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10. Exercise sheet Algebraic Geometry I

All solutions have to be completely justified.

Aufgabe 1 Let $\{X_i\}_{i\in I}$ be a family of schemes. For each $i, j \in I$ suppose given an open subset $X_{i,j} \subseteq X_i$, and let it have the induced scheme structure. Suppose also given for each $i, j \in I$ an isomorphism of schemes $\varphi_{i,j} : X_{i,j} \to X_{j,i}$. Assume that
(1) $X_{i,i} = X_i$ for all $i \in I$,
(2) $\varphi_{i,j} = \varphi_{j,i}^{-1}$ for all $i, j \in I$,
(3) $\varphi_{i,j}(X_{i,j} \cap X_{i,k}) = X_{j,i} \cap X_{j,k}$ and $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$ for all $i, j, k \in I$.
Then show that there exists a scheme X, together with morphisms $\psi_i : X_i \to X$ for all $i \in I$ such that
(i) ψ_i is an isomorphism of X_i onto an open subscheme of X for all $i \in I$,
(ii) the family $\{\psi_i(X_i)\}_{i\in I}$ is an open covering of X,
(iii) $\psi_i(X_{i,j}) = \psi_i(X_i) \cap \psi_j(X_j)$ and $\psi_i = \psi_j \circ \varphi_{i,j}$ on $X_{i,j}$ for all $i, j \in I$.

Aufgabe 2 (a) Let K be a field. Show that there is a bijection $\mathrm{Hom}(\mathrm{Spec}\, K, \mathbb{P}^n_K) \simeq (K^{n+1} \setminus \{0\})/K^\times$.
(b) Let X be a scheme and R a local ring. Show that every morphism $\mathrm{Spec}\, R \to X$ factors through the canonical morphism $\mathrm{Spec}\, \mathcal{O}_{X,x} \to X$, where x is the image of the closed point of $\mathrm{Spec}\, R$. Prove that in this way one obtains a bijection between $\mathrm{Hom}(\mathrm{Spec}\, R, X)$ and the set of pairs (x, φ), where $x \in X$ and $\varphi : \mathcal{O}_{X,x} \to R$ is a local homomorphism.

Aufgabe 3 (a) Show that an affine scheme $X = \mathrm{Spec}\, A$ is reduced if and only if $\sqrt{(0)} = (0)$.
(b) Show that a scheme X is reduced if and only if for every $x \in X$ the local ring $\mathcal{O}_{X,x}$ has no nonzero nilpotent elements.
(c) Show that an affine scheme $X = \mathrm{Spec}\, A$ is integral if and only if A is an integral domain.