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3. Exercise sheet Algebraic Geometry I

Note: all solutions have to be completely justified.

Aufgabe 1 (a) Show that, in the category of commutative rings with 1 with ring homomorphisms that send 1 to 1, the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphisms, but is not surjective.
(b) Given an abelian group $(A, *)$, for any positive integer n and any $a \in A$, we define $na := \underbrace{a * \cdots * a}_{n \text{ times}}$. We say that A is divisible if for all $a \in A$ and all positive integers n, there is an element $b \in A$ such that $nb = a$.
Show that, in the category of divisible abelian groups with group homomorphisms, the quotient $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphisms, but is not injective. Here $\mathbb{Q}$ is the additive group of rational numbers and $\mathbb{Z}$ is the additive group of integers.

Aufgabe 2 Let $f \in k[T_1]$ be a non-constant polynomial. Show that $X_1 := V(T_2 - f(T_1)) \subset \mathbb{A}^2(k)$ is isomorphic to $\mathbb{A}^1(k)$ and show that $X_2 := V(1 - f(T_1)T_2) \subset \mathbb{A}^2(k)$ is isomorphic to $A^1(k) \setminus \{x_1, \ldots, x_n\}$ for some $n \geq 1$. Show that X_1 and X_2 are not isomorphic.

Aufgabe 3 Given an irreducible affine algebraic set X, we say that a rational function $f \in K(X)$ is regular at a point $x \in X$ if $f = \frac{g}{h}$ for some $g, h \in \Gamma(X)$ such that $h \notin \mathfrak{m}_x$.
(a) At which points of $V(X^2 + Y^2 - 1)$ is the rational function $\frac{1-Y}{X}$ regular?
(b) At which points of $V(Y^2 - X^2 - X^3)$ is the rational function $\frac{X^2}{Y}$ regular?

Aufgabe 4 (a) Calculate $\mathcal{O}_{\mathbb{A}^1(k)}(\mathbb{A}^1(k) \setminus \{0\}) \subset K(\mathbb{A}^1(k)) = k(T_1)$.
(b) Calculate $\mathcal{O}_{\mathbb{A}^2(k)}(\mathbb{A}^2(k) \setminus \{(0,0)\}) \subset K(\mathbb{A}^2(k)) = k(T_1, T_2)$.