Prof. W. Bley
Marta Pieropan 24.10.2013

2. Exercise sheet Algebraic Geometry I

Aufgabe 1 (a) Let K be an infinite field with $\mathrm{char}(K) \neq 2$. Consider $Z_1 = V(U(T-1)-1) \subseteq \mathbb{A}^2(K)$ and $Z_2 = V(Y^2 - X^2(X+1)) \subseteq \mathbb{A}^2(K)$. Show that the morphism $\varphi : Z_1 \to Z_2$ defined by $(t, u) \mapsto (t^2 - 1, t(t^2 - 1))$ is a bijection but not an isomorphism.
(b) Prove that if $K \cong \mathbb{Z}/3\mathbb{Z}$ is a field with three elements, then the morphism φ defined in (a) is an isomorphism.

Aufgabe 2 (a) Consider the *twisted cubic curve* $C = \{(t, t^2, t^3) : t \in k\} \subseteq \mathbb{A}^3(k)$. Show that C is an irreducible closed subset of $\mathbb{A}^3(k)$. Find generators for the ideal $I(C)$.
(b) Let $V = V(X^2 - YZ, XZ - X) \subseteq \mathbb{A}^3(k)$. Show that V consists of three irreducible components and determine the corresponding prime ideals.

Aufgabe 3 Let $f \in k[X_1, \ldots, X_n]$ be a non-costant polynomial. Write $f = \prod_{i=1}^r f_i^{n_i}$ with irreducible polynomials f_i such that $(f_i) \neq (f_j)$ for all $i \neq j$ and integers $n_i \geq 1$. Show that $\mathrm{rad}(f) = (f_1 \ldots f_r)$ and that the irreducible components of $V(f) \subseteq \mathbb{A}^n(k)$ are the closed subsets $V(f_i)$, $i = 1, \ldots, r$.

Aufgabe 4 Let $\mathfrak{a} \subseteq k[T_1, \ldots, T_n]$ be a radical ideal, i.e., $\mathfrak{a} = \mathrm{rad}(\mathfrak{a})$. Then $\mathfrak{a}$ is the intersection of a finite number of prime ideals that do not contain each other. The set of these prime ideals is uniquely determined by $\mathfrak{a}$.