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# THE CONSTRUCTION PROBLEM FOR HODGE NUMBERS MODULO AN INTEGER IN POSITIVE CHARACTERISTIC

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ABSTRACT. Let  $k$  be an algebraically closed field of positive characteristic. For any integer  $m \geq 2$ , we show that the Hodge numbers of a smooth projective  $k$ -variety can take on any combination of values modulo  $m$ , subject only to Serre duality. In particular, there are no non-trivial polynomial relations between the Hodge numbers.

## 1. INTRODUCTION

The Hodge numbers  $h^{p,q}(X) = \dim_{\mathbf{C}} H^q(X, \Omega_X^p)$  of an  $n$ -dimensional smooth projective variety  $X$  over  $\mathbf{C}$  satisfy the following conditions:

- (1)  $h^{0,0}(X) = 1$  (connectedness);
- (2)  $h^{p,q}(X) = h^{n-p,n-q}(X)$  for all  $0 \leq p, q \leq n$  (Serre duality);
- (3)  $h^{p,q}(X) = h^{q,p}(X)$  for all  $0 \leq p, q \leq n$  (Hodge symmetry).

Kotschick and Schreieder showed [KS13, Thm. 1, consequence (2)] that the only linear relations among the Hodge numbers that are satisfied by *all* smooth projective  $\mathbf{C}$ -varieties of dimension  $n$  are the ones induced by (1), (2), and (3).

In positive characteristic, Hodge symmetry (3) does not always hold [Ser58, Prop. 16], but Serre duality (2) is still true. The first author proved that (1) and (2) are indeed the only universal *linear* relations among the Hodge numbers of  $n$ -dimensional smooth projective  $k$ -varieties if  $\text{char } k > 0$  [vDdB20, Thm. 1].

In [PS19, Thm. 2], the second author and Schreieder solved the construction problem over  $\mathbf{C}$  for Hodge diamonds modulo an arbitrary integer  $m \geq 2$ . This means that for any dimension  $n$  and any collection of integers satisfying the conditions (1), (2), and (3), there exists a smooth projective  $\mathbf{C}$ -variety of dimension  $n$  whose Hodge numbers agree with the given integers modulo  $m$ . As a corollary, there are no non-trivial *polynomial* relations among the Hodge numbers, which strengthens the result from [KS13] on linear relations.

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In this paper, we solve the construction problem for Hodge diamonds modulo  $m$  in positive characteristic:

**Theorem 1.** *Let  $k$  be an algebraically closed field of positive characteristic, and let  $m \geq 2$  and  $n \geq 0$  be integers. Let  $(a^{p,q})_{0 \leq p, q \leq n}$  be any collection of integers such that  $a^{0,0} = 1$  and  $a^{p,q} = a^{n-p, n-q}$  for all  $0 \leq p, q \leq n$ . Then there exists a smooth projective  $k$ -variety  $X$  of dimension  $n$  such that*

$$h^{p,q}(X) \equiv a^{p,q} \pmod{m}$$

for all  $0 \leq p, q \leq n$ .

In analogy to [PS19, Cor. 3], it follows that there are no polynomial relations among the Hodge numbers in positive characteristic besides (1) and (2) (see Corollary 5.1). This extends the result from [vDdB20, Thm. 1] on linear relations.

Theorem 1 also shows that Hodge symmetry may fail arbitrarily badly in positive characteristic. For any dimension  $n$  and all  $0 \leq p < q \leq n$  with  $p + q \neq n$ , the Hodge numbers  $h^{p,q}$  and  $h^{q,p}$  can not only be different, but can even be incongruent modulo any integer  $m \geq 2$ . Note that Hodge symmetry (3) is a consequence of Serre duality (2) if  $p + q = n$ , and thus always holds in the middle row of the Hodge diamond.

A complete classification of the possible Hodge diamonds of smooth projective  $k$ -varieties, i. e. a version of Theorem 1 without the “modulo  $m$ ” part, seems to be very hard already when Hodge symmetry is true; see [Sch15] for strong partial results on this in characteristic zero.

The structure of our proof is similar to [PS19], with some improvements. First we solve the construction problem modulo  $m$  for the outer Hodge numbers, i. e. the Hodge numbers  $h^{p,q}$  with  $p \in \{0, n\}$  or  $q \in \{0, n\}$  (see Proposition 3.1). Then we prove that for any smooth projective  $k$ -variety, there exists a sequence of blowups in smooth centres such that the inner Hodge numbers of the blowup, i. e. the Hodge numbers  $h^{p,q}$  with  $1 \leq p, q \leq n - 1$ , attend any given values in  $\mathbf{Z}/m$  satisfying Serre duality (2). Hence we obtain the following result, which might be of independent interest:

**Theorem 2.** *Let  $k$  be an algebraically closed field of positive characteristic, and let  $m \geq 2$  and  $n \geq 0$  be integers. Let  $X$  be a smooth projective  $k$ -variety of dimension  $n$  and let  $(a^{p,q})_{1 \leq p, q \leq n-1}$  be any collection of integers such that  $a^{p,q} = a^{n-p, n-q}$  for all  $1 \leq p, q \leq n - 1$ . Then there exists a smooth projective  $k$ -variety  $\tilde{X}$  birational to  $X$  such that*

$$h^{p,q}(\tilde{X}) \equiv a^{p,q} \pmod{m}$$

for all  $1 \leq p, q \leq n - 1$ .

The analogous statement in characteristic zero was obtained in [PS19, Thm. 5]. The fact that all outer Hodge numbers are birational invariants in positive characteristic was proven by Chatzistamatiou and Rülling [CR11, Thm. 1], so Theorem 2 is the best possible statement. Again, it follows that the result from [vDdB20, Thm. 3] on linear birational invariants extends to polynomials (see Corollary 5.2).

In analogy with [vDdB20, Thm. 2], our constructions only need Serre’s counterexample [Ser58, Prop. 16] to generate all Hodge asymmetry. While the structure of our argument is similar to [PS19], the absence of condition (3) in positive characteristic raises new difficulties for both the inner and the outer Hodge numbers. There is a quick proof of Theorem 2 assuming embedded resolution of singularities in positive characteristic, see Remark 4.4. The proof we present is similar, but does a little more work to avoid using embedded resolution.

In Section 2, we state and prove some lemmas on Hodge numbers that are used later. The constructions for outer and inner Hodge numbers are carried out in Section 3 and Section 4, respectively. Finally, we deduce corollaries on polynomial relations in Section 5.

**Notation.** Throughout this paper, we fix an algebraically closed field  $k$  of positive characteristic and an integer  $m \geq 2$ .

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## 2. SOME LEMMAS ON HODGE NUMBERS

In this section, we collect some standard results on Hodge numbers that we will use repeatedly in the arguments. The only difference between the situation in characteristic zero [KS13, PS19] and positive characteristic [vDdB20] comes from asymmetry of Hodge diamonds, and as in [vDdB20] the only example we need is Serre’s surface:

**Theorem 2.1.** *There exists a smooth projective surface  $S$  over  $k$  such that  $h^{1,0}(S) = 0$  and  $h^{0,1}(S) = 1$ .*

*Proof.* See [Ser58, Prop. 16], or [vDdB20, Prop. 1.3] for a short modern account.  $\square$

We use the following well-known formula for Hodge numbers under blowups.

**Lemma 2.2.** *Let  $X$  be a smooth projective  $k$ -variety, let  $Z \subseteq X$  be a smooth subvariety of codimension  $r$ , and let  $\tilde{X} \rightarrow X$  be the blowup of  $X$  at  $Z$ . Then the Hodge numbers of  $\tilde{X}$  satisfy*

$$h^{p,q}(\tilde{X}) = h^{p,q}(X) + \sum_{i=1}^{r-1} h^{p-i,q-i}(Z).$$

A consequence that will be used repeatedly is that any blowup construction carried out  $m$  times does not change the Hodge numbers modulo  $m$ .

*Proof of Lemma 2.2.* See for example [Gro85, Cor. IV.1.1.11]. As noted by Achinger and Zdanowicz [AZ17, Cor. 2.8], it is also an immediate consequence of Voevodsky's motivic blowup formula [Voe00, Prop. 3.5.3] and Chatzistamatiou–Rülling's action of Chow groups on Hodge cohomology [CR11].  $\square$

The next lemma gives a weak Lefschetz theorem for sufficiently ample hypersurfaces.

**Lemma 2.3.** *Let  $X$  be a smooth projective  $k$ -variety of dimension  $n+1$  with a very ample line bundle  $\mathcal{L} = \mathcal{O}_X(H)$ . Let  $d_0 \in \mathbf{Z}_{>0}$  such that  $H^q(X, \Omega_X^p(-dH)) = 0$  when  $d \geq d_0$  and  $p+q \leq n$ . Then any smooth divisor  $Y \in |dH|$  with  $d \geq d_0$  satisfies  $h^{p,q}(Y) = h^{p,q}(X)$  when  $p+q \leq n-1$ .*

*Proof.* The short exact sequence

$$0 \rightarrow \Omega_X^p(-dH) \rightarrow \Omega_X^p \rightarrow \Omega_X^p|_Y \rightarrow 0$$

shows that for all  $p+q \leq n-1$  and all  $e \geq 0$ , we have

$$H^q\left(X, \Omega_X^p(-eH)\right) = H^q\left(Y, \Omega_X^p(-eH)|_Y\right). \quad (2.1)$$

We will prove by induction on  $p$  that  $H^q(Y, \Omega_X^p(-eH)|_Y) = H^q(Y, \Omega_Y^p(-eH))$  for all  $e \geq 0$  and  $p+q \leq n-1$ . Together with (2.1) this proves the result by taking  $e=0$ . The base case  $p=0$  is trivial since  $\mathcal{O}_X|_Y = \mathcal{O}_Y$ . For  $p > 0$ , the inductive hypothesis, (2.1), and the assumption on  $d_0$  give

$$H^q\left(Y, \Omega_Y^i(-eH)\right) = H^q\left(Y, \Omega_X^i(-eH)|_Y\right) = H^q\left(X, \Omega_X^i(-eH)\right) = 0 \quad (2.2)$$

for  $i+q \leq n-1$ ,  $e \geq d_0$ , and  $i < p$ . The conormal sequence

$$0 \rightarrow \mathcal{O}_Y(-Y) \rightarrow \Omega_X^1|_Y \rightarrow \Omega_Y^1 \rightarrow 0$$

gives a filtration  $0 = \mathcal{F}_{-1} \subseteq \dots \subseteq \mathcal{F}_p = \Omega_X^p|_Y$  with short exact sequences

$$0 \rightarrow \mathcal{F}_{i-1} \rightarrow \mathcal{F}_i \rightarrow \Omega_Y^i((i-p)Y) \rightarrow 0. \quad (2.3)$$

Now (2.2) gives

$$H^q\left(Y, \Omega_Y^i((i-p)Y - eH)\right) = H^q\left(Y, \Omega_Y^i((d(i-p) - e)H)\right) = 0$$

for  $i < p$ ,  $e \geq 0$ , and  $i+q \leq n-1$ ; since  $d(i-p) - e \leq -d$ . Thus, (2.3) gives isomorphisms  $H^q(Y, \mathcal{F}_{i-1}(-eH)) = H^q(Y, \mathcal{F}_i(-eH))$  for  $i < p$ ,  $e \geq 0$ , and  $i+q \leq n-1$  (in particular when  $p+q \leq n$  since  $i \leq p-1$ ). Then induction on  $i$  gives  $H^q(Y, \mathcal{F}_{p-1}(-eH)) = 0$  for any  $e \geq 0$  and  $p+q \leq n$ , so (2.3) shows that the natural map

$$H^q\left(Y, \Omega_X^p(-eH)|_Y\right) \rightarrow H^q\left(Y, \Omega_Y^p(-eH)\right)$$

is an isomorphism for  $p+q \leq n-1$  and  $e \geq 0$ , as claimed.  $\square$

**Corollary 2.4.** *Let  $X$  be a smooth projective  $k$ -variety of dimension  $n+1$  with a very ample line bundle  $\mathcal{L} = \mathcal{O}_X(H)$ . Then any smooth divisor  $Y \in |dH|$  with  $d \gg 0$  satisfies  $h^{p,q}(Y) = h^{p,q}(X)$  when  $p+q \leq n-1$ .*

*Proof.* By Serre vanishing, there exists  $d_0 \in \mathbf{Z}$  such that  $H^q(X, \Omega_X^p(-dH)) = 0$  for all  $d \geq d_0$  and  $q \leq n$ . Then [Lemma 2.3](#) gives the result.  $\square$

**Remark 2.5.** If  $\text{char } k = 0$ , then by Nakano vanishing we may take  $d_0 = 1$  in [Lemma 2.3](#). This recovers the usual proof of weak Lefschetz from Nakano vanishing, although usually the implication goes in the other direction. Similarly, if  $\text{char } k > 0$  and Nakano vanishing holds for  $X$ , then we may take  $d_0 = 1$ , but in general already Kodaira vanishing may fail in positive characteristic [[Ray78](#)].

For our application, it's useful to have some control over the Euler characteristic of  $\mathcal{L}^{-1}$ .

**Lemma 2.6.** *Let  $X$  be a smooth projective  $k$ -variety of dimension  $n + 1$  and let  $e \in \mathbf{Z}$ . Then, up to modifying  $X$  by blowups in smooth centres that do not change its Hodge numbers modulo  $m$ , we may assume that  $X$  admits a very ample line bundle  $\mathcal{L} = \mathcal{O}_X(H)$  such that  $\chi(X, \mathcal{L}^{-1}) \equiv e \pmod{m}$  and such that any smooth divisor  $Y \in |H|$  satisfies  $h^{p,q}(Y) = h^{p,q}(X)$  when  $p + q \leq n - 1$ .*

*Proof.* Let  $\pi: \tilde{X} \rightarrow X$  be a blowup in  $m$  distinct points  $p_1, \dots, p_m \in X$ . Then the blowup formula for Hodge numbers ([Lemma 2.2](#)) gives  $h^{p,q}(\tilde{X}) \equiv h^{p,q}(X) \pmod{m}$ . Let  $E_i = \pi^{-1}(p_i)$  be the exceptional divisors, and for  $r \in \{0, \dots, m\}$  write  $E_{\leq r} = E_1 + \dots + E_r$ . Then the short exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(-E_{\leq r}) \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow \mathcal{O}_{E_{\leq r}} \rightarrow 0$$

shows that

$$\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}(-E_{\leq r})) = \chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) - \sum_{i=1}^r \chi(E_i, \mathcal{O}_{E_i}) = \chi(X, \mathcal{O}_X) - r.$$

Take  $r \in \{0, \dots, m-1\}$  with  $r \equiv \chi(X, \mathcal{O}_X) - e \pmod{m}$ .

Let  $\mathcal{M}$  be an ample line bundle on  $\tilde{X}$ . By Serre vanishing there exists  $a_0 \in \mathbf{Z}$  such that for all  $a \geq a_0$ , the line bundle  $\mathcal{L} = \mathcal{M}^{\otimes a} \otimes \mathcal{O}_{\tilde{X}}(E_{\leq r})$  is very ample and satisfies

$$H^q(X, \Omega_X^p \otimes \mathcal{L}^{-d}) = 0 \tag{2.4}$$

for  $d > 0$  and  $q \leq n$ . Taking  $a$  divisible by the product of  $m$  and the denominators of the coefficients of the Hilbert polynomial  $P(t) = \chi(\tilde{X}, \mathcal{M}^{\otimes t} \otimes \mathcal{O}_{\tilde{X}}(-E_{\leq r}))$ , we see that

$$\chi(\tilde{X}, \mathcal{L}^{-1}) \equiv \chi(\tilde{X}, \mathcal{O}_{\tilde{X}}(-E_{\leq r})) \equiv e \pmod{m}.$$

Finally,  $\mathcal{L}$  satisfies weak Lefschetz by [Lemma 2.3](#) and (2.4).  $\square$

### 3. OUTER HODGE NUMBERS

In this section, we solve the construction problem for the outer Hodge numbers. Because of Serre duality and the fact that  $h^{0,0} = 1$ , it suffices to consider the Hodge numbers  $h^{p,q}$  with  $(p, q) \in J_n$ , where

$$J_n = \{(1, 0), \dots, (n, 0), (0, 1), \dots, (0, n)\}.$$

The main result of this section is the following:

**Proposition 3.1.** *Let  $n \geq 0$ . For any given integers  $a^{1,0}, \dots, a^{n,0}$  and  $a^{0,1}, \dots, a^{0,n}$  with  $a^{n,0} = a^{0,n}$ , there exists a smooth projective  $k$ -variety  $X$  of dimension  $n$  such that*

$$h^{p,q}(X) \equiv a^{p,q} \pmod{m}$$

for all  $(p, q) \in J_n$ .

The construction will be carried out by induction on the dimension, using the weak Lefschetz results from [Corollary 2.4](#) and [Lemma 2.6](#).

**Lemma 3.2.** *Let  $n, d \geq 0$  be integers such that  $d \geq n - 1$ . If [Proposition 3.1](#) holds for  $a^{1,0}, \dots, a^{d,0}$  and  $a^{0,1}, \dots, a^{0,d}$  in dimension  $d$ , then it also holds for  $a^{1,0}, \dots, a^{n-1,0}, b$  and  $a^{0,1}, \dots, a^{0,n-1}, b$  in dimension  $n$  for any  $b \in \mathbf{Z}$ .*

*Proof.* Let  $X$  be a smooth projective  $k$ -variety of dimension  $d$  with the given Hodge numbers  $a^{p,q}$ . We may assume that  $d \geq n + 1$  by multiplying  $X$  with  $\mathbf{P}^2$ , which does not change its outer Hodge numbers in degree  $\leq n - 1$ . By repeatedly replacing  $X$  by a smooth hyperplane section of sufficiently high degree, we may further assume that  $d = n + 1$  by [Corollary 2.4](#). By [Lemma 2.6](#), after possibly replacing  $X$  by a blowup that does not change its Hodge numbers modulo  $m$ , there exists a very ample line bundle  $\mathcal{L}$  on  $X$  such that

$$\chi(X, \mathcal{L}^{-1}) \equiv (-1)^n (a^{0,n} - a^{0,n+1} - b) \pmod{m} \quad (3.1)$$

and such that a smooth section  $Y$  of  $\mathcal{L}$  satisfies  $h^{p,q}(Y) \equiv a^{p,q} \pmod{m}$  for  $p+q \leq n-1$ . The short exact sequence

$$0 \rightarrow \mathcal{L}^{-1} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0$$

gives  $\chi(X, \mathcal{L}^{-1}) = \chi(X, \mathcal{O}_X) - \chi(Y, \mathcal{O}_Y)$ . Since  $h^{0,q}(X) = h^{0,q}(Y)$  for  $q \leq n - 1$ , we conclude that

$$\begin{aligned} \chi(X, \mathcal{L}^{-1}) &= (-1)^n h^{0,n}(X) + (-1)^{n+1} h^{0,n+1}(X) - (-1)^n h^{0,n}(Y) \\ &\equiv (-1)^n (a^{0,n} - a^{0,n+1} - h^{0,n}(Y)) \pmod{m} \end{aligned}$$

Then [\(3.1\)](#) gives  $h^{0,n}(Y) \equiv b \pmod{m}$ , so Serre duality gives  $h^{n,0}(Y) \equiv b \pmod{m}$ .  $\square$

Note that in characteristic zero, [Lemma 3.2](#) immediately implies [Proposition 3.1](#), giving an alternative approach to a variant of [\[PS19, Prop. 4\]](#). In positive characteristic, however, the failure of Hodge symmetry raises new difficulties, since e. g.  $h^{n-1,0} = h^{0,n-1}$  is true for varieties of dimension  $n-1$  but not for all varieties of dimension  $n$ . This problem is solved in the following construction, which together with [Lemma 3.2](#) implies [Proposition 3.1](#).

**Lemma 3.3.** *Let  $n \geq 2$ . For any given integers  $a^{0,1}, \dots, a^{0,n-1}$  and  $a^{1,0}, \dots, a^{n-1,0}$ , there exists a smooth projective  $k$ -variety  $X$  of dimension  $\geq n - 1$  such that*

$$h^{p,q}(X) \equiv a^{p,q} \pmod{m}$$

for all  $(p, q) \in J_{n-1}$ .

Note that we do not assume  $a^{0,n-1} = a^{n-1,0}$  here, so we typically need  $\dim X \geq n$ .

*Proof of Lemma 3.3.* First we consider the case  $n = 2$ . Let  $E$  be an elliptic curve and let  $S$  be the surface from Theorem 2.1. Choose  $i, j \geq 1$  such that  $i \equiv a^{0,1} - a^{1,0} \pmod{m}$  and  $j \equiv a^{1,0} \pmod{m}$ , and set  $X = S^i \times E^j$ . Then it follows from Künneth's formula that  $h^{0,1}(X) \equiv i + j \equiv a^{0,1} \pmod{m}$  and  $h^{1,0}(X) \equiv j \equiv a^{1,0} \pmod{m}$ .

Now assume  $n \geq 3$ . By Lemma 3.2, we may assume inductively that Proposition 3.1 holds in dimensions  $\leq n - 1$ . Therefore, there exists a smooth projective variety  $Y$  of dimension  $n - 1$  with outer Hodge numbers

$$h^{p,q}(Y) \equiv \begin{cases} (-1)^q, & p = 0, 0 \leq q < n - 1, \\ 0, & p = 0, q = n - 1, \\ 0, & p > 0, q = 0. \end{cases} \pmod{m}.$$

By Proposition 3.1 in dimension 2, there exists a smooth projective surface  $S$  with outer Hodge numbers  $h^{1,0}(S) \equiv h^{2,0}(S) \equiv h^{0,2}(S) \equiv 0 \pmod{m}$  and  $h^{0,1}(S) \equiv 1 \pmod{m}$ . The Künneth formula shows that  $S \times Y$  has outer Hodge numbers  $h^{p,q}(S \times Y) \equiv 0 \pmod{m}$  for  $(p, q) \in J_{n-1}$ , except  $h^{0,0}(S \times Y) = 1$  and  $h^{0,n-1}(S \times Y) \equiv (-1)^n \pmod{m}$ .

Finally, by Proposition 3.1 in dimension  $n - 1$ , there exists a smooth projective variety  $Z$  with outer Hodge numbers given by

$$h^{p,q}(Z) \equiv \begin{cases} a^{p,q}, & (p, q) \in J_{n-1} \setminus \{(0, n-1)\}, \\ a^{n-1,0}, & (p, q) = (0, n-1). \end{cases} \pmod{m}$$

Taking  $X = Z \times (S \times Y)^i$  for  $i \geq 0$  gives outer Hodge numbers

$$h^{p,q}(X) \equiv \begin{cases} a^{p,q}, & (p, q) \in J_{n-1} \setminus \{(0, n-1)\}, \\ a^{n-1,0} + (-1)^n i, & (p, q) = (0, n-1). \end{cases} \pmod{m}$$

Taking  $i \equiv (-1)^n (a^{0,n-1} - a^{n-1,0}) \pmod{m}$  gives the result.  $\square$

#### 4. INNER HODGE NUMBERS

The aim of this section is to prove Theorem 2, i. e. to modify the inner Hodge numbers of a smooth projective  $k$ -variety via successive blowups. We first show how to produce certain subvarieties with asymmetric Hodge numbers that we will blow up later.

**Lemma 4.1.** *Let  $X$  be a smooth projective  $k$ -variety of dimension  $n$ , let  $b, c \in \mathbf{Z}$ , and let  $d \in \{2, \dots, n - 2\}$ . Then there exists a smooth projective variety  $\tilde{X}$  and a birational morphism  $\tilde{X} \rightarrow X$  obtained as a composition of blowups in smooth centres that does not change the Hodge numbers modulo  $m$  such that  $\tilde{X}$  contains a smooth subvariety  $W$  of dimension  $d$  satisfying*

$$h^{d,0}(W) = h^{0,d}(W) \equiv 0 \pmod{m} \tag{4.1}$$

and

$$h^{d-1,0}(W) \equiv b, \quad h^{0,d-1}(W) \equiv c \pmod{m}. \tag{4.2}$$

*Proof.* Let  $X_1 \rightarrow X$  be the blowup of  $X$  in a point. The assumption on  $d$  implies  $n \geq 4$ , so the exceptional divisor of  $X_1$  contains  $\mathbf{P}^3$ . By [Proposition 3.1](#), there exists a smooth projective surface  $\tilde{S}$  such that  $h^{2,0}(\tilde{S}) = h^{0,2}(\tilde{S}) \equiv 0 \pmod{m}$  and

$$h^{1,0}(\tilde{S}) \equiv b, \quad h^{0,1}(\tilde{S}) \equiv c \pmod{m}.$$

Choose a possibly singular surface  $S' \subseteq \mathbf{P}^3$  birational to  $\tilde{S}$ . By embedded resolution of surfaces [[Abh66](#), Thm. 9.1.3] (see also [[Cut09](#), Thm. 1.2]), there exists a birational morphism  $X_2 \rightarrow X_1$  obtained as a composition of blowups in smooth centres contained in  $\mathbf{P}^3$  such that the strict transform  $S$  of  $S'$  is smooth. Since  $S$  is also birational to  $\tilde{S}$ , we have  $h^{2,0}(S) = h^{0,2}(S) \equiv 0 \pmod{m}$  and

$$h^{1,0}(S) \equiv b, \quad h^{0,1}(S) \equiv c \pmod{m}.$$

Now consider the blowup  $X_3 \rightarrow X_2$  in  $S$ . The exceptional divisor is a projective bundle  $\mathbf{P}_S(\mathcal{E})$  over  $S$ . By Maruyama's theory of elementary transformations [[Mar73](#), Thm. 1.12] (see also [[Mar82](#), Thm. 2.3]), there exists a diagram

$$\begin{array}{ccc} & \tilde{P} & \\ f \swarrow & & \searrow g \\ \mathbf{P}_S(\mathcal{E}) & & S \times \mathbf{P}^{n-3}, \end{array}$$

where  $f$  and  $g$  are blowups in smooth centres  $Z(f)$  and  $Z(g)$  respectively<sup>1</sup>. Let  $Y \subseteq \mathbf{P}^{n-3}$  be a smooth hypersurface of degree  $d$  in a linear subspace  $\mathbf{P}^{d-1} \subseteq \mathbf{P}^{n-3}$ . In particular,  $Y$  satisfies  $h^{d-2,0}(Y) = h^{0,d-2}(Y) = 1$ . Then the blowup  $X_4 \rightarrow X_3$  in  $Z(f)$  contains the strict transform  $W = \widetilde{S \times Y}$  under  $g$  of  $S \times Y$ . Birational invariance of outer Hodge numbers (in the case of a blowup this is [Lemma 2.2](#)) and the Künneth formula give

$$\begin{aligned} h^{d,0}(W) &= h^{0,d}(W) = h^{d,0}(S \times Y) = h^{2,0}(S)h^{d-2,0}(Y) \equiv 0 \pmod{m}, \\ h^{d-1,0}(W) &= h^{d-1,0}(S \times Y) = h^{2,0}(S)h^{d-3,0}(Y) + h^{1,0}(S)h^{d-2,0}(Y) \equiv b \pmod{m}, \\ h^{0,d-1}(W) &= h^{0,d-1}(S \times Y) = h^{0,2}(S)h^{0,d-3}(Y) + h^{0,1}(S)h^{0,d-2}(Y) \equiv c \pmod{m}. \end{aligned}$$

Blowing up  $m-1$  more points coming from  $X$  and repeating the above construction  $m-1$  more times in each exceptional  $\mathbf{P}^{n-1}$  separately, the blowup formula of [Lemma 2.2](#) shows that the Hodge numbers of  $X$  do not change modulo  $m$ .  $\square$

**Corollary 4.2.** *Let  $X$  be a smooth projective  $k$ -variety of dimension  $n$ , let  $b, c \in \mathbf{Z}$ , and let  $r \in \{1, \dots, n-1\}$ . Assume that  $b = c$  if  $r = 1$  or  $r = n-1$ . Then there exists a birational morphism  $\tilde{X} \rightarrow X$  obtained by a sequence of blowups in smooth centres such that*

$$h^{r,1}(\tilde{X}) \equiv h^{r,1}(X) + b, \quad h^{1,r}(\tilde{X}) \equiv h^{1,r}(X) + c \pmod{m}$$

and

$$h^{p,1}(\tilde{X}) \equiv h^{p,1}(X), \quad h^{1,p}(\tilde{X}) \equiv h^{1,p}(X) \pmod{m}$$

for all  $p > r$ .

<sup>1</sup>In fact,  $Z(f)$  and  $Z(g)$  are projective subbundles of relative dimensions 0 and  $n-4$  inside  $\mathbf{P}_C(\mathcal{E}|_C)$  and  $C \times \mathbf{P}^{n-3}$  respectively, where  $C \subseteq S$  is a smooth curve. However, we won't use this.



*Proof.* If  $r \in \{2, \dots, n-2\}$ , then [Lemma 4.1](#) shows that there exists a successive blowup  $X' \rightarrow X$  that does not change the Hodge numbers modulo  $m$  such that  $X'$  contains a subvariety  $W$  of dimension  $r$  satisfying [\(4.1\)](#) and [\(4.2\)](#). Letting  $\tilde{X} \rightarrow X'$  be the blowup in  $W$  gives the result by [Lemma 2.2](#).

For  $r = 1$ , we consider the blowup in  $i \geq 0$  points where  $i \equiv b = c \pmod{m}$ . Then the statement follows again from [Lemma 2.2](#).

For  $r = n-1$ , we first blow up  $X$  in  $i \geq 0$  points where  $i \equiv b = c \pmod{m}$ . Then, in each exceptional  $\mathbf{P}^{n-1}$  we blow up a smooth hypersurface  $Y$  of degree  $n$ . Since  $h^{n-2,0}(Y) = h^{0,n-2}(Y) = 1$ , the result follows from [Lemma 2.2](#).  $\square$

We are now able to solve the construction problem modulo  $m$  for the second outer Hodge numbers, i. e. the inner Hodge numbers  $h^{p,q}$  with  $p \in \{1, n-1\}$  or  $q \in \{1, n-1\}$ , via repeated blowups in smooth centres. By Serre duality, it is enough to consider the Hodge numbers  $h^{p,q}$  with  $(p, q) \in I_n$ , where

$$I_n = \left\{ (1, q) \mid q \in \{1, \dots, n-1\} \right\} \cup \left\{ (p, 1) \mid p \in \{1, \dots, n-1\} \right\}.$$

**Corollary 4.3.** *Let  $X$  be a smooth projective  $k$ -variety of dimension  $n$ . For any given integers  $(a^{p,q})_{(p,q) \in I_n}$  with  $a^{n-1,1} = a^{1,n-1}$ , there exists a birational morphism  $\tilde{X} \rightarrow X$  obtained by a sequence of blowups in smooth centres such that*

$$h^{p,q}(\tilde{X}) \equiv a^{p,q} \pmod{m}$$

for all  $(p, q) \in I_n$ .

*Proof.* For  $r \in \{1, \dots, n-1\}$ , let  $b = a^{r,1} - h^{r,1}(X)$  and  $c = a^{1,r} - h^{1,r}(X)$ . We see that  $b = c$  if  $r = 1$  or  $r = n-1$ . Hence, we may apply [Corollary 4.2](#) for all  $r \in \{1, \dots, n-1\}$  in descending order to obtain the result.  $\square$

Finally, we are ready to prove [Theorem 2](#), which together with [Proposition 3.1](#) implies our main result [Theorem 1](#).

*Proof of Theorem 2.* We will proceed by induction on  $n$ . The case  $n \leq 1$  is vacuous, as there are no inner Hodge numbers. Let  $n \geq 2$ , and assume the result is known in all dimensions  $\leq n-1$ . By [Corollary 4.3](#), there exists a birational morphism  $X_1 \rightarrow X$  obtained by a sequence of blowups in smooth centres such that for  $(p, q) \in I_n$  we have

$$h^{p,q}(X_1) \equiv a^{p,q} - h^{p-1,q-1}(\mathbf{P}^{n-2}) \pmod{m}.$$

Let  $X_2 \rightarrow X_1$  be the blowup in a point, and let  $\mathbf{P}^{n-2} \subseteq X_2$  be a hyperplane in the exceptional divisor. By the induction hypothesis, there exists a birational morphism  $\tilde{P} \rightarrow \mathbf{P}^{n-2}$  obtained by a sequence of blowups in smooth centres such that the Hodge numbers of  $\tilde{P}$  are given by

$$h^{p,q}(\tilde{P}) \equiv \begin{cases} h^{p,q}(\mathbf{P}^{n-2}), & p \in \{0, n-2\} \text{ or } q \in \{0, n-2\}, \\ a^{p+1,q+1} - h^{p+1,q+1}(X_1), & \text{else.} \end{cases} \pmod{m}$$

Since  $\tilde{P} \rightarrow \mathbf{P}^{n-2}$  is a sequence of blowups in smooth centres, we can blow up the (strict transforms of) the same centres in  $X_2$  to get a birational morphism  $X_3 \rightarrow X_2$  such that

the strict transform of  $\mathbf{P}^{n-2}$  is  $\tilde{P}$ . Blowing up  $m - 1$  more points coming from  $X_1$  and applying the same construction in each of the exceptional divisors separately gives a birational morphism  $X_4 \rightarrow X_1$  that does not change the Hodge numbers modulo  $m$  by the blowup formula of [Lemma 2.2](#). Finally, if we let  $\tilde{X} \rightarrow X_4$  be the blowup in one of the  $\tilde{P}$  obtained in this way, we get

$$h^{p,q}(\tilde{X}) = h^{p,q}(X_1) + h^{p-1,q-1}(\tilde{P}) \equiv a^{p,q} \pmod{m}$$

for all  $(p, q)$  with  $1 \leq p, q \leq n - 1$ , which finishes the induction step.  $\square$

**Remark 4.4.** The proof above can be simplified if one assumes embedded resolution of singularities in arbitrary dimension. Indeed, by blowing up a finite number of points, we may assume that  $h^{1,1}(X) \equiv a^{1,1} - 1 \pmod{m}$  and  $X$  contains  $\mathbf{P}^{n-1}$ . Now we claim that we can construct an  $(n - 2)$ -dimensional subvariety  $Y$  in a blowup  $X' \rightarrow X$  with  $h^{p,q}(X') \equiv h^{p,q}(X) \pmod{m}$  such that  $h^{p,q}(Y) \equiv a^{p+1,q+1} - h^{p+1,q+1}(X) \pmod{m}$ . Then the blowup  $\tilde{X} \rightarrow X'$  in  $Y$  has the required Hodge numbers.

To construct  $Y$ , first construct any smooth projective variety  $Z$  of dimension  $n - 2$  with the correct outer Hodge numbers using [Proposition 3.1](#). Then  $Z$  is birational to a (possibly singular) hypersurface  $Z' \subseteq \mathbf{P}^{n-1}$ . Embedded resolution of  $Z' \subseteq \mathbf{P}^{n-1}$  gives a birational map  $X' \rightarrow X$  such that the strict transform of  $Z'$  is smooth, so  $Z'$  has the desired outer Hodge numbers by [[CR11](#), Thm. 1]. By the induction hypothesis we may blow up further to get the inner Hodge numbers we want. Repeating this construction  $m - 1$  more times, as usual, gives  $h^{p,q}(X') \equiv h^{p,q}(X) \pmod{m}$ .

However, because resolution of singularities is currently unknown in positive characteristic beyond dimension 3, we have developed the above approach using embedded resolution of surfaces, Maruyama's theory of elementary transformations of projective bundles, and the fortuitous fact that the failure of Hodge symmetry is 'generated' by surfaces (see also [[vDdB20](#), Thm. 2]).

**Remark 4.5.** Both the proof of [Theorem 2](#) above (replacing [Lemma 4.1](#) by an easy case of [[PS19](#), Lem. 6]) and the alternative argument of [Remark 4.4](#) using resolution of singularities give new methods to prove the characteristic zero result [[PS19](#), Thm. 5].

Conversely, it is possible to adapt the methods of [[PS19](#), §3] to prove [Theorem 2](#), using the subvarieties from [[PS19](#), Lem. 6] as well as projective bundles over the subvarieties from [Lemma 4.1](#), but the analysis is a bit more intricate.

## 5. POLYNOMIAL RELATIONS

**Corollary 5.1.** *There are no polynomial relations among the Hodge numbers of smooth projective  $k$ -varieties of the same dimension besides the ones induced by Serre duality.*

*Proof.* Using [[PS19](#), Lem. 8], this follows from [Theorem 1](#) in the same way as [[PS19](#), Cor. 3], except that we now consider the Hodge numbers  $h^{p,q}$  with  $0 \leq p \leq q \leq n$  and  $(p, q) \neq (0, 0), (n, n)$ .  $\square$

**Corollary 5.2.** *There are no polynomial relations among the inner Hodge numbers of smooth projective  $k$ -varieties of any fixed birational equivalence class besides the ones induced by Serre duality.*

*Proof.* This follows from [Theorem 2](#) in a similar fashion.  $\square$

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