CHAPTER 8

Toolbox

6. Coalgebras

Definition 8.6.1. A $\mathbb{K}$-*coalgebra* is a $\mathbb{K}$-module C together with a *comultiplication* or *diagonal* $\Delta : C \longrightarrow C \otimes C$ that is coassociative:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes \mathrm{id}"] \\ C \otimes C \arrow[r, "\mathrm{id} \otimes \Delta"'] & C \otimes C \otimes C \end{tikzcd}$$

and a *counit* or *augmentation* $\epsilon : C \longrightarrow \mathbb{K}$:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "\mathrm{id}"] & C \otimes C \arrow[d, "\mathrm{id} \otimes \epsilon"] \\ C \otimes C \arrow[r, "\epsilon \otimes \mathrm{id}"'] & \mathbb{K} \otimes C \cong C \cong C \otimes \mathbb{K}. \end{tikzcd}$$

A $\mathbb{K}$-coalgebra C is *cocommutative* if the following diagram commutes

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[rr, "\tau"] & & C \otimes C \end{tikzcd}$$

Let C and D be $\mathbb{K}$-coalgebras. A *homomorphism of coalgebras* $f : C \longrightarrow D$ is a $\mathbb{K}$-linear map such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f \otimes f"'] & D \otimes D \end{tikzcd}$$

and

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "\epsilon_C"'] & D. \arrow[d, "\epsilon_D"] \\ & \mathbb{K} \end{tikzcd}$$

Remark 8.6.2. Obviously the composition of two homomorphisms of coalgebras is again a homomorphism of coalgebras. Furthermore the identity map is a homomorphism of coalgebras. Hence the $\mathbb{K}$-coalgebras form a category $\mathbb{K}$-**Coalg**. The category of cocommutative $\mathbb{K}$-coalgebras will be denoted by $\mathbb{K}$-**cCoalg**.

Problem 8.6.1. 1. Show that $V \otimes V^*$ is a coalgebra for every finite dimensional vector space V over a field $\mathbb{K}$ if the comultiplication is defined by $\Delta(v \otimes v^*) := \sum_{i=1}^n v \otimes v_i^* \otimes v_i \otimes v^*$ where $\{v_i\}$ and $\{v_i^*\}$ are dual bases of V resp. V^*.

2. Show that the free $\mathbb{K}$-modules $\mathbb{K}X$ with the basis X and the comultiplication $\Delta(x) = x \otimes x$ is a coalgebra. What is the counit? Is the counit unique?

3. Show that $\mathbb{K} \oplus V$ with $\Delta(1) = 1 \otimes 1$, $\Delta(v) = v \otimes 1 + 1 \otimes v$ defines a coalgebra.

4. Let C and D be coalgebras. Then $C \otimes D$ is a coalgebra with the comultiplication $\Delta_{C\otimes D} := (1_C \otimes \tau \otimes 1_D)(\Delta_C \otimes \Delta_D) : C \otimes D \otimes C \otimes D \to C \otimes D$ and counit $\varepsilon = \varepsilon_{C\otimes D} : C \otimes D \to \mathbb{K} \otimes K \to \mathbb{K}$. (The proof is analogous to the proof of Lemma 8.5.3.)

To describe the comultiplication of a $\mathbb{K}$-coalgebra in terms of elements we introduce a notation first introduced by Sweedler similar to the notation $\nabla(a \otimes b) = ab$ used for algebras. Instead of $\Delta(c) = \sum c_i \otimes c_i'$ we write

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}.$$

Observe that only the complete expression on the right hand side makes sense, not the components $c_{(1)}$ or $c_{(2)}$ which are *not* considered as families of elements of C. This notation alone does not help much in the calculations we have to perform later on. So we introduce a more general notation.

Definition 8.6.3. (Sweedler Notation) Let M be an arbitrary $\mathbb{K}$-module and C be a $\mathbb{K}$-coalgebra. Then there is a bijection between all multilinear maps

$$f : C \times \ldots \times C \to M$$

and all linear maps

$$f' : C \otimes \ldots \otimes C \to M.$$

These maps are associated to each other by the formula

$$f(c_1, \ldots, c_n) = f'(c_1 \otimes \ldots \otimes c_n).$$

For $c \in C$ we define

$$\sum f(c_{(1)}, \ldots, c_{(n)}) := f'(\Delta^{n-1}(c)),$$

where Δ^{n-1} denotes the $n-1$-fold application of Δ, for example $\Delta^{n-1} = (\Delta \otimes 1 \otimes \ldots \otimes 1) \circ (\Delta \otimes 1) \circ \Delta$.

In particular we obtain for the bilinear map $\otimes : C \times C \ni (c, d) \mapsto c \otimes d \in C \otimes C$

$$\sum c_{(1)} \otimes c_{(2)} = \Delta(c),$$

and for the multilinear map $\otimes^2 : C \times C \times C \to C \otimes C \otimes C$

$$\sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)} = (\Delta \otimes 1)\Delta(c) = (1 \otimes \Delta)\Delta(c).$$

With this notation one verifies easily

$$\sum c_{(1)} \otimes \ldots \otimes \Delta(c_{(i)}) \otimes \ldots \otimes c_{(n)} = \sum c_{(1)} \otimes \ldots \otimes c_{(n+1)}$$

and

$$\begin{aligned}\sum c_{(1)} \otimes \ldots \otimes \epsilon(c_{(i)}) \otimes \ldots \otimes c_{(n)} &= \sum c_{(1)} \otimes \ldots \otimes 1 \otimes \ldots \otimes c_{(n-1)} \\ &= \sum c_{(1)} \otimes \ldots \otimes c_{(n-1)}\end{aligned}$$

This notation and its application to multilinear maps will also be used in more general contexts like comodules.

Proposition 8.6.4. *Let C be a coalgebra and A an algebra. Then the composition $f * g := \nabla_A(f \otimes g)\Delta_C$ defines a multiplication*

$$\mathrm{Hom}(C, A) \otimes \mathrm{Hom}(C, A) \ni f \otimes g \mapsto f * g \in \mathrm{Hom}(C, A),$$

such that $\mathrm{Hom}(C, A)$ becomes an algebra. The unit element is given by $\mathbb{K} \ni \alpha \mapsto (c \mapsto \eta(\alpha\epsilon(c))) \in \mathrm{Hom}(C, A)$.

Proof. The multiplication of $\mathrm{Hom}(C, A)$ obviously is a bilinear map. The multiplication is associative since $(f * g) * h = \nabla_A((\nabla_A(f \otimes g)\Delta_C) \otimes h)\Delta_C = \nabla_A(\nabla_A \otimes 1)((f \otimes g) \otimes h)(\Delta_C \otimes 1)\Delta_C = \nabla_A(1 \otimes \nabla_A)(f \otimes (g \otimes h))(1 \otimes \Delta_C)\Delta_C = \nabla_A(f \otimes (\nabla_A(g \otimes h)\Delta_C))\Delta_C = f * (g * h)$. Furthermore it is unitary with unit $1_{\mathrm{Hom}(C,A)} = \eta_A\epsilon_C$ since $\eta_A\epsilon_C * f = \nabla_A(\eta_A\epsilon_C \otimes f)\Delta_C = \nabla_A(\eta_A \otimes 1_A)(1_K \otimes f)(\epsilon_C \otimes 1_C)\Delta_C = f$ and similarly $f * \eta_A\epsilon_C = f$. □

Definition 8.6.5. The multiplication $* : \mathrm{Hom}(C, A) \otimes \mathrm{Hom}(C, A) \longrightarrow \mathrm{Hom}(C, A)$ is called *convolution*.

Corollary 8.6.6. *Let C be a $\mathbb{K}$-coalgebra. Then $C^* = \mathrm{Hom}_K(C, \mathbb{K})$ is an $\mathbb{K}$-algebra.*

Proof. Use that $\mathbb{K}$ itself is a $\mathbb{K}$-algebra. □

Remark 8.6.7. If we write the evaluation as $C^* \otimes C \ni a \otimes c \mapsto \langle a, c\rangle \in \mathbb{K}$ then an element $a \in C^*$ is completely determined by the values of $\langle a, c\rangle$ for all $c \in C$. So the product of a and b in C^* is uniquely determined by the formula

$$\langle a * b, c\rangle = \langle a \otimes b, \Delta(c)\rangle = \sum a(c_{(1)})b(c_{(2)}).$$

The unit element of C^* is $\epsilon \in C^*$.

Lemma 8.6.8. *Let $\mathbb{K}$ be a field and A be a finite dimensional $\mathbb{K}$-algebra. Then $A^* = \mathrm{Hom}_K(A, \mathbb{K})$ is a $\mathbb{K}$-coalgebra.*

Proof. Define the comultiplication on C^* by

$$\Delta : A^* \xrightarrow{\nabla^*} (A \otimes A)^* \xrightarrow{\mathrm{can}^{-1}} A^* \otimes A^*.$$

The canonical map $\mathrm{can} : A^* \otimes A^* \longrightarrow (A \otimes A)^*$ is invertible, since A is finite dimensional. By a diagrammatic proof or by calculation with elements it is easy to show that A^* becomes a $\mathbb{K}$-coalgebra. □

Remark 8.6.9. If $\mathbb{K}$ is an arbitrary commutative ring, then $A^* = \mathrm{Hom}_K(A, \mathbb{K})$ is a $\mathbb{K}$-coalgebra if A is a finitely generated projective $\mathbb{K}$-module.

Problem 8.6.2. Find sufficient conditions for an algebra A resp. a coalgebra C such that $\mathrm{Hom}(A,C)$ becomes a coalgebra with co-convolution as comultiplication.

Definition 8.6.10. Let C be a $\mathbb{K}$-coalgebra. A *left C-comodule* is a $\mathbb{K}$-module M together with a homomorphism $\delta_M : M \to C \otimes M$, such that the diagrams

$$\begin{tikzcd} M \arrow[r, "\delta"] \arrow[d, "\delta"'] & C \otimes M \arrow[d, "\Delta\otimes\mathrm{id}"] \\ C \otimes M \arrow[r, "\mathrm{id}\otimes\delta"'] & C \otimes C \otimes M \end{tikzcd}$$

and

$$\begin{tikzcd} M \arrow[d, "\delta"'] \arrow[dr, "\mathrm{id}"] & \\ C \otimes M \arrow[r, "\epsilon\otimes\mathrm{id}"'] & \mathbb{K} \otimes M \cong M. \end{tikzcd}$$

commute.

Let ${}^CM$ and ${}^CN$ be C-comodules and let $f : M \to N$ be a $\mathbb{K}$-linear map. The map f is called a *homomorphism of comodules* if the diagram

$$\begin{tikzcd} M \arrow[r, "\delta_M"] \arrow[d, "f"'] & C \otimes M \arrow[d, "1\otimes f"] \\ N \arrow[r, "\delta_N"'] & C \otimes N \end{tikzcd}$$

commutes.

The left C-comodules and their homomorphisms form the *category ${}^C\mathcal{M}$ of comodules.*

Let N be an arbitrary $\mathbb{K}$-module and M be a C-comodule. Then there is a bijection between all multilinear maps

$$f : C \times \ldots \times M \to N$$

and all linear maps

$$f' : C \otimes \ldots \otimes M \to N.$$

These maps are associated to each other by the formula

$$f(c_1, \ldots, c_n, m) = f'(c_1 \otimes \ldots \otimes c_n \otimes m).$$

For $m \in M$ we define

$$\sum f(m_{(1)}, \ldots, m_{(n)}, m_{(M)}) := f'(\delta^n(m)),$$

where δ^n denotes the n-fold application of δ, i.e. $\delta^n = (1 \otimes \ldots \otimes 1 \otimes \delta) \circ (1 \otimes \delta) \circ \delta$.

In particular we obtain for the bilinear map $\otimes : C \times M \to C \otimes M$

$$\sum m_{(1)} \otimes m_{(M)} = \delta(m),$$

and for the multilinear map $\otimes^2 : C \times C \times M \to C \otimes C \otimes M$

$$\sum m_{(1)} \otimes m_{(2)} \otimes m_{(M)} = (1 \otimes \delta)\delta(c) = (\Delta \otimes 1)\delta(m).$$

Problem 8.6.3. Show that a finite dimensional vector space V is a comodule over the coalgebra $V \otimes V^*$ as defined in problem 8.11.1 with the coaction $\delta(v) := \sum v \otimes v_i^* \otimes v_i \in (V \otimes V^*) \otimes V$ where $\sum v_i^* \otimes v_i$ is the dual basis of V in $V^* \otimes V$.

Theorem 8.6.11. *(Fundamental Theorem for Comodules) Let $\mathbb{K}$ be a field. Let M be a left C-comodule and let $m \in M$ be given. Then there exists a finite dimensional subcoalgebra $C' \subseteq C$ and a finite dimensional C'-comodule M' with $m \in M' \subseteq M$ where $M' \subseteq M$ is a $\mathbb{K}$-submodule, such that the diagram*

$$\begin{tikzcd} M' \arrow[r, "\delta'"] \arrow[d] & C' \otimes M' \arrow[d] \\ M \arrow[r, "\delta"'] & C \otimes M \end{tikzcd}$$

commutes.

Corollary 8.6.12. *1. Each element $c \in C$ of a coalgebra is contained in a finite dimensional subcoalgebra of C.*

2. Each element $m \in M$ of a comodule is contained in a finite dimensional subcomodule of M.

Corollary 8.6.13. *1. Each finite dimensional subspace V of a coalgebra C is contained in a finite dimensional subcoalgebra C' of C.*

2. Each finite dimensional subspace V of a comodule M is contained in a finite dimensional subcomodule M' of M.

Corollary 8.6.14. *1. Each coalgebra is a union of finite dimensional subcoalgebras.*

2. Each comodule is a union of finite dimensional subcomodules.

PROOF. (*of the Theorem*) We can assume that $m \neq 0$ for else we can use $M' = 0$ and $C' = 0$.

Under the representations of $\delta(m) \in C \otimes M$ as finite sums of decomposable tensors pick one

$$\delta(m) = \sum_{i=1}^{s} c_i \otimes m_i$$

of shortest length s. Then the families $(c_i|i=1,\dots,s)$ and $(m_i|i=1,\dots,s)$ are linearly independent. Choose coefficients $c_{ij}\in C$ such that

$$\Delta(c_j)=\sum_{i=1}^{t} c_i\otimes c_{ij},\quad \forall j=1,\dots,s,$$

by suitably extending the linearly independent family $(c_i|i=1,\dots,s)$ to a linearly independent family $(c_i|i=1,\dots,t)$ and $t\geq s$.

We first show that we can choose $t=s$. By coassociativity we have $\sum_{i=1}^{s} c_i\otimes \delta(m_i)=\sum_{j=1}^{s}\Delta(c_j)\otimes m_j=\sum_{j=1}^{s}\sum_{i=1}^{t} c_i\otimes c_{ij}\otimes m_j$. Since the c_i and the m_j are linearly independent we can compare coefficients and get

$$\delta(m_i)=\sum_{j=1}^{s} c_{ij}\otimes m_j,\quad \forall i=1,\dots,s \tag{1}$$

and $0=\sum_{j=1}^{s} c_{ij}\otimes m_j$ for $i>s$. The last statement implies

$$c_{ij}=0,\quad \forall i>s, j=1,\dots,s.$$

Hence we get $t=s$ and

$$\Delta(c_j)=\sum_{i=1}^{s} c_i\otimes c_{ij},\quad \forall j=1,\dots,s.$$

Define finite dimensional subspaces $C'=\langle c_{ij}|i,j=1,\dots,s\rangle\subseteq C$ and $M'=\langle m_i|i=1,\dots,s\rangle\subseteq M$. Then by (1) we get $\delta: M'\longrightarrow C'\otimes M'$. We show that $m\in M'$ and that the restriction of Δ to C' gives a linear map $\Delta: C'\longrightarrow C'\otimes C'$ so that the required properties of the theorem are satisfied. First observe that $m=\sum\varepsilon(c_i)m_i\in M'$ and $c_j=\sum\varepsilon(c_i)c_{ij}\in C'$. Using coassociativity we get

$$\begin{aligned}\textstyle\sum_{i,j=1}^{n} c_i\otimes\Delta(c_{ij})\otimes m_j &= \textstyle\sum_{k,j=1}^{s}\Delta(c_k)\otimes c_{kj}\otimes m_j\\ &= \textstyle\sum_{i,j,k=1}^{s} c_i\otimes c_{ik}\otimes c_{kj}\otimes m_j\end{aligned}$$

hence

$$\Delta(c_{ij})=\sum_{k=1}^{s} c_{ik}\otimes c_{kj}. \tag{2}$$

□

Remark 8.6.15. We give a sketch of a second proof which is somewhat more technical. Since C is a $\mathbb{K}$-coalgebra, the dual C^* is an algebra. The comodule structure $\delta: M\longrightarrow C\otimes M$ leads to a module structure by $\rho=(\mathrm{ev}\otimes 1)(1\otimes\delta): C^*\otimes M\longrightarrow C^*\otimes C\otimes M\longrightarrow M$. Consider the submodule $N:=C^*m$. Then N is finite dimensional, since $c^*m=\sum_{i=1}^{n}\langle c^*,c_i\rangle m_i$ for all $c^*\in C^*$ where $\sum_{i=1}^{n} c_i\otimes m_i=\delta(m)$. Observe that C^*m is a subspace of the space generated by the m_i. But it does not depend on the choice of the m_i. Furthermore if we take $\delta(m)=\sum c_i\otimes m_i$ with a shortest representation then the m_i are in C^*m since $c^*m = \sum \langle c^*, c_i\rangle m_i = m_i$ for c^* an element of a dual basis of the c_i.

N is a C-comodule since $\delta(c^*m) = \sum \langle c^*, c_i\rangle \delta(m_i) = \sum \langle c^*, c_{i(1)}\rangle c_{i(2)} \otimes m_i \in C \otimes C^*m$.

Now we construct a subcoalgebra D of C such that N is a D-comodule with the induced coaction. Let $D := N \otimes N^*$. By 8.13 N is a comodule over the coalgebra $N \otimes N^*$. Construct a linear map $\phi : D \longrightarrow C$ by $n \otimes n^* \mapsto \sum n_{(1)}\langle n^*, n_{(N)}\rangle$. By definition of the dual basis we have $n = \sum n_i \langle n_i^*, n\rangle$. Thus we get

$$\begin{aligned}(\phi \otimes \phi)\Delta_D(n \otimes n^*) &= (\phi \otimes \phi)(\textstyle\sum n \otimes n_i^* \otimes n_i \otimes n^*)\\ &= \textstyle\sum n_{(1)}\langle n_i^*, n_{(N)}\rangle \otimes n_{i(1)}\langle n^*, n_{i(N)}\rangle\\ &= \textstyle\sum n_{(1)} \otimes n_{i(1)}\langle n^*, n_{i(N)}\rangle\langle n_i^*, n_{(N)}\rangle\\ &= \textstyle\sum n_{(1)} \otimes n_{(2)}\langle n^*, n_{(N)}\rangle = \sum \Delta_C(n_{(1)})\langle n^*, n_{(N)}\rangle\\ &= \Delta_C\phi(n \otimes n^*).\end{aligned}$$

Furthermore $\varepsilon_C\phi(n \otimes n^*) = \varepsilon(\sum n_{(1)}\langle n^*, n_{(N)}\rangle = \langle n^*, \sum \varepsilon(n_{(1)})n_{(N)}\rangle = \langle n^*, n\rangle = \varepsilon(n \otimes n^*)$. Hence $\phi : D \longrightarrow C$ is a homomorphism of coalgebras, D is finite dimensional and the image $C' := \phi(D)$ is a finite dimensional subcoalgebra of C. Clearly N is also a C'-comodule, since it is a D-comodule.

Finally we show that the D-comodule structure on N if lifted to the C-comodule structure coincides with the one defined on M. We have

$$\begin{aligned}\delta_C(c^*m) &= \delta_C(\textstyle\sum\langle c^*, m_{(1)}\rangle m_{(M)}) = \sum\langle c^*, m_{(1)}\rangle m_{(2)} \otimes m_{(M)}\\ &= \textstyle\sum\langle c^*, m_{(1)}\rangle m_{(2)} \otimes m_i\langle m_i^*, m_{(M)}\rangle = \sum\langle c^*, m_{(1)}\rangle m_{(2)}\langle m_i^*, m_{(M)}\rangle \otimes m_i\\ &= (\phi \otimes 1)(\textstyle\sum\langle c^*, m_{(1)}\rangle m_{(M)} \otimes m_i^* \otimes m_i) = (\phi \otimes 1)(\sum c^*m \otimes m_i^* \otimes m_i)\\ &= (\phi \otimes 1)\delta_D(c^*m).\end{aligned}$$