APPENDIX 8

Toolbox

10. Limits and Colimits, Products and Equalizers

Limit constructions are a very important tool in category theory. We will introduce the basic facts on limits and colimits in this section.

Definition 8.10.1. A *diagramscheme* $\mathcal{D}$ is a small category (i. e. the class of objects is a set). Let $\mathcal{C}$ be an arbitrary category. A *diagram* in $\mathcal{C}$ over the diagramscheme $\mathcal{D}$ is a covariant functor $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$.

Example 8.10.2. (for diagramschemes)

1. The empty category $\mathcal{D}$.
2. The category with precisely one object D and precisely one morphism 1_D.
3. The category with two objects D_1, D_2 and one morphism $f : D_1 \longrightarrow D_2$ (apart from the two identities).
4. The category with two objects D_1, D_2 and two morphisms $f, g : D_1 \longrightarrow D_2$ between them.
5. The category with a family of objects $(D_i | i \in I)$ and the associated identities.
6. The category with four objects $D_1, \ldots, D_4$ and morphisms f, g, h, k such that the diagram

$$\begin{tikzcd} D_1 \arrow[r, "f"] \arrow[d, "g"'] & D_2 \arrow[d, "k"] \\ D_3 \arrow[r, "h"'] & D_4 \end{tikzcd}$$

commutes, i. e. $kf = hg$.

Definition 8.10.3. Let $\mathcal{D}$ be a diagramscheme and $\mathcal{C}$ a category. Each object $C \in \mathcal{C}$ defines a *constant diagram* $\mathcal{K}_C : \mathcal{D} \longrightarrow \mathcal{C}$ with $\mathcal{K}_C(D) := C$ for all $D \in \mathcal{D}$ and $\mathcal{K}(f) := 1_C$ for all morphisms in $\mathcal{D}$. Each morphism $f : C \longrightarrow C'$ in $\mathcal{C}$ defines a *constant natural transformation* $\mathcal{K}_f : \mathcal{K}_C \longrightarrow \mathcal{K}_{C'}$ with $\mathcal{K}_f(D) = f$. This defines a *constant functor* $\mathcal{K} : \mathcal{C} \longrightarrow \mathrm{Funct}(\mathcal{D}, \mathcal{C})$ from the category $\mathcal{C}$ into the category of diagrams $\mathrm{Funct}(\mathcal{D}, \mathcal{C})$.

Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be a diagram. An object C together with a natural transformation $\pi : \mathcal{K}_C \longrightarrow \mathcal{F}$ is called a *limit* or a *projective limit* of the diagram $\mathcal{F}$ with the *projection* π if for each object $C' \in \mathcal{C}$ and for each natural transformation $\varphi : \mathcal{K}_{C'} \longrightarrow \mathcal{F}$ there is a unique morphism $f : C' \longrightarrow C$ such that

$$\begin{tikzcd} \mathcal{K}_{C'} \arrow[d, "\mathcal{K}_f"'] \arrow[dr, "\varphi"] & \\ \mathcal{K}_C \arrow[r, "\pi"'] & \mathcal{F} \end{tikzcd}$$

commutes, i. e. the diagrams

$$\begin{tikzcd} C \arrow[r, "\pi_i"] \arrow[dr, "\pi_j"'] & \mathcal{F}(D_i) \arrow[d, "\mathcal{F}(g)"] \\ & \mathcal{F}(D_j) \end{tikzcd}$$

commute for all morphisms $g : D_i \longrightarrow D_j$ in $\mathcal{D}$ (π is a natural transformation) and the diagrams

$$\begin{tikzcd} C' \arrow[d, "f"'] \arrow[dr, "\varphi_i"] & \\ C \arrow[r, "\pi_i"'] & \mathcal{F}(D_i). \end{tikzcd}$$

commute for all objects D_i in $\mathcal{D}$.

A category $\mathcal{C}$ *has limits for diagrams over a diagramscheme* $\mathcal{D}$ if for each diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ over $\mathcal{D}$ there is a limit in $\mathcal{C}$. A category $\mathcal{C}$ is called *complete* if each diagram in $\mathcal{C}$ has a limit.

Example 8.10.4. 1. Let $\mathcal{D}$ be a diagramscheme consisting of two objects D_1, D_2 and the identities. A diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ is defined by giving two objects C_1 and C_2 in $\mathcal{C}$. An object $C_1 \times C_2$ together with two morphisms $\pi_1 : C_1 \times C_2 \longrightarrow C_1$ and $\pi_2 : C_1 \times C_2 \longrightarrow C_2$ is called a *product* of the two objects if $C_1 \times C_2, \pi : \mathcal{K}_{C_1 \times C_2} \longrightarrow \mathcal{F}$ is a limit, i. e. if for each object C' in $\mathcal{C}$ and for any two morphisms $\varphi_1 : C' \longrightarrow C_1$ and $\varphi_2 : C' \longrightarrow C_2$ there is a unique morphism $f : C' \longrightarrow C_1 \times C_2$ such that

$$\begin{tikzcd} & C' \arrow[dl, "\varphi_1"'] \arrow[d, "f"] \arrow[dr, "\varphi_2"] & \\ C_1 & C_1 \times C_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C_2 \end{tikzcd}$$

commutes. The two morphisms $\pi_1 : C_1 \times C_2 \longrightarrow C_1$ and $\pi_2 : C_1 \times C_2 \longrightarrow C_2$ are called the *projections* from the product to the two factors.

2. Let $\mathcal{D}$ a diagramscheme consisting of a finite (non empty) set of objects $D_1, \ldots, D_n$ and the associated identities. A limit of a diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ is called a *finite product* of the objects $C_1 := \mathcal{F}(D_1), \ldots, C_n := \mathcal{F}(D_n)$ and is denoted by $C_1 \times \ldots \times C_n = \prod_{i=1}^n C_i$.

3. A limit over a discrete diagram (i. e. $\mathcal{D}$ has only the identities as morphisms) is called *product* of the $C_i := \mathcal{F}(D_i)$, $i \in I$ and is denoted by $\prod_I C_i$.

4. Let $\mathcal{D}$ be the empty diagramscheme and $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ the (only possible) empty diagram. The limit $C, \pi : \mathcal{K}_C \longrightarrow \mathcal{F}$ of $\mathcal{F}$ is called the *final object*. It has the property that for each object C' in $\mathcal{C}$ (the uniquely determined natural transformation $\varphi : \mathcal{K}_{C'} \longrightarrow \mathcal{F}$ does not have to be mentioned) there is a unique morphism $f : C' \longrightarrow C$. In

Set the one-point set is a final object. In **Ab**, **Gr**, **Vec** the zero group 0 is a final object.

5. Let $\mathcal{D}$ be the diagramscheme from 8.10.2 4. with two objects and two morphisms (different from the two identities). A diagram over $\mathcal{D}$ consists of two objects C_1 and C_2 and two morphisms $g, h : C_1 \to C_2$. The limit of such a diagram is called *Equalizer* of the two morphisms and is given by an object $\mathrm{Ker}(g,h)$ and a morphism $\pi_1 : \mathrm{Ker}(g,h) \to C_1$. The second morphism to C_2 arises from the composition $\pi_2 = g\pi_1 = h\pi_1$. The equalizer has the following universal property. For each object C' and each morphism $\varphi_1 : C' \to C_1$ with $g\varphi_1 = h\varphi_1 (= \varphi_2)$ there is a unique morphism $f : C' \to \mathrm{Ker}(g,h)$ with $\pi_1 f = \varphi_1$ (and thus $\pi_2 f = \varphi_2$, i. e. the diagram

$$\begin{tikzcd}
& C' \arrow[dl, "f"'] \arrow[d, "\varphi_1"] & \\
\mathrm{Ker}(g,h) \arrow[r, "\pi_1"] & C_1 \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C_2
\end{tikzcd}$$

commutes.

Problem 8.10.1. 1. Let $\mathcal{F} : \mathcal{D} \to \mathbf{Set}$ be a discrete diagram. Show that the cartesian product over $\mathcal{F}$ coincides with the categorical product.

2. Let $\mathcal{D}$ be a pair of morphisms as in 8.10.4 5. and let $\mathcal{F} : \mathcal{D} \to \mathbf{Set}$ be a diagram. Show that the set $\{x \in \mathcal{F}(D_1) | \mathcal{F}(f)(x) = \mathcal{F}(g)(x)\}$ with the inclusion map into $\mathcal{F}(D_1)$ is an equalizer of $\mathcal{F} : \mathcal{D} \to \mathbf{Set}$.

3. Let $\mathcal{F} : \mathcal{D} \to \mathbf{Set}$ be a diagram. Show that the set

$$\{(x_D | D \in \mathrm{Ob}\, D, x_D \in \mathcal{F}(D)) | \forall (f : D \to D') \in \mathcal{D} : \mathcal{F}(f)(x_D) = x_{D'}\}$$

with the projections into the single of the families is the limit of $\mathcal{F}$.

Definition 8.10.5. Let $\mathcal{F} : \mathcal{D} \to \mathcal{C}$ be a diagram. An object C and a natural transformation $\iota : \mathcal{F} \to \mathcal{K}_C$ is called *colimit* or *inductive limit* of the diagram $\mathcal{F}$ with the *injection* ι if for each object $C' \in \mathcal{C}$ and for each natural transformation $\varphi : \mathcal{F} \to \mathcal{K}_{C'}$ there is a unique morphism $f : C \to C'$ such that

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathcal{K}_C \arrow[d, "\mathcal{K}_f"] \\
& \mathcal{K}_{C'}
\end{tikzcd}$$

commutes, i. e. the diagram

$$\begin{tikzcd}
\mathcal{F}(D_i) \arrow[d, "\mathcal{F}(g)"'] \arrow[dr, "\iota_i"] & \\
\mathcal{F}(D_j) \arrow[r, "\iota_j"'] & C
\end{tikzcd}$$

commutes for all morphisms $g : D_i \longrightarrow D_j$ in $\mathcal{D}$ (ι is a natural transformation) and the diagram

$$\begin{tikzcd} \mathcal{F}(D_i) \arrow[r, "\iota_i"] \arrow[dr, "\varphi_i"'] & C \arrow[d, "f"] \\ & C'. \end{tikzcd}$$

commutes for all objects D_i in $\mathcal{D}$.

The special colimits that can be formed over the diagrams as in Example 8.10.4 are called *coproduct, initial object,* resp. *coequalizer.*

Example 8.10.6. In **Vec** the object 0 is an initial object. In $\mathbb{K}$-**Alg** the object $\mathbb{K}$ is an initial object. In Geom the one-element functor $A \mapsto \{*\}$ is a final object. In $\mathbb{K}$-**Alg** the object $\{a \in A | f(a) = g(a)\}$ is the equalizer of the two algebra homomorphisms $f : A \longrightarrow B$ and $g : A \longrightarrow B$. In $\mathbb{K}$**Alg** the cartesian (set of pairs) and the categorical products coincide.

Remark 8.10.7. A colimit of a diagram $\mathcal{C}$ is a limit of the corresponding (dual) diagram in the dual category $\mathcal{C}^{op}$. Thus theorems about limits in arbitrary categories automatically also produce (dual) theorems about colimits. However, observe that theorems about limits in a particular category (for example the category of vector spaces) translate only into theorems about colimits in the *dual* category, which most often is not too useful.

Proposition 8.10.8. *Limits and colimits of diagrams are unique up to isomorphism.*

PROOF. Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be a diagram and let C, π and $\tilde{C}, \tilde{\pi}$ be limits of $\mathcal{F}$. Then there are unique morphisms $f : \tilde{C} \longrightarrow C$ and $g : C \longrightarrow \tilde{C}$ with $\pi\mathcal{K}_f = \tilde{\pi}$ and $\tilde{\pi}\mathcal{K}_g = \pi$. This implies $\pi\mathcal{K}_{1_C} = \pi\,\mathrm{id}_{\mathcal{K}_C} = \pi = \tilde{\pi}\mathcal{K}_g = \pi\mathcal{K}_f\mathcal{K}_g = \pi\mathcal{K}_{fg}$ and analogously $\tilde{\pi}\mathcal{K}_{1_{\tilde{C}}} = \tilde{\pi}\mathcal{K}_{gf}$. Because of the uniqueness this implies $1_C = fg$ and $1_{\tilde{C}} = gf$. ∎

Remark 8.10.9. Now that we have the uniqueness of the limit resp. colimit (up to isomorphism) we can introduce a unified notation. The limit of a diagram $\mathcal{F} : \mathcal{D} \to \mathcal{C}$ will be denoted by $\varprojlim(\mathcal{F})$, the colimit by $\varinjlim(\mathcal{F})$.

Theorem 8.10.10. *If $\mathcal{C}$ has arbitrary products and equalizers then $\mathcal{C}$ has arbitrary limits, i. e. $\mathcal{C}$ is complete.*

PROOF. Let $\mathcal{D}$ be a diagramscheme and $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ a diagram. First we form the products $\prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D)$ and $\prod_{f \in \mathrm{Mor}\mathcal{D}} \mathcal{F}(\mathrm{Codom}(f))$ where $\mathrm{Codom}(f)$ is the codomain (range) of the morphism $f : D' \longrightarrow D''$ in $\mathcal{D}$ so in this case $\mathrm{Codom}(f) = D''$. We define for each morphism $f : D' \longrightarrow D''$ two morphisms as follows

$$p_f := \pi_{\mathcal{F}(D'')} : \prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \longrightarrow \mathcal{F}(D'') = \mathcal{F}(\mathrm{Codom}(f))$$

and

$$q_f := \mathcal{F}(f)\pi_{\mathcal{F}(D')} : \prod_{D\in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \to \mathcal{F}(D') \to \mathcal{F}(D'') = \mathcal{F}(\mathrm{Codom}(f)).$$

These two families of morphisms induce two morphisms into the corresponding product

$$p, q : \prod_{D\in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \to \prod_{f\in \mathrm{Mor}\mathcal{D}} \mathcal{F}(\mathrm{Codom}(f))$$

with $\pi_f q = q_f$ and $\pi_f p = p_f$. Now we show that the equalizer of these two morphisms

$$\begin{tikzcd} \mathrm{Ker}(p,q) \arrow[r, "\psi"] & \prod_{D\in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{f\in \mathrm{Mor}\mathcal{D}} \mathcal{F}(\mathrm{Codom}(f)) \end{tikzcd}$$

is the limit of the diagram $\mathcal{F} : \mathcal{D} \to \mathcal{C}$. We have $p\psi = q\psi$. The morphism $\rho(D) := \pi_{\mathcal{F}(D)}\psi : \mathrm{Ker}(p,q) \to \prod_{D\in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \to \mathcal{F}(D)$ defines a family of morphisms for $D \in \mathrm{Ob}\,\mathcal{D}$. If $f : D' \to D''$ is in $\mathcal{D}$ then the diagram

$$\begin{tikzcd} \mathrm{Ker}(p,q) \arrow[r, "\rho(D')"] \arrow[dr, "\rho(D'')"'] & \mathcal{F}(D') \arrow[d, "\mathcal{F}(f)"] \\ & \mathcal{F}(D'') \end{tikzcd}$$

is commutative because of $\mathcal{F}(f)\rho(D') = \mathcal{F}(f)\pi_{\mathcal{F}(D')}\psi = q_f\psi = \pi_f q\psi = \pi_f p\psi = p_f\psi = \pi_{\mathcal{F}(D'')}\psi = \rho(D'')$. Thus we have obtained a natural transformation $\rho : \mathcal{K}_{\mathrm{Ker}(p,q)} \to \mathcal{F}$.

Now let an object C' and a natural transformation $\varphi : \mathcal{K}_{C'} \to \mathcal{F}$ be given. Then this defines a unique morphism $g : C' \to \prod_{D\in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D)$ with $\pi_{\mathcal{F}(D)} g = \varphi(D)$ for all $D \in \mathcal{D}$. Since φ is a natural transformation we have $\varphi(D'') = \mathcal{F}(f)\varphi(D')$ for each morphism $f : D' \to D''$. Thus we obtain $\pi_f pg = p_f g = \pi_{\mathcal{F}(D'')} g = \varphi(D'') = \mathcal{F}(f)\varphi(D') = \mathcal{F}(f)\pi_{\mathcal{F}(D')} g = q_f g = \pi_f qg$ for all morphisms $f : D' \to D''$ hence $pg = qg$. Thus g can be uniquely factorized through the equalizer $\psi : \mathrm{Ker}(p,q) \to \prod_{D\in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D)$ in the form $g = \psi h$ with $h : C' \to \mathrm{Ker}(p,q)$. Then we have $\rho(D)h = \pi_{\mathcal{F}(D)}\psi h = \pi_{\mathcal{F}(D)} g = \varphi(D)$ for all $D \in \mathcal{D}$ hence $\rho\mathcal{K}_h = \varphi$.

Finally let another morphism $h' : C' \to \mathrm{Ker}(p,q)$ with $\rho\mathcal{K}_{h'} = \varphi$ be given. Then we have $\pi_{\mathcal{F}(D)}\psi h' = \rho(D)h' = \varphi(D) = \rho(D)h = \pi_{\mathcal{F}(D)}\psi h$ hence $\psi h' = \psi h = g$. Because of the uniqueness of the factorization of g through ψ we get $h = h'$. Thus $(\mathrm{Ker}(p,q), \rho)$ is the limit of $\mathcal{F}$. ∎

Remark 8.10.11. The proof of the preceding Theorem gives an explicit construction of the limit of $\mathcal{F}$ as an equalizer

$$\begin{tikzcd} \mathrm{Ker}(p,q) \arrow[r, "\psi"] & \prod_{D\in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{f\in \mathrm{Mor}\mathcal{D}} \mathcal{F}(\mathrm{Codom}(f)) \end{tikzcd}$$

Hence the limit can be represented as a subobject of a suitable product. Dually the colimit can be represented as a quotient object of a suitable coproduct. This construction will be used in chapter 3.

Another fact is very important for us, the fact that certain functors preserve limits resp. colimits. We say that a functor $\mathcal{G} : \mathcal{C} \longrightarrow \mathcal{C}'$ *preserves* limits over the diagramscheme $\mathcal{D}$ if $\varprojlim(\mathcal{G}\mathcal{F}) \cong \mathcal{G}(\varprojlim(\mathcal{F}))$ for each diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$.

Proposition 8.10.12. *Covariant representable functors preserve limits. Contravariant representable functors map colimits into limits.*

PROOF. We only prove the first assertion. The second assertion is dual to the first one. For a diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathbf{Set}$ the set

$$\{(x_D | D \in \operatorname{Ob}\mathcal{D}, x_D \in \mathcal{F}(D)) | \forall (f : D \longrightarrow D') \in \mathcal{D} : \mathcal{F}(f)(x_D) = x_{D'}\}$$

is a limit of $\mathcal{F}$ by Problem 8.1. Now let a diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be given and let $\varprojlim(\mathcal{F})$ be the limit. Furthermore let $\operatorname{Mor}_{\mathcal{C}}(C', \text{-}) : \mathcal{C} \longrightarrow \mathbf{Set}$ be a representable functor. By the definition of the limit of $\mathcal{F}$ there is a unique morphism $f : C' \longrightarrow \varprojlim(\mathcal{F})$ with $\pi\mathcal{K}_f = \varphi$ for each natural transformation $\varphi : \mathcal{K}_{C'} \longrightarrow \mathcal{F}$. This defines an isomorphism $\operatorname{Nat}(\mathcal{K}_{C'}, \mathcal{F}) \cong \operatorname{Mor}_{\mathcal{C}}(C', \varprojlim(\mathcal{F}))$. Hence we have

$$\begin{aligned} &\varprojlim(\operatorname{Mor}_{\mathcal{C}}(C, \mathcal{F})) \cong \\ &\{(\varphi(D) : C' \longrightarrow \mathcal{F}(D) | D \in \mathcal{D}) | \forall (f : D \longrightarrow D') \in \mathcal{D} : \mathcal{F}(f)\varphi(D) = \varphi(D')\} \\ &= \operatorname{Nat}(\mathcal{K}_{C'}, \mathcal{F}) \cong \operatorname{Mor}_{\mathcal{C}}(C', \varprojlim(\mathcal{F})). \blacksquare \end{aligned}$$

Corollary 8.10.13. *Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{C}'$ be left adjoint to $\mathcal{G} : \mathcal{C}' \longrightarrow \mathcal{C}$. Then $\mathcal{F}$ preserves colimits and $\mathcal{G}$ preserves limits.*

PROOF. For a diagram $\mathcal{H} : \mathcal{D} \longrightarrow \mathcal{C}$ we have

$$\begin{aligned} \operatorname{Mor}_{\mathcal{C}}(\text{-}, \varprojlim(\mathcal{G}\mathcal{H})) \cong \varprojlim \operatorname{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\mathcal{H}) \cong \varprojlim \operatorname{Mor}_{\mathcal{C}'}(\mathcal{F}\text{-}, \mathcal{H}) \cong \\ \operatorname{Mor}_{\mathcal{C}'}(\mathcal{F}\text{-}, \varprojlim(\mathcal{H})) \cong \operatorname{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}(\varprojlim(\mathcal{H}))), \end{aligned}$$

hence $\varprojlim(\mathcal{G}\mathcal{H}) \cong \mathcal{G}(\varprojlim(\mathcal{H}))$ as representing objects. The proof for the left adjoint functor is analogous. ∎