CHAPTER 8

Toolbox

7. Bialgebras

Definition 8.7.1. 1. A *bialgebra* $(B, \nabla, \eta, \Delta, \epsilon)$ consists of an algebra (B, ∇, η) and a coalgebra (B, Δ, ϵ) such that the diagrams

$$\begin{tikzcd}
B \otimes B \arrow[r, "\Delta\otimes\Delta"] \arrow[dd, "\nabla"'] & B \otimes B \otimes B \otimes B \arrow[dr, "1\otimes\tau\otimes 1"] & \\
& & B \otimes B \otimes B \otimes B \arrow[d, "\nabla\otimes\nabla"] \\
B \arrow[rr, "\Delta"] & & B \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \mathbb{K} \arrow[dl, "\eta"'] \arrow[dr, "\eta\otimes\eta"] & \\
B \arrow[rr, "\Delta"] & & B \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \otimes B \arrow[rr, "\nabla"] \arrow[dr, "\epsilon\otimes\epsilon"'] & & B \arrow[dl, "\epsilon"] \\
& \mathbb{K} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K} \arrow[rr, "\mathrm{id}"] \arrow[dr, "\eta"'] & & \mathbb{K} \\
& B \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

commute, i.e. Δ and ϵ are homomorphisms of algebras resp. ∇ and η are homomorphisms of coalgebras.

2. Given bialgebras A and B. A map $f : A \longrightarrow B$ is called a *homomorphism of bialgebras* if it is a homomorphism of algebras and a homomorphism of coalgebras.

3. The category of bialgebras is denoted by $\mathbb{K}$-**Bialg**.

Problem 8.7.1. 1. Let (B, ∇, η) be an algebra and (B, Δ, ε) be a coalgebra. The following are equivalent:

a) $(B, \nabla, \eta, \Delta, \varepsilon)$ is a bialgebra.
b) $\Delta : B \longrightarrow B \otimes B$ and $\varepsilon : B \longrightarrow \mathbb{K}$ are homomorphisms of $\mathbb{K}$-algebras.
c) $\nabla : B \otimes B \longrightarrow B$ and $\eta : \mathbb{K} \longrightarrow B$ are homomorphisms of $\mathbb{K}$-coalgebras.

2. Let B be a finite dimensional bialgebra over field $\mathbb{K}$. Show that the dual space B^* is a bialgebra.

One of the most important properties of bialgebras B is that the tensor product over $\mathbb{K}$ of two B-modules or two B-comodules is again a B-module.

Proposition 8.7.2. 1. *Let B be a bialgebra. Let M and N be left B-modules. Then $M \otimes_{\mathbb{K}} N$ is a B-module by the map*

$$B \otimes M \otimes N \xrightarrow{\Delta\otimes 1} B \otimes B \otimes M \otimes N \xrightarrow{1\otimes\tau\otimes 1} B \otimes M \otimes B \otimes N \xrightarrow{\mu\otimes\mu} M \otimes N.$$

2. *Let B be a bialgebra. Let M and N be left B-comodules. Then $M \otimes_{\mathbb{K}} N$ is a B-comodule by the map*

$$M \otimes N \xrightarrow{\delta\otimes\delta} B \otimes M \otimes B \otimes N \xrightarrow{1\otimes\tau\otimes 1} B \otimes B \otimes M \otimes N \xrightarrow{\nabla\otimes 1} B \otimes M \otimes N.$$

3. $\mathbb{K}$ *is a B-module by the map* $B \otimes \mathbb{K} \cong B \xrightarrow{\varepsilon} \mathbb{K}$.
4. $\mathbb{K}$ *is a B-comodule by the map* $\mathbb{K} \xrightarrow{\eta} B \cong B \otimes \mathbb{K}$.

PROOF. We give a diagrammatic proof for 1. The associativity law is given by

$$\begin{tikzcd}
B\otimes B\otimes M\otimes N \arrow[r, "1\otimes\Delta\otimes1\otimes1"] \arrow[ddd, "\nabla\otimes1\otimes1"'] & B\otimes B\otimes B\otimes M\otimes N \arrow[r, "1\otimes1\otimes\tau\otimes1"] \arrow[d, "\Delta\otimes1\otimes1\otimes1\otimes1"] & B\otimes B\otimes M\otimes B\otimes N \arrow[r, "1\otimes\mu\otimes\mu"] \arrow[d, "\Delta\otimes1\otimes1\otimes1\otimes1"] & B\otimes M\otimes N \arrow[d, "\Delta\otimes1\otimes1"] \\
 & B\otimes B\otimes B\otimes B\otimes M\otimes N \arrow[r, "1\otimes1\otimes1\otimes\tau\otimes1"] \arrow[d, "1\otimes\tau\otimes1\otimes1\otimes1"] & B\otimes B\otimes B\otimes M\otimes B\otimes N \arrow[r, "1\otimes1\otimes\mu\otimes\mu"] \arrow[d, "1\otimes\tau(B,B\otimes M)\otimes1\otimes1"] & B\otimes B\otimes M\otimes N \arrow[d, "1\otimes\tau\otimes1"] \\
 & B\otimes B\otimes B\otimes B\otimes M\otimes N \arrow[r, "1\otimes1\otimes\tau(B\otimes B,M)\otimes1"] \arrow[d, "\nabla\otimes\nabla\otimes1\otimes1"] & B\otimes B\otimes M\otimes B\otimes B\otimes N \arrow[r, "1\otimes\mu\otimes1\otimes\mu"] \arrow[d, "\nabla\otimes1\otimes\nabla\otimes1"] & B\otimes M\otimes B\otimes N \arrow[d, "\mu\otimes\mu"] \\
B\otimes M\otimes N \arrow[r, "\Delta\otimes1\otimes1"] & B\otimes B\otimes M\otimes N \arrow[r, "1\otimes\tau\otimes1"] & B\otimes M\otimes B\otimes N \arrow[r, "\mu\otimes\mu"] & M\otimes N
\end{tikzcd}$$

The unit law is the commutativity of

$$\begin{tikzcd}
M\otimes N \cong \arrow[dd, "="'] & \mathbb{K}\otimes M\otimes N \arrow[r, "\eta\otimes1\otimes1"] \arrow[d, "\cong"] & B\otimes M\otimes N \arrow[d, "\Delta\otimes1\otimes1"] \\
 & \mathbb{K}\otimes\mathbb{K}\otimes M\otimes N \arrow[r, "\eta\otimes\eta\otimes1\otimes1"] \arrow[d, "1\otimes\tau\otimes1"] & B\otimes B\otimes M\otimes N \arrow[d, "1\otimes\tau\otimes1"] \\
M\otimes N \cong & \mathbb{K}\otimes M\otimes\mathbb{K}\otimes N \arrow[r, "\eta\otimes1\otimes\eta\otimes1"] \arrow[dr, "1"'] & B\otimes M\otimes B\otimes N \arrow[d, "\mu\otimes\mu"] \\
 & & M\otimes N
\end{tikzcd}$$

The corresponding properties for comodules follows from the dualized diagrams. The module and comodule properties of $\mathbb{K}$ are easily checked. $\square$

Definition 8.7.3. 1. Let $(B, \nabla, \eta, \Delta, \epsilon)$ be a bialgebra. Let A be a left B-module with structure map $\mu : B \otimes A \to A$. Let furthermore (A, ∇_A, η_A) be an algebra such that ∇_A and η_A are homomorphisms of B-modules. Then $(A, \nabla_A, \eta_A, \mu)$ is called a *B-module algebra.*

2. Let $(B, \nabla, \eta, \Delta, \epsilon)$ be a bialgebra. Let C be a left B-module with structure map $\mu : B \otimes C \to C$. Let furthermore $(C, \Delta_C, \varepsilon_C)$ be a coalgebra such that Δ_C and ε_C are homomorphisms of B- modules. Then $(C, \Delta_C, \varepsilon_C, \mu)$ is called a *B-module coalgebra.*

3. Let $(B, \nabla, \eta, \Delta, \epsilon)$ be a bialgebra. Let A be a left B-comodule with structure map $\delta : A \to B \otimes A$. Let furthermore (A, ∇_A, η_A) be an algebra such that ∇_A and η_A are homomorphisms of B-comodules. Then $(A, \nabla_A, \eta_A, \delta)$ is called a *B-comodule algebra.*

4. Let $(B, \nabla, \eta, \Delta, \epsilon)$ be a bialgebra. Let C be a left B-comodule with structure map $\delta : C \to B \otimes C$. Let furthermore $(C, \Delta_C, \varepsilon_C)$ be a coalgebra such that Δ_C and ε_C are homomorphisms of B-comodules. Then $(C, \Delta_C, \varepsilon_C, \delta)$ is called a *B-comodule coalgebra*.

Remark 8.7.4. If $(C, \Delta_C, \varepsilon_C)$ is a $\mathbb{K}$-coalgebra and (C, μ) is a B-module, then $(C, \Delta_C, \varepsilon_C, \mu)$ is a B-module coalgebra iff μ is a homomorphism of $\mathbb{K}$-coalgebras.

If (A, ∇_A, η_A) is a $\mathbb{K}$-algebra and (A, δ) is a B-comodule, then $(A, \nabla_A, \eta_A, \delta)$ is a B-comodule algebra iff δ is a homomorphism of $\mathbb{K}$-algebras.

Similar statement for module algebras or comodule coalgebras do *not* hold.