sheet Algebraic Geometry I

Note: all solutions have to be completely justified.

Aufgabe 1 Let $(X, \mathcal{O}_X)$ be a prevariety. Show that the topological space X is noetherian and irreducible.

Aufgabe 2 (a) Let $X \subset \mathbb{A}^n(k)$ be an irreducible affine algebraic set, Z an irreducible closed subset of X and U an open subset of Z. Show that the image of the inclusion $\mathcal{O}_Z(U) \to \text{Map}(U, k)$ is the set of functions $f \in \text{Map}(U, k)$ with the following property: for every $x \in U$ there exists an open neighborhood V of x in X and $g \in \mathcal{O}_X(V)$ such that $f|_{U \cap V} = g|_{U \cap V}$.
(b) Let X be a prevariety and Z an irreducible closed subset of X. For all open subsets U of Z we define $\mathcal{O}_Z(U)$ to be the set of functions $f \in \text{Map}(U, k)$ with the following property: for every $x \in U$ there exists an open neighborhood V of x in X and $g \in \mathcal{O}_X(V)$ such that $f|_{U \cap V} = g|_{U \cap V}$. Show that $(Z, \mathcal{O}_Z)$ is a prevariety and the inclusion $Z \subseteq X$ is a morphism of prevarieties.

Aufgabe 3 (a) Let X be an irreducible affine algebraic set and I a proper ideal of $\Gamma(X)$. Show that $V(I) \neq \emptyset$.
(b) Let $X := \mathbb{A}^n(k) \setminus V(T_1, \dots, T_n)$. Show that $(X, \mathcal{O}_{\mathbb{A}^n(k)}|_X)$ is a prevariety. Is $(X, \mathcal{O}_{\mathbb{A}^n(k)}|_X)$ an affine variety?

Aufgabe 4 Let X be a prevariety and let Y be an affine variety. Show that the map

$$Hom(X, Y) \to Hom_{k-alg}(\Gamma(Y), \Gamma(X)), \quad f \mapsto f^* : \varphi \mapsto \varphi \circ f,$$

is bijective. Deduce that $Hom(X, \mathbb{A}^n(k)) = \Gamma(X)^n$.