CHAPTER 8

Toolbox

8. Representable Functors

Definition 8.8.1. Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathbf{Set}$ be a covariant functor. A pair (A, x) with $A \in \mathcal{C}, x \in \mathcal{F}(A)$ is called a *representing (generic, universal) object* for $\mathcal{F}$ and $\mathcal{F}$ is called a *representable functor*, if for each $B \in \mathcal{C}$ and $y \in \mathcal{F}(B)$ there exists a unique $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ such that $\mathcal{F}(f)(x) = y$:

$$\begin{tikzcd} A \arrow[d, "f"'] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(f)"'] & \ni & x \arrow[d, mapsto] \\ B & \mathcal{F}(B) & \ni & y \end{tikzcd}$$

Proposition 8.8.2. *Let (A, x) and (B, y) be representing objects for $\mathcal{F}$. Then there exists a unique isomorphism $f : A \longrightarrow B$ such that $\mathcal{F}(f)(x) = y$.*

$$\begin{tikzcd} A \arrow[d, "h"] \arrow[dd, bend right=60, "1_A"'] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(h)"'] \arrow[dd, bend left=60, "1_{\mathcal{F}(A)}"] & x \arrow[d, mapsto] \\ B \arrow[d, "k"] \arrow[dd, bend right=60, "1_B"'] & \mathcal{F}(B) \arrow[d, "\mathcal{F}(k)"'] \arrow[dd, bend left=60, "1_{\mathcal{F}(B)}"] & y \arrow[d, mapsto] \\ A \arrow[d, "h"] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(h)"'] & x \arrow[d, mapsto] \\ B & \mathcal{F}(B) & y \end{tikzcd}$$

Examples 8.8.3. 1. Let $X \in \mathbf{Set}$ and let R be a ring. $\mathcal{F} : R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Set}$, $\mathcal{F}(M) := \mathrm{Map}(X, M)$ is a covariant functor. A representing object for $\mathcal{F}$ is given by $(RX, x : X \longrightarrow RX)$ with the property, that for all $(M, y : X \longrightarrow M)$ there exists a unique $f \in \mathrm{Hom}_R(RX, M)$ such that $\mathcal{F}(f)(x) = \mathrm{Map}(X, f)(x) = fx = y$

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[dr, "y"'] & RX \arrow[d, "f"] \\ & M. \end{tikzcd}$$

2. Given modules M_R and ${}_RN$. Define $\mathcal{F} : \mathbf{Ab} \longrightarrow \mathbf{Set}$ by $\mathcal{F}(A) := \mathrm{Bil}_R(M, N; A)$. Then $\mathcal{F}$ is a covariant functor. A representing object for $\mathcal{F}$ is given by $(M \otimes_R N, \otimes : M \times N \longrightarrow M \otimes_R N)$ with the property that for all $(A, f : M \times N \longrightarrow A)$ there exists

a unique $g \in \mathrm{Hom}(M \otimes_R N, A)$ such that $\mathcal{F}(g)(\otimes) = \mathrm{Bil}_R(M, N; g)(\otimes) = g\otimes = f$

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[dr, "f"'] & M \otimes_R N \arrow[d, "g"] \\ & A. \end{tikzcd}$$

3. Given a $\mathbb{K}$-module V. Define $\mathcal{F} : \mathbf{Alg} \to \mathbf{Set}$ by $\mathcal{F}(A) := \mathrm{Hom}(V, A)$. Then $\mathcal{F}$ is a covariant functor. A representing object for $\mathcal{F}$ is given by $(T(V), \iota : V \to T(V))$ with the property that for all $(A, f : V \to A)$ the exists a unique $g \in \mathrm{Mor}_{\mathbf{Alg}}(T(V), A)$ such that $\mathcal{F}(g)(\iota) = \mathrm{Hom}(V, g)(\iota) = g\iota = f$

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & T(V) \arrow[d, "g"] \\ & A. \end{tikzcd}$$

4. Given a $\mathbb{K}$-module V. Define $\mathcal{F} : \mathbf{cAlg} \to \mathbf{Set}$ by $\mathcal{F}(A) := \mathrm{Hom}(V, A)$. Then $\mathcal{F}$ is a covariant functor. A representing object for $\mathcal{F}$ is given by $(S(V), \iota : V \to S(V))$ with the property that for all $(A, f : V \to A)$ the exists a unique $g \in \mathrm{Mor}_{\mathbf{Alg}}(S(V), A)$ such that $\mathcal{F}(g)(\iota) = \mathrm{Hom}(V, g)(\iota) = g\iota = f$

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & S(V) \arrow[d, "g"] \\ & A. \end{tikzcd}$$

Proposition 8.8.4. *$\mathcal{F}$ has a representing object (A, a) if and only if there is a natural isomorphism $\varphi : \mathcal{F} \cong \mathrm{Mor}_{\mathcal{C}}(A, -)$ (with $a = \varphi(A)^{-1}(1_A)$).*

PROOF. $\Longrightarrow$: The map

$$\varphi(B) : \mathcal{F}(B) \ni y \mapsto f \in \mathrm{Mor}_{\mathcal{C}}(A, B) \text{ with } \mathcal{F}(f)(a) = y$$

is bijective with the inverse map

$$\psi(B) : \mathrm{Mor}_{\mathcal{C}}(A, B) \ni f \mapsto \mathcal{F}(f)(a) \in \mathcal{F}(B).$$

In fact we have $y \mapsto f \mapsto \mathcal{F}(f)(a) = y$ and $f \mapsto y := \mathcal{F}(f)(a) \mapsto g : \mathcal{F}(g)(a) = y = \mathcal{F}(f)(a)$. By uniqueness we get $f = g$. Hence all $\varphi(B)$ are bijective with inverse map $\psi(B)$. It is sufficient to show that ψ is a natural transformation.

Given $g : B \to C$. Then the following diagram commutes

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(A,B) \arrow[r, "\psi(B)"] \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(A,g)"'] & \mathcal{F}(B) \arrow[d, "\mathcal{F}(g)"] \\
\mathrm{Mor}_{\mathcal{C}}(A,C) \arrow[r, "\psi(C)"'] & \mathcal{F}(C)
\end{tikzcd}$$

since $\psi(C)\mathrm{Mor}_{\mathcal{C}}(A,g)(f) = \psi(C)(gf) = \mathcal{F}(gf)(a) = \mathcal{F}(g)\mathcal{F}(f)(a) = \mathcal{F}(g)\psi(B)(f)$.

$\Leftarrow$: Let A be given. Let $a := \varphi(A)^{-1}(1_A)$. For $y \in \mathcal{F}(B)$ we get $y = \varphi(B)^{-1}(f) = \varphi(B)^{-1}(f1_A) = \varphi(B)^{-1}\mathrm{Mor}_{\mathcal{C}}(A,f)(1_A) = \mathcal{F}(f)\varphi(A)^{-1}(1_A) = \mathcal{F}(f)(a)$ for a uniquely determined $f \in \mathrm{Mor}_{\mathcal{C}}(A,B)$. $\square$

Proposition 8.8.5. *Given a representable functor $\mathcal{F}_X : \mathcal{C} \to$ **Set** for each $X \in \mathcal{D}$. Given a natural transformation $\mathcal{F}_g : \mathcal{F}_Y \to \mathcal{F}_X$ for each $g : X \to Y$ (contravariant!) such that $\mathcal{F}$ depends functorially on X, i.e. $\mathcal{F}_{1_X} = 1_{\mathcal{F}_X}, \mathcal{F}_{hg} = \mathcal{F}_g\mathcal{F}_h$. Then the representing objects (A_X, a_X) for $\mathcal{F}_X$ depend functorially on X, i.e. for each $g : X \to Y$ there is a unique homomorphism $A_g : A_X \to A_Y$ (with $\mathcal{F}_X(A_g)(a_X) = \mathcal{F}_g(A_Y)(a_Y)$) and the following identities hold $A_{1_X} = 1_{A_X}, A_{hg} = A_hA_g$.*

Proof. Choose a representing object (A_X, a_X) for $\mathcal{F}_X$ for each $X \in \mathcal{C}$ (by the axiom of choice). Then there is a unique homomorphism $A_g : A_X \to A_Y$ with

$$\mathcal{F}_X(A_g)(a_X) = \mathcal{F}_g(A_Y)(a_Y) \in \mathcal{F}_X(A_Y),$$

for each $g : X \to Y$ because $\mathcal{F}_g(A_Y) : \mathcal{F}_Y(A_Y) \to \mathcal{F}_X(A_Y)$ is given. We have $\mathcal{F}_X(A_1)(a_X) = \mathcal{F}_1(A_X)(a_X) = a_X = \mathcal{F}_X(1)(a_X)$ hence $A_1 = 1$, and $\mathcal{F}_X(A_{hg})(a_X) = \mathcal{F}_{hg}(A_Z)(a_Z) = \mathcal{F}_g(A_Z)\mathcal{F}_h(A_Z)(a_Z) = \mathcal{F}_g(A_Z)\mathcal{F}_Y(A_h)(a_Y) = \mathcal{F}_X(A_h)\mathcal{F}_g(A_Y)(a_Y) = \mathcal{F}_X(A_h)\mathcal{F}_X(A_g)(a_X) = \mathcal{F}_X(A_hA_g)(a_X)$ hence $A_hA_g = A_{hg}$ for $g : X \to Y$ and $h : Y \to Z$ in $\mathcal{D}$. $\square$

Corollary 8.8.6. 1. $\mathrm{Map}(X,M) \cong \mathrm{Hom}_R(RX,M)$ *is a natural transformation in M (and in X!). In particular* **Set** $\ni X \mapsto RX \in R$-**Mod** *is a functor.*

2. $\mathrm{Bil}_R(M,N;A) \cong \mathrm{Hom}(M \otimes_R N, A)$ *is a natural transformation in A (and in $(M,N) \in$* **Mod**-$R \times R$-**Mod**). *In particular* **Mod**-$R \times R$-**Mod** $\ni M,N \mapsto M \otimes_r N \in$ **Ab** *is a functor.*

3. R-**Mod**-$S \times S$-**Mod**-$T \ni (M,N) \mapsto M \otimes_S N \in R$-**Mod**-$T$ *is a functor.*