CHAPTER 8

Toolbox

5. Algebras

Let $\mathbb{K}$ be a commutative ring. In most of our applications $\mathbb{K}$ will be a field. Tensor products of $\mathbb{K}$-modules will be simply written as $M \otimes N := M \otimes_K N$. Every such tensor product is again a $\mathbb{K}$-bimodule since each $\mathbb{K}$-module M resp. N is a $\mathbb{K}$-bimodule (see 8.4.14).

Definition 8.5.1. A $\mathbb{K}$-*algebra* is a vector space A together with a *multiplication* $\nabla : A \otimes A \to A$ that is associative:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{id} \otimes \nabla"] \arrow[d, "\nabla \otimes \mathrm{id}"'] & A \otimes A \arrow[d, "\nabla"] \\
A \otimes A \arrow[r, "\nabla"'] & A
\end{tikzcd}$$

and a *unit* $\eta : \mathbb{K} \to A$:

$$\begin{tikzcd}
\mathbb{K} \otimes A \cong A \cong A \otimes \mathbb{K} \arrow[r, "\mathrm{id} \otimes \eta"] \arrow[d, "\eta \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & A \otimes A \arrow[d, "\nabla"] \\
A \otimes A \arrow[r, "\nabla"'] & A.
\end{tikzcd}$$

A $\mathbb{K}$-algebra A is *commutative* if the following diagram commutes

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\tau"] \arrow[dr, "\nabla"'] & & A \otimes A \arrow[dl, "\nabla"] \\
& A. &
\end{tikzcd}$$

Let A and B be $\mathbb{K}$-algebras. A *homomorphism of algebras* $f : A \to B$ is a $\mathbb{K}$-linear map such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\nabla_A"'] & B \otimes B \arrow[d, "\nabla_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \mathbb{K} \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_B"] & \\
A \arrow[rr, "f"] & & B.
\end{tikzcd}$$

Remark 8.5.2. Every $\mathbb{K}$-algebra A is a ring with the multiplication

$$A \times A \overset{\otimes}{\longrightarrow} A \otimes A \overset{\nabla}{\longrightarrow} A.$$

The unit element is $\eta(1)$, where 1 is the unit element of $\mathbb{K}$.

Obviously the composition of two homomorphisms of algebras is again a homomorphism of algebras. Furthermore the identity map is a homomorphism of algebras. Hence the $\mathbb{K}$-algebras form a category $\mathbb{K}$-**Alg**. The category of commutative $\mathbb{K}$-algebras will be denoted by $\mathbb{K}$-**cAlg**.

Problem 8.5.1. 1. Show that $\mathrm{End}_K(V)$ is a $\mathbb{K}$-algebra.

2. Show that $(A, \nabla : A \otimes A \longrightarrow A, \eta : \mathbb{K} \longrightarrow A)$ is a $\mathbb{K}$-algebra if and only if A with the multiplication $A \times A \overset{\otimes}{\longrightarrow} A \otimes A \overset{\nabla}{\longrightarrow} A$ and the unit $\eta(1)$ is a ring and $\eta : \mathbb{K} \longrightarrow \mathrm{Cent}(A)$ is a ring homomorphism into the center of A.

3. Let V be a $\mathbb{K}$-module. Show that $D(V) := \mathbb{K} \times V$ with the multiplication $(r_1, v_1)(r_2, v_2) := (r_1 r_2, r_1 v_2 + r_2 v_1)$ is a commutative $\mathbb{K}$-algebra.

Lemma 8.5.3. *Let A and B be algebras. Then $A \otimes B$ is an algebra with the multiplication $(a_1 \otimes b_1)(a_2 \otimes b_2) := a_1 a_2 \otimes b_1 b_2$.*

Proof. Certainly the algebra properties can easily be checked by a simple calculation with elements. We prefer for later applications a diagrammatic proof.

Let $\nabla_A : A \otimes A \longrightarrow A$ and $\nabla_B : B \otimes B \longrightarrow B$ denote the multiplications of the two algebras. Then the new multiplication is $\nabla_{A \otimes B} := (\nabla_A \otimes \nabla_B)(1_A \otimes \tau \otimes 1_B) : A \otimes B \otimes A \otimes B \longrightarrow A \otimes B$ where $\tau : B \otimes A \longrightarrow A \otimes B$ is the symmetry map from Theorem 8.4.15. Now the following diagrams commute

$$\begin{tikzcd}
A \otimes B \otimes A \otimes B \otimes A \otimes B \arrow[rr, "1\otimes\tau\otimes 1^3"] \arrow[dd, "1^3\otimes\tau\otimes 1"'] & & A \otimes A \otimes B \otimes B \otimes A \otimes B \arrow[rr, "\nabla\otimes\nabla\otimes 1^2"] \arrow[dl, "1^3\otimes\tau\otimes 1"'] \arrow[dd, "1\otimes\tau_{B\otimes B,A}\otimes 1"] & & A \otimes B \otimes A \otimes B \arrow[dd, "1\otimes\tau\otimes 1"] \\
& A \otimes A \otimes B \otimes A \otimes B \otimes B \arrow[dr, "1^2\otimes\tau\otimes 1^2"] & & & \\
A \otimes B \otimes A \otimes A \otimes B \otimes B \arrow[ur, "1\otimes\tau\otimes 1^3"] \arrow[rr, "1\otimes\tau_{B,A\otimes A}\otimes 1^3"'] \arrow[d, "1\otimes\nabla\otimes\nabla"'] & & A \otimes A \otimes A \otimes B \otimes B \otimes B \arrow[rr, "\nabla\otimes 1\otimes\nabla\otimes 1"] \arrow[d, "1\otimes\nabla\otimes 1\otimes\nabla"] & & A \otimes A \otimes B \otimes B \arrow[d, "\nabla\otimes\nabla"] \\
A \otimes B \otimes A \otimes B \arrow[rr, "1\otimes\tau\otimes 1"] & & A \otimes A \otimes B \otimes B \arrow[rr, "\nabla\otimes\nabla"] & & A \otimes B
\end{tikzcd}$$

In the left upper rectangle of the diagram the quadrangle commutes by the properties of the tensor product and the two triangles commute by inner properties of τ. The right upper and left lower rectangles commute since τ is a natural transformation and the right lower rectangle commutes by the associativity of the algebras A and B.

Furthermore we use the homomorphism $\eta = \eta_{A\otimes B} : \mathbb{K} \longrightarrow \mathbb{K}\otimes K \longrightarrow A\otimes B$ in the following commutative diagram

$$\begin{tikzcd}
\mathbb{K}\otimes A\otimes B \cong A\otimes B \cong A\otimes B\otimes\mathbb{K} \arrow[rr] \arrow[d] \arrow[dr] \arrow[drrr, "1"] \arrow[drr] & & A\otimes B\otimes\mathbb{K}\otimes\mathbb{K} \arrow[r, "1^2\otimes\eta\otimes\eta"] \arrow[d, "1\otimes\tau\otimes 1"] & A\otimes B\otimes A\otimes B \arrow[d, "1\otimes\tau\otimes 1"] \\
\mathbb{K}\otimes\mathbb{K}\otimes A\otimes B \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "\eta\otimes\eta\otimes 1^2"'] & \mathbb{K}\otimes A\otimes\mathbb{K}\otimes B \arrow[d, "\eta\otimes 1\otimes\eta\otimes 1"'] & A\otimes\mathbb{K}\otimes B\otimes\mathbb{K} \arrow[r, "1\otimes\eta\otimes 1\otimes\eta"'] & A\otimes A\otimes B\otimes B \arrow[d, "\nabla\otimes\nabla"] \\
A\otimes B\otimes A\otimes B \arrow[r, "1\otimes\tau\otimes 1"] & A\otimes A\otimes B\otimes B \arrow[rr, "\nabla\otimes\nabla"] & & A\otimes B.
\end{tikzcd}$$

$\square$

Definition 8.5.4. Let $\mathbb{K}$ be a commutative ring. Let V be a $\mathbb{K}$-module. A $\mathbb{K}$-algebra $T(V)$ together with a homomorphism of $\mathbb{K}$-modules $\iota : V \longrightarrow T(V)$ is called a *tensor algebra over* V if for each $\mathbb{K}$-algebra A and for each homomorphism of $\mathbb{K}$-modules $f : V \longrightarrow A$ there exists a unique homomorphism of $\mathbb{K}$-algebras $g : T(V) \longrightarrow A$ such that the diagram

$$\begin{tikzcd}
V \arrow[r, "\iota"] \arrow[dr, "f"'] & T(V) \arrow[d, "g"] \\
& A
\end{tikzcd}$$

commutes.

Note: If you want to define a homomorphism $g : T(V) \longrightarrow A$ with a tensor algebra as domain you should define it by giving a homomorphism of $\mathbb{K}$-modules defined on V.

Lemma 8.5.5. *A tensor algebra $(T(V), \iota)$ defined by V is unique up to a unique isomorphism.*

Proof. Let $(T(V), \iota)$ and $(T'(V), \iota')$ be tensor algebras over V. Then

$$\begin{tikzcd}
& & V \arrow[dll, "\iota"'] \arrow[dl, "\iota'"] \arrow[dr, "\iota"] \arrow[drr, "\iota'"] & & \\
T(V) \arrow[r, "h"] & T'(V) \arrow[rr, "k"] & & T(V) \arrow[r, "h"] & T'(V)
\end{tikzcd}$$

implies $k = h^{-1}$. $\square$

Proposition 8.5.6. (Rules of computation in a tensor algebra) *Let $(T(V), \iota)$ be the tensor algebra over V. Then we have*

1. *$\iota : V \longrightarrow T(V)$ is injective (so we may identify the elements $\iota(v)$ and v for all $v \in V$),*
2. *$T(V) = \{\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$,*
3. *if $f : V \longrightarrow A$ is a homomorphism of $\mathbb{K}$-modules, A is a $\mathbb{K}$-algebra, and $g : T(V) \longrightarrow A$ is the induced homomorphism of $\mathbb{K}$-algebras, then*

$$g(\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n}) = \sum_{n,\bar{i}} f(v_{i_1}) \cdot \ldots \cdot f(v_{i_n}).$$

Proof. 1. Use the embedding homomorphism $j : V \longrightarrow D(V)$, where $D(V)$ is defined as in 8.5.3. to construct $g : T(V) \longrightarrow D(V)$ such that $g \circ \iota = j$. Since j is injective so is ι.

2. Let $B := \{\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$. Obviously B is the subalgebra of $T(V)$ generated by the elements of V. Let $j : B \longrightarrow T(V)$ be the embedding homomorphism. Then $\iota : V \longrightarrow T(V)$ factors through a linear map $\iota' : V \longrightarrow B$. In the following diagram

$$\begin{tikzcd}
V \arrow[r, "\iota'"] \arrow[dr, "\iota'"'] & B \arrow[r, "j"] \arrow[d, "\mathrm{id}_B"] & T(V) \arrow[dl, "p"] \arrow[d, "jp"] \\
 & B \arrow[r, "j"] & T(V)
\end{tikzcd}$$

we have $\mathrm{id}_B \circ \iota' = \iota'$. p with $p \circ j \circ \iota' = p \circ \iota = \iota'$ exists since ι' is a homomorphism of $\mathbb{K}$-modules. Because of $jp \circ \iota = j \circ \iota' = \iota = \mathrm{id}_{T(V)} \circ \iota$ we get $jp = \mathrm{id}_{T(V)}$, hence the embedding j is surjective and thus j is the identity.

3. is precisely the definition of the induced homomorphism. $\square$

Proposition 8.5.7. *Given a $\mathbb{K}$-module V. Then there exists a tensor algebra $(T(V), \iota)$.*

Proof. Define $T^n(V) := V \otimes \ldots \otimes V = V^{\otimes n}$ to be the n-fold tensor product of V. Define $T^0(V) := \mathbb{K}$ and $T^1(V) := V$. We define

$$T(V) := \bigoplus_{i \geq 0} T^i(V) = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \ldots .$$

The components $T^n(V)$ of $T(V)$ are called *homogeneous components.*

The canonical isomorphisms $T^m(V) \otimes T^n(V) \cong T^{m+n}(V)$ taken as multiplication

$$\nabla : T^m(V) \otimes T^n(V) \to T^{m+n}(V)$$
$$\nabla : T(V) \otimes T(V) \to T(V)$$

and the embedding $\eta : \mathbb{K} = T^0(V) \longrightarrow T(V)$ induce the structure of a $\mathbb{K}$-algebra on $T(V)$. Furthermore we have the embedding $\iota : V \longrightarrow T^1(V) \subseteq T(V)$.

We have to show that $(T(V), \iota)$ is a tensor algebra. Let $f : V \to A$ be a homomorphism of $\mathbb{K}$-modules. Each element in $T(V)$ is a sum of decomposable tensors $v_1 \otimes \ldots \otimes v_n$. Define $g : T(V) \to A$ by $g(v_1 \otimes \ldots \otimes v_n) := f(v_1) \ldots f(v_n)$ (and $(g : T^0(V) \to A) = (\eta : \mathbb{K} \to A)$). By induction one sees that g is a homomorphism of algebras. Since $(g : T^1(V) \to A) = (f : V \to A)$ we get $g \circ \iota = f$. If $h : T(V) \to A$ is a homomorphism of algebras with $h \circ \iota = f$ we get $h(v_1 \otimes \ldots \otimes v_n) = h(v_1) \ldots h(v_n) = f(v_1) \ldots f(v_n)$ hence $h = g$. □

Proposition 8.5.8. *The construction of tensor algebras $T(V)$ defines a functor $T : \mathbb{K}\text{-}\mathbf{Mod} \to \mathbb{K}\text{-}\mathbf{Alg}$ that is left adjoint to the underlying functor $U : \mathbb{K}\text{-}\mathbf{Alg} \to \mathbb{K}\text{-}\mathbf{Mod}$.*

PROOF. Follows from the universal property and 8.9.16. □

Problem 8.5.2. 1. Let X be a set and $V := \mathbb{K}X$ be the free $\mathbb{K}$-module over X. Show that $X \to V \to T(V)$ defines a *free algebra* over X, i.e. for every $\mathbb{K}$-algebra A and every map $f : X \to A$ there is a unique homomorphism of $\mathbb{K}$-algebras $g : T(V) \to A$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & T(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.

We write $\mathbb{K}\langle X\rangle := T(\mathbb{K}X)$ and call it the *polynomial ring over $\mathbb{K}$ in the non-commuting variables X*.

2. Let $T(V)$ and $\iota : V \to T(V)$ be a tensor algebra. Regard V as a subset of $T(V)$ by ι. Show that there is a unique homomorphism $\Delta : T(V) \to T(V) \otimes T(V)$ with $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$.

3. Show that $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta : T(V) \to T(V) \otimes T(V) \otimes T(V)$.

4. Show that there is a unique homomorphism of algebras $\varepsilon : T(V) \to \mathbb{K}$ with $\varepsilon(v) = 0$ for all $v \in V$.

5. Show that $(\varepsilon \otimes 1)\Delta = (1 \otimes \varepsilon)\Delta = \mathrm{id}_{T(V)}$.

6. Show that there is a unique homomorphism of algebras $S : T(V) \to T(V)^{op}$ with $S(v) = -v$. ($T(V)^{op}$ is the *opposite algebra* of $T(V)$ with multiplication $s*t := ts$ for all $s, t \in T(V) = T(V)^{op}$ and where st denotes the product in $T(V)$.)

7. Show that the diagrams

$$\begin{tikzcd} T(V) \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & T(V) \\ T(V) \otimes T(V) \arrow[rr, "1\otimes S"', "S\otimes 1"'] & & T(V) \otimes T(V) \arrow[u, "\nabla"'] \end{tikzcd}$$

commute.

Definition 8.5.9. Let $\mathbb{K}$ be a commutative ring. Let V be a $\mathbb{K}$-module. A $\mathbb{K}$-algebra $S(V)$ together with a homomorphism of $\mathbb{K}$-modules $\iota : V \longrightarrow S(V)$, such that $\iota(v) \cdot \iota(v') = \iota(v') \cdot \iota(v)$ for all $v, v' \in V$, is called a *symmetric algebra over* V if for each $\mathbb{K}$-algebra A and for each homomorphism of $\mathbb{K}$-modules $f : V \longrightarrow A$, such that $f(v) \cdot f(v') = f(v') \cdot f(v)$ for all $v, v' \in V$, there exists a unique homomorphism of $\mathbb{K}$-algebras $g : S(V) \longrightarrow A$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & S(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.

Note: If you want to define a homomorphism $g : S(V) \longrightarrow A$ with a symmetric algebra as domain you should define it by giving a homomorphism of $\mathbb{K}$-modules $f : V \longrightarrow A$ satisfying $f(v) \cdot f(v') = f(v') \cdot f(v)$ for all $v, v' \in V$.

Lemma 8.5.10. *A symmetric algebra $(S(V), \iota)$ defined by V is unique up to a unique isomorphism.*

Proof. Let $(S(V), \iota)$ and $(S'(V), \iota')$ be symmetric algebras over V. Then

$$\begin{tikzcd} & & V \arrow[dll, "\iota"'] \arrow[dl, "\iota'"'] \arrow[dr, "\iota"] \arrow[drr, "\iota'"] & & \\ S(V) \arrow[r, "h"] & S'(V) \arrow[rr, "k"] & & S(V) \arrow[r, "h"] & S'(V) \end{tikzcd}$$

implies $k = h^{-1}$. □

Proposition 8.5.11. (Rules of computation in a symmetric algebra) *Let $(S(V), \iota)$ be the symmetric algebra over V. Then we have*

1. *$\iota : V \longrightarrow S(V)$ is injective (we will identify the elements $\iota(v)$ and v for all $v \in V$),*
2. *$S(V) = \{\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$,*
3. *if $f : V \longrightarrow A$ is a homomorphism of $\mathbb{K}$-modules satisfying $f(v) \cdot f(v') = f(v') \cdot f(v)$ for all $v, v' \in V$, A is a $\mathbb{K}$-algebra, and $g : S(V) \longrightarrow A$ is the induced homomorphism $\mathbb{K}$-algebras, then*

$$g(\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n}) = \sum_{n,\bar{i}} f(v_{i_1}) \cdot \ldots \cdot f(v_{i_n}).$$

Proof. 1. Use the embedding homomorphism $j : V \longrightarrow D(V)$, where $D(V)$ is the commutative algebra defined in 8.5.3. to construct $g : S(V) \longrightarrow D(V)$ such that $g \circ \iota = j$. Since j is injective so is ι.

2. Let $B := \{\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$. Obviously B is the subalgebra of $S(V)$ generated by the elements of V. Let $j : B \to S(V)$ be the embedding homomorphism. Then $\iota : V \to S(V)$ factors through a linear map $\iota' : V \to B$. In the following diagram

$$\begin{tikzcd}
V \arrow[r, "\iota'"] \arrow[dr, "\iota'"'] & B \arrow[r, "j"] \arrow[d, "\mathrm{id}_B"'] & S(V) \arrow[dl, "p"] \arrow[d, "jp"] \\
& B \arrow[r, "j"] & S(V)
\end{tikzcd}$$

we have $\mathrm{id}_B \circ \iota' = \iota'$, p with $p \circ j \circ \iota' = p \circ \iota = \iota'$ exists since ι' is a homomorphism of $\mathbb{K}$-modules satisfying $\iota'(v) \cdot \iota'(v') = \iota'(v') \cdot \iota'(v)$ for all $v, v' \in V$. Because of $jp \circ \iota = j \circ \iota' = \iota = \mathrm{id}_{S(V)} \circ \iota$ we get $jp = \mathrm{id}_{S(V)}$, hence the embedding j is surjective and thus the identity.

3. is precisely the definition of the induced homomorphism. □

Proposition 8.5.12. *Let V be a $\mathbb{K}$-module. The symmetric algebra $(S(V), \iota)$ is commutative and satisfies the following universal property:*

for each commutative $\mathbb{K}$-algebra A and for each homomorphism of $\mathbb{K}$-modules $f : V \to A$ there exists a unique homomorphism of $\mathbb{K}$-algebras $g : S(V) \to A$ such that the diagram

$$\begin{tikzcd}
V \arrow[r, "\iota"] \arrow[dr, "f"'] & S(V) \arrow[d, "g"] \\
& A
\end{tikzcd}$$

commutes.

Proof. Commutativity follows from the commutativity of the generators: $vv' = v'v$ which carries over to the elements of the form $\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n}$. The universal property follows since the defining condition $f(v) \cdot f(v') = f(v') \cdot f(v)$ for all $v, v' \in V$ is automatically satisfied. □

Proposition 8.5.13. *Given a $\mathbb{K}$-module V. Then there exists a symmetric algebra $(S(V), \iota)$.*

Proof. Define $S(V) := T(V)/I$ where $I = \langle vv' - v'v | v, v' \in V \rangle$ is the two-sided ideal generated by the elements $vv' - v'v$. Let ι be the canonical map $V \to T(V) \to S(V)$. Then the universal property is easily verified by the homomorphism theorem for algebras. □

Proposition 8.5.14. *The construction of symmetric algebras $S(V)$ defines a functor $S : \mathbb{K}\text{-}\mathbf{Mod} \to \mathbb{K}\text{-}\mathbf{cAlg}$ that is left adjoint to the underlying functor $U : \mathbb{K}\text{-}\mathbf{cAlg} \to \mathbb{K}\text{-}\mathbf{Mod}$.*

PROOF. Follows from the universal property and 8.9.16. $\square$

Problem 8.5.3. Let X be a set and $V := \mathbb{K}X$ be the free $\mathbb{K}$-module over X. Show that $X \longrightarrow V \longrightarrow S(V)$ defines a *free commutative algebra* over X, i.e. for every commutative $\mathbb{K}$-algebra A and every map $f : X \longrightarrow A$ there is a unique homomorphism of $\mathbb{K}$-algebras $g : S(V) \longrightarrow A$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & S(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.

The algebra $\mathbb{K}[X] := S(\mathbb{K}X)$ is called the *polynomial ring over* $\mathbb{K}$ *in the (commuting) variables* X.

2. Let $S(V)$ and $\iota : V \longrightarrow S(V)$ be a symmetric algebra. Show that there is a unique homomorphism $\Delta : S(V) \longrightarrow S(V) \otimes S(V)$ with $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$.

3. Show that $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta : S(V) \longrightarrow S(V) \otimes S(V) \otimes S(V)$.

4. Show that there is a unique homomorphism of algebras $\varepsilon : S(V) \longrightarrow \mathbb{K}$ with $\varepsilon(v) = 0$ for all $v \in V$.

5. Show that $(\varepsilon \otimes 1)\Delta = (1 \otimes \varepsilon)\Delta = \mathrm{id}_{S(V)}$.

6. Show that there is a unique homomorphism of algebras $S : S(V) \longrightarrow S(V)$ with $S(v) = -v$.

7. Show that the diagrams

$$\begin{tikzcd} S(V) \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & S(V) \\ S(V) \otimes S(V) \arrow[rr, "1\otimes S", "S\otimes 1"'] & & S(V) \otimes S(V) \arrow[u, "\nabla"'] \end{tikzcd}$$

commute.

Definition 8.5.15. Let $\mathbb{K}$ be a commutative ring. Let V be a $\mathbb{K}$-module. A $\mathbb{K}$-algebra $E(V)$ together with a homomorphism of $\mathbb{K}$-modules $\iota : V \longrightarrow E(V)$, such that $\iota(v)^2 = 0$ for all $v \in V$, is called an *exterior algebra or Grassmann algebra over* V if for each $\mathbb{K}$-algebra A and for each homomorphism of $\mathbb{K}$-modules $f : V \longrightarrow A$, such that $f(v)^2 = 0$ for all $v \in V$, there exists a unique homomorphism of $\mathbb{K}$-algebras $g : E(V) \longrightarrow A$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & E(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.

The multiplication in $E(V)$ is usually denoted by $u \wedge v$.

Note: If you want to define a homomorphism $g : E(V) \longrightarrow A$ with an exterior algebra as domain you should define it by giving a homomorphism of $\mathbb{K}$-modules defined on V satisfying $f(v)^2 = 0$ for all $v, v' \in V$.

Problem 8.5.4. 1. Let $f : V \longrightarrow A$ be a linear map satisfying $f(v)^2 = 0$ for all $v \in V$. Then $f(v)f(v') = -f(v')f(v)$ for all $v, v' \in V$.

2. Let 2 be invertible in $\mathbb{K}$ (e.g. $\mathbb{K}$ a field of characteristic $\neq 2$). Let $f : V \longrightarrow A$ be a linear map satisfying $f(v)f(v') = -f(v')f(v)$ for all $v, v' \in V$. Then $f(v)^2 = 0$ for all $v \in V$.

Lemma 8.5.16. *An exterior algebra $(E(V), \iota)$ defined by V is unique up to a unique isomorphism.*

Proof. Let $(E(V), \iota)$ and $(E'(V), \iota')$ be exterior algebras over V. Then

$$\begin{tikzcd}
 & & V \arrow[dll, "\iota"'] \arrow[dl, "\iota'"] \arrow[dr, "\iota"] \arrow[drr, "\iota'"] & & \\
E(V) \arrow[r, "h"] & E'(V) \arrow[rr, "k"] & & E(V) \arrow[r, "h"] & E'(V)
\end{tikzcd}$$

implies $k = h^{-1}$. □

Proposition 8.5.17. (Rules of computation in an exterior algebra) *Let $(E(V), \iota)$ be the exterior algebra over V. Then we have*

1. *$\iota : V \longrightarrow E(V)$ is injective (we will identify the elements $\iota(v)$ and v for all $v \in V$),*
2. *$E(V) = \{\sum_{n,\bar{i}} v_{i_1} \wedge \ldots \wedge v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$,*
3. *if $f : V \longrightarrow A$ is a homomorphism of $\mathbb{K}$-modules satisfying $f(v) \cdot f(v') = -f(v') \cdot f(v)$ for all $v, v' \in V$, A is a $\mathbb{K}$-algebra, and $g : E(V) \longrightarrow A$ is the induced homomorphism $\mathbb{K}$-algebras, then*

$$g(\sum_{n,\bar{i}} v_{i_1} \wedge \ldots \wedge v_{i_n}) = \sum_{n,\bar{i}} f(v_{i_1}) \cdot \ldots \cdot f(v_{i_n}).$$

Proof. 1. Use the embedding homomorphism $j : V \longrightarrow D(V)$, where $D(V)$ is the algebra defined in 8.5.3. to construct $g : E(V) \longrightarrow D(V)$ such that $g \circ \iota = j$. Since j is injective so is ι.

2. Let $B := \{\sum_{n,\bar{i}} v_{i_1} \wedge \ldots \wedge v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$. Obviously B is the subalgebra of $E(V)$ generated by the elements of V. Let $j : B \longrightarrow E(V)$ be the embedding homomorphism. Then $\iota : V \longrightarrow E(V)$ factors through a linear map

$\iota' : V \to B$. In the following diagram

$$\begin{tikzcd} V \arrow[r, "\iota'"] \arrow[dr, "\iota'"'] & B \arrow[r, "j"] \arrow[d, "\mathrm{id}_B"] & E(V) \arrow[dl, "p"] \arrow[d, "jp"] \\ & B \arrow[r, "j"] & E(V) \end{tikzcd}$$

we have $\mathrm{id}_B \circ \iota' = \iota'$, p with $p \circ j \circ \iota' = p \circ \iota = \iota'$ exists since ι' is a homomorphism of $\mathbb{K}$-modules satisfying $\iota'(v) \cdot \iota'(v') = -\iota'(v') \cdot \iota'(v)$ for all $v, v' \in V$. Because of $jp \circ \iota = j \circ \iota' = \iota = \mathrm{id}_{E(V)} \circ \iota$ we get $jp = \mathrm{id}_{E(V)}$, hence the embedding j is surjective and thus j is the identity.

3. is precisely the definition of the induced homomorphism. □

Proposition 8.5.18. *Given a $\mathbb{K}$-module V. Then there exists an exterior algebra $(E(V), \iota)$.*

Proof. Define $E(V) := T(V)/I$ where $I = \langle v^2 | v \in V \rangle$ is the two-sided ideal generated by the elements v^2. Let ι be the canonical map $V \to T(V) \to E(V)$. Then the universal property is easily verified by the homomorphism theorem for algebras. □

Problem 8.5.5. 1. Let V be a finite dimensional vector space of dimension n. Show that $E(V)$ is finite dimensional of dimension 2^n. (Hint: The homogeneous components $E^i(V)$ have dimension $\binom{n}{i}$.

2. Show that the symmetric group S_n operates (from the left) on $T^n(V)$ by $\sigma(v_1 \otimes \ldots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}$ with $\sigma \in S_n$ and $v_i \in V$.

3. A tensor $a \in T^n(V)$ is called *a symmetric tensor* if $\sigma(a) = a$ for all $\sigma \in S_n$. Let $\hat{S}^n(V)$ be the subspace of symmetric tensors in $T^n(V)$.

a) Show that $\mathcal{S} : T^n(V) \ni a \mapsto \sum_{\sigma \in S_n} \sigma(a) \in T^n(V)$ is a linear map.

b) Show that $\mathcal{S}$ has its image in $\hat{S}^n(V)$.

c) Show that $\mathrm{Im}(\mathcal{S}) = \hat{S}^n(V)$ if $n!$ is invertible in $\mathbb{K}$.

d) Show that $\hat{S}^n(V) \hookrightarrow T^n(V) \xrightarrow{\nu} S^n(V)$ is an isomorphism if $n!$ is invertible in $\mathbb{K}$ and $\nu : T^n(V) \to S^n(V)$ is the restriction of $\nu : T(V) \to S(V)$, the symmetric algebra.

4. A tensor $a \in T^n(V)$ is called *an antisymmetric tensor* if $\sigma(a) = \varepsilon(\sigma)a$ for all $\sigma \in S_n$ where $\varepsilon(\sigma)$ is the sign of the permutation σ. Let $\hat{E}^n(V)$ be the subspace of antisymmetric tensors in $T^n(V)$.

a) Show that $\mathcal{E} : T^n(V) \ni a \mapsto \sum_{\sigma \in S_n} \varepsilon(\sigma)\sigma(a) \in T^n(V)$ is a linear map.

b) Show that $\mathcal{E}$ has its image in $\hat{E}^n(V)$.

c) Show that $\mathrm{Im}(\mathcal{E}) = \hat{E}^n(V)$ if $n!$ is invertible in $\mathbb{K}$.

d) Show that $\hat{E}^n(V) \hookrightarrow T^n(V) \xrightarrow{\nu} E^n(V)$ is an isomorphism if $n!$ is invertible in $\mathbb{K}$ and $\nu : T^n(V) \to E^n(V)$ is the restriction of $\nu : T(V) \to E(V)$, the exterior algebra.

Definition 8.5.19. Let A be a $\mathbb{K}$-algebra. A *left A-module* is a $\mathbb{K}$-module M together with a homomorphism $\mu_M : A \otimes M \to M$, such that the diagrams

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "\mathrm{id} \otimes \mu"] \arrow[d, "\nabla \otimes \mathrm{id}"'] & A \otimes M \arrow[d, "\mu"] \\ A \otimes M \arrow[r, "\mu"'] & M \end{tikzcd}$$

and

$$\begin{tikzcd} M \cong \mathbb{K} \otimes M \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[dr, "\mathrm{id}"] & A \otimes M \arrow[d, "\mu"] \\ & M \end{tikzcd}$$

commute.

Let ${}_AM$ and ${}_AN$ be A-modules and let $f : M \to N$ be a $\mathbb{K}$-linear map. The map f is called a *homomorphism of modules* if the diagram

$$\begin{tikzcd} A \otimes M \arrow[r, "\mu_M"] \arrow[d, "1 \otimes f"'] & M \arrow[d, "f"] \\ A \otimes N \arrow[r, "\mu_N"'] & N \end{tikzcd}$$

commutes.

The left A-modules and their homomorphisms form the *category ${}_A\mathcal{M}$ of A-modules*.

Problem 8.5.6. Show that an abelian group M is a left module over the ring A if and only if M is a $\mathbb{K}$-module and an A-module in the sense of Definition 8.5.19.