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12. Exercise sheet Algebraic Geometry I

All solutions have to be completely justified.

Aufgabe 1 Prove the following facts directly from the universal property of the tensor product of algebras:

(a) For any R-algebra we have $R \otimes_R S = S$.
(b) If S, T are R-algebras and $I \subseteq S$ is an ideal, then

$$(S/I) \otimes_R T = (S \otimes_R T)/(I \otimes 1)(S \otimes_R T).$$

(c) If $x_1, \dots, x_n$ and $y_1, \dots, y_m$ are indeterminates, then

$$R[x_1, \dots, x_n] \otimes_R R[y_1, \dots, y_m] = R[x_1, \dots, x_n, y_1, \dots, y_m].$$

Use these principles to solve the remainder of this exercise.

(d) Let m, n be integers. Compute the fibred product

$$\mathrm{Spec}(\mathbb{Z}/m\mathbb{Z}) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}/n\mathbb{Z}).$$

(e) Compute the fibred product $\mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$.

Note that in (d) the underlying set of the fibred product is the fibred product of the underlying sets. Note that this is not true in (e).

(f) Consider the ring homomorphisms

$$R[x] \longrightarrow R, \quad x \mapsto 0$$

and

$$R[x] \longrightarrow R[y], \quad x \mapsto y^2.$$

Show that with respect to these maps we have

$$\mathrm{Spec}(R[y]) \times_{\mathrm{Spec}(R[x])} \mathrm{Spec}(R) = \mathrm{Spec}(R[y]/(y^2)).$$

Aufgabe 2 Let $A = k[x, y]$, $X = \mathrm{Spec}\, A$.
(a) Show that $\mathfrak{a} = (xy)$ defines a reducible closed subscheme of X.
(b) Let $\mathfrak{a}_1 = (x^2)$ and $\mathfrak{a}_2 = (x^2, xy)$. Show that $V(\mathfrak{a}_1) = V(\mathfrak{a}_2)$. Let Z_1, Z_2 be the closed subschemes of X defined by $\mathfrak{a}_1, \mathfrak{a}_2$ respectively. Show that for all $z \in V(\mathfrak{a}_1)$ the local ring $\mathcal{O}_{Z_1,z}$ is not reduced. Show that the local ring $\mathcal{O}_{Z_2,z}$ is reduced for all $z \in V(\mathfrak{a}_2) \setminus \{(0,0)\}$.

Aufgabe 3 Let $f : X \to S$ and $g : S' \to S$ be morphisms of schemes. Let $X' := X \times_S S'$, and let $f' : X' \to S'$ be the induced morphism. Show that if f is of finite type, then f' is of finite type.

Aufgabe 4 Let $X \to S$ and $S' \to S$ be morphisms of schemes, and $X' := X \times_S S'$.
(a) Assume that X, S and S' are reduced. Show that X' is not reduced in general.
(b) Assume that X, S and S' are irreducible. Show that X' is not irreducible in general.