CHAPTER 8

# Toolbox

## 3. Natural Transformations

**Definition 8.3.1.** Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$ be two functors. A *natural transformation* or a *functorial morphism* $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a family of morphisms $\{\varphi(A) : \mathcal{F}(A) \longrightarrow \mathcal{G}(A) | A \in \mathcal{C}\}$ such that the diagram

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\varphi(A)"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}(A) \arrow[d, "\mathcal{G}(f)"] \\
\mathcal{F}(B) \arrow[r, "\varphi(B)"'] & \mathcal{G}(B)
\end{tikzcd}$$

commutes for all $f : A \longrightarrow B$ in $\mathcal{C}$, i.e. $\mathcal{G}(f)\varphi(A) = \varphi(B)\mathcal{F}(f)$.

**Lemma 8.3.2.** *Given covariant functors* $\mathcal{F} = \mathrm{Id}_{\mathbf{Set}} : \mathbf{Set} \longrightarrow \mathbf{Set}$ *and* $\mathcal{G} = \mathrm{Mor}_{\mathbf{Set}}(\mathrm{Mor}_{\mathbf{Set}}(-, A), A) : \mathbf{Set} \longrightarrow \mathbf{Set}$ *for a set* $A$*. Then* $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ *with*

$$\varphi(B) : B \ni b \mapsto (\mathrm{Mor}_{\mathbf{Set}}(B, A) \ni f \mapsto f(b) \in A) \in \mathcal{G}(B)$$

*is a natural transformation.*

Proof. Given $g : B \longrightarrow C$. Then the following diagram commutes

$$\begin{tikzcd}
B \arrow[r, "\varphi(B)"] \arrow[d, "g"'] & \mathrm{Mor}_{\mathbf{Set}}(\mathrm{Mor}_{\mathbf{Set}}(B, A), A) \arrow[d, "\mathrm{Mor}_{\mathbf{Set}}(\mathrm{Mor}_{\mathbf{Set}}(g, A), A)"] \\
C \arrow[r, "\varphi(C)"] & \mathrm{Mor}_{\mathbf{Set}}(\mathrm{Mor}_{\mathbf{Set}}(C, A), A)
\end{tikzcd}$$

since

$$\begin{aligned}
\varphi(C)\mathcal{F}(g)(b)(f) &= \varphi(C)g(b)(f) = fg(b) = \varphi(B)(b)(fg) \\
&= [\varphi(B)(b)\mathrm{Mor}_{\mathbf{Set}}(g, A)](f) = [\mathrm{Mor}_{\mathbf{Set}}(\mathrm{Mor}_{\mathbf{Set}}(g, A), A)\varphi(A)(b)](f).
\end{aligned}$$

□

**Lemma 8.3.3.** *Let* $f : A \longrightarrow B$ *be a morphism in* $\mathcal{C}$*. Then* $\mathrm{Mor}_{\mathcal{C}}(f, \text{-}) : \mathrm{Mor}_{\mathcal{C}}(B, \text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(A, \text{-})$ *given by* $\mathrm{Mor}_{\mathcal{C}}(f, C) : \mathrm{Mor}_{\mathcal{C}}(B, C) \ni g \mapsto gf \in \mathrm{Mor}_{\mathcal{C}}(A, C)$ *is a natural transformation of covariant functors.*

*Let* $f : A \longrightarrow B$ *be a morphism in* $\mathcal{C}$*. Then* $\mathrm{Mor}_{\mathcal{C}}(\text{-}, f) : \mathrm{Mor}_{\mathcal{C}}(\text{-}, A) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(\text{-}, B)$ *given by* $\mathrm{Mor}_{\mathcal{C}}(C, f) : \mathrm{Mor}_{\mathcal{C}}(C, A) \ni g \mapsto fg \in \mathrm{Mor}_{\mathcal{C}}(C, B)$ *is a natural transformation of contravariant functors.*

PROOF. Let $h : C \longrightarrow C'$ be a morphism in $\mathcal{C}$. Then the diagrams

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(B, C) \arrow[r, "\mathrm{Mor}_{\mathcal{C}}(f,C)"] \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(B,h)"'] & \mathrm{Mor}_{\mathcal{C}}(A, C) \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(A,h)"] \\
\mathrm{Mor}_{\mathcal{C}}(B, C') \arrow[r, "\mathrm{Mor}_{\mathcal{C}}(f,C')"'] & \mathrm{Mor}_{\mathcal{C}}(A, C')
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(C', A) \arrow[r, "\mathrm{Mor}_{\mathcal{C}}(C',f)"] \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(h,A)"'] & \mathrm{Mor}_{\mathcal{C}}(C', B) \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(h,B)"] \\
\mathrm{Mor}_{\mathcal{C}}(C, A) \arrow[r, "\mathrm{Mor}_{\mathcal{C}}(C,f)"'] & \mathrm{Mor}_{\mathcal{C}}(C, B)
\end{tikzcd}$$

commute. $\square$

**Remark 8.3.4.** The composition of two natural transformations is again a natural transformation. The identity $\mathrm{id}_{\mathcal{F}}(A) := 1_{\mathcal{F}(A)}$ is also a natural transformation.

**Definition 8.3.5.** A natural transformation $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is called a *natural isomorphism* if there exists a natural transformation $\psi : \mathcal{G} \longrightarrow \mathcal{F}$ such that $\varphi \circ \psi = \mathrm{id}_{\mathcal{G}}$ and $\psi \circ \varphi = \mathrm{id}_{\mathcal{F}}$. The natural transformation $\psi$ is uniquely determined by $\varphi$. We write $\varphi^{-1} := \psi$.

A functor $\mathcal{F}$ is said to be *isomorphic* to a functor $\mathcal{G}$ if there exists a natural isomorphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$.

**Problem 8.3.1.** 1. Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$ be functors. Show that a natural transformation $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a natural isomorphism if and only if $\varphi(A)$ is an isomorphism for all objects $A \in \mathcal{C}$.

2. Let $(A \times B, p_A, p_B)$ be the product of $A$ and $B$ in $\mathcal{C}$. Then there is a natural isomorphism

$$\mathrm{Mor}(\text{-}, A \times B) \cong \mathrm{Mor}_{\mathcal{C}}(\text{-}, A) \times \mathrm{Mor}_{\mathcal{C}}(\text{-}, B).$$

3. Let $\mathcal{C}$ be a category with finite products. For each object $A$ in $\mathcal{C}$ show that there exists a morphism $\Delta_A : A \longrightarrow A \times A$ satisfying $p_1 \Delta_A = 1_A = p_2 \Delta_A$. Show that this defines a natural transformation. What are the functors?

4. Let $\mathcal{C}$ be a category with finite products. Show that there is a bi*functor* $\text{-} \times \text{-} : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ such that $(\text{-} \times \text{-})(A, B)$ is the object of a product of $A$ and $B$. We denote elements in the image of this functor by $A \times B := (\text{-} \times \text{-})(A, B)$ and similarly $f \times g$.

5. With the notation of the preceding problem show that there is a natural transformation $\alpha(A, B, C) : (A \times B) \times C \cong A \times (B \times C)$. Show that the diagram

(*coherence or constraints*)

$$\begin{tikzcd}((A \times B) \times C) \times D \arrow[r, "\alpha(A,B,C)\times 1"] \arrow[d, "\alpha(A\times B,C,D)"] & (A \times (B \times C)) \times D \arrow[r, "\alpha(A,B\times C,D)"] & A \times ((B \times C) \times D) \arrow[d, "1\times\alpha(B,C,D)"'] \\ (A \times B) \times (C \times D) \arrow[rr, "\alpha(A,B,C\times D)"] & & A \times (B \times (C \times D))\end{tikzcd}$$

commutes.

6. With the notation of the preceding problem show that there are a natural transformations $\lambda(A) : E \times A \longrightarrow A$ and $\rho(A) : A \times E \longrightarrow A$ such that the diagram (*coherence or constraints*)

$$\begin{tikzcd}(A \times E) \times B \arrow[rr, "\alpha(A,E,B)"] \arrow[dr, "\rho(A)\times 1"'] & & A \times (E \times B) \arrow[dl, "1\times\lambda(B)"] \\ & A \times B & \end{tikzcd}$$

**Definition 8.3.6.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is called an *equivalence of categories* if there exists a covariant functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ and natural isomorphisms $\varphi : \mathcal{GF} \cong \mathrm{Id}_{\mathcal{C}}$ and $\psi : \mathcal{FG} \cong \mathrm{Id}_{\mathcal{D}}$.

A contravariant functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is called a *duality of categories* if there exists a contravariant functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ and natural isomorphisms $\varphi : \mathcal{GF} \cong \mathrm{Id}_{\mathcal{C}}$ and $\psi : \mathcal{FG} \cong \mathrm{Id}_{\mathcal{D}}$.

A category $\mathcal{C}$ is said to be *equivalent* to a category $\mathcal{D}$ if there exists an equivalence $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$. A category $\mathcal{C}$ is said to be *dual* to a category $\mathcal{D}$ if there exists a duality $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$.

**Problem 8.3.2.** 1. Show that the dual category $\mathcal{C}^{op}$ is dual to the category $\mathcal{C}$.

2. Let $\mathcal{D}$ be a category dual to the category $\mathcal{C}$. Show that $\mathcal{D}$ is equivalent to the dual category $\mathcal{C}^{op}$.

3. Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be an equivalence with respect to $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$, $\varphi : \mathcal{GF} \cong \mathrm{Id}_{\mathcal{C}}$, and $\psi : \mathcal{FG} \cong \mathrm{Id}_{\mathcal{D}}$. Show that $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ is an equivalence. Show that $\mathcal{G}$ is uniquely determined by $\mathcal{F}$ up to a natural isomorphism.