CHAPTER 8

Toolbox

1. Categories

Definition 8.1.1. Let $\mathcal{C}$ consist of

1. a class $\mathrm{Ob}\,\mathcal{C}$ whose elements $A, B, C, \ldots \in \mathrm{Ob}\,\mathcal{C}$ are called *objects*,
2. a family $\{\mathrm{Mor}_{\mathcal{C}}(A,B) | A, B \in \mathrm{Ob}\,\mathcal{C}\}$ of mutually disjoint sets whose elements $f, g, \ldots \in \mathrm{Mor}_{\mathcal{C}}(A,B)$ are called *morphisms*, and
3. a family $\{\mathrm{Mor}_{\mathcal{C}}(A,B) \times \mathrm{Mor}_{\mathcal{C}}(B,C) \ni (f,g) \mapsto gf \in \mathrm{Mor}_{\mathcal{C}}(A,C) | A, B, C \in \mathrm{Ob}\,\mathcal{C}\}$ of maps called *compositions*.

$\mathcal{C}$ is called a *category* if the following axioms hold for $\mathcal{C}$

1. Associative Law:
$\forall A, B, C, D \in \mathrm{Ob}\,\mathcal{C}, f \in \mathrm{Mor}_{\mathcal{C}}(A,B), g \in \mathrm{Mor}_{\mathcal{C}}(B,C), h \in \mathrm{Mor}_{\mathcal{C}}(C,D):$
$$h(gf) = (hg)f;$$
2. Identity Law:
$\forall A \in \mathrm{Ob}\,\mathcal{C}\ \exists 1_A \in \mathrm{Mor}_{\mathcal{C}}(A,A)\ \forall B, C \in \mathrm{Ob}\,\mathcal{C},\ \forall f \in \mathrm{Mor}_{\mathcal{C}}(A,B),\ \forall g \in \mathrm{Mor}_{\mathcal{C}}(C,A):$
$$1_A g = g \qquad \text{and} \qquad f 1_A = f.$$

Examples 8.1.2. 1. The category of sets **Set**.

2. The categories of R-modules R-**Mod**, k-vector spaces k-**Vec** or k-**Mod**, groups **Gr**, abelian groups **Ab**, monoids **Mon**, commutative monoids **cMon**, rings **Ri**, fields **Fld**, topological spaces **Top**.

Since modules are highly important for all what follows, we recall the definition and some basic properties.

Definition and Remark 8.1.3. Let R be a ring (always associative with unit). A *left R-module* ${}_RM$ is an (additively written) abelian group M together with an operation $R \times M \ni (r,m) \mapsto rm \in M$ such that

1. $(rs)m = r(sm)$,
2. $(r+s)m = rm + sm$,
3. $r(m+m') = rm + rm'$,
4. $1m = m$

for all $r, s \in R$, $m, m' \in M$.

Each abelian group is a $\mathbb{Z}$-module in a unique way.

A *homomorphism of left R-modules* $f : {}_RM \longrightarrow {}_RN$ is a group homomorphism such that $f(rm) = rf(m)$.

Analogously we define right R-modules M_R and their homomorphisms.

We denote by $\mathrm{Hom}_R(.M, .N)$ the set of homomorphisms of left R-modules ${}_RM$ and ${}_RN$. Similarly $\mathrm{Hom}_R(M., N.)$ denotes the set of homomorphisms of right R-modules M_R and N_R. Both sets are abelian groups by $(f + g)(m) := f(m) + g(m)$.

For arbitrary categories we adopt many of the customary notations.

Notation 8.1.4. $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ will be written as $f : A \longrightarrow B$ or $A \xrightarrow{f} B$. A is called the *domain*, B the *range* of f.

The *composition* of two morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ is written as $gf : A \longrightarrow C$ or as $g \circ f : A \longrightarrow C$.

Definition and Remark 8.1.5. A morphism $f : A \longrightarrow B$ is called an *isomorphism* if there exists a morphism $g : B \longrightarrow A$ in $\mathcal{C}$ such that $fg = 1_B$ and $gf = 1_A$. The morphism g is uniquely determined by f since $g' = g'fg = g$. We write $f^{-1} := g$.

An object A is said to be *isomorphic* to an object B if there exists an isomorphism $f : A \longrightarrow B$. If f is an isomorphism the so is f^{-1}. If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are isomorphisms in $\mathcal{C}$ then so is $gf : A \longrightarrow C$. We have: $(f^{-1})^{-1} = f$ and $(gf)^{-1} = f^{-1}g^{-1}$. The relation of being isomorphic between objects is an equivalence relation.

Example 8.1.6. In the categories **Set**, R-**Mod**, k-**Vec**, **Gr**, **Ab**, **Mon**, **cMon**, **Ri**, **Fld** the isomorphisms are exactly those morphisms which are bijective as set maps.

In **Top** the set $M = \{a, b\}$ with $\mathfrak{T}_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and with $\mathfrak{T}_2 = \{\emptyset, M\}$ defines two different topological spaces. The map $f = \mathrm{id} : (M, \mathfrak{T}_1) \longrightarrow (M, \mathfrak{T}_2)$ is bijective and continuous. The inverse map, however, is not continuous, hence f is no isomorphism (homeomorphism).

Many well known concepts can be defined for arbitrary categories. We are going to apply some of them. Here are two examples.

Definition 8.1.7. 1. A morphism $f : A \longrightarrow B$ is called a *monomorphism* if $\forall C \in \mathrm{Ob}\,\mathcal{C},\ \forall g, h \in \mathrm{Mor}_{\mathcal{C}}(C, A)$:

$$fg = fh \Longrightarrow g = h \quad (f \text{ is left cancellable}).$$

2. A morphism $f : A \longrightarrow B$ is called an *epimorphism* if $\forall C \in \mathrm{Ob}\,\mathcal{C},\ \forall g, h \in \mathrm{Mor}_{\mathcal{C}}(B, C)$:

$$gf = hf \Longrightarrow g = h \quad (f \text{ is right cancellable}).$$

Definition 8.1.8. Given $A, B \in \mathcal{C}$. An object $A \times B$ in $\mathcal{C}$ together with morphisms $p_A : A \times B \longrightarrow A$ and $p_B : A \times B \longrightarrow B$ is called a (categorical) *product* of A and B if for every object $T \in \mathcal{C}$ and every pair of morphisms $f : T \longrightarrow A$ and

$g : T \to B$ there exists a unique morphism $(f, g) : T \to A \times B$ such that the diagram

$$\begin{tikzcd} & T \arrow[dl, "f"'] \arrow[d, "(f,g)"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "p_A"'] \arrow[r, "p_B"] & B \end{tikzcd}$$

commutes.

An object $E \in \mathcal{C}$ is called a *final object* if for every object $T \in \mathcal{C}$ there exists a unique morphism $e : T \to E$ (i.e. $\mathrm{Mor}_{\mathcal{C}}(T, E)$ consists of exactly one element).

A category $\mathcal{C}$ which has a product for any two objects A and B and which has a final object is called a category with finite products.

Remark 8.1.9. If the product $(A \times B, p_A, p_B)$ of two objects A and B in $\mathcal{C}$ exists then it is unique up to isomorphism.

If the final object E in $\mathcal{C}$ exists then it is unique up to isomorphism.

Problem 8.1.1. Let $\mathcal{C}$ be a category with finite products. Give a definition of a product of a family $A_1, \ldots, A_n$ $(n \geq 0)$. Show that products of such families exist in $\mathcal{C}$.

Definition and Remark 8.1.10. Let $\mathcal{C}$ be a category. Then $\mathcal{C}^{op}$ with the following data $\mathrm{Ob}\, \mathcal{C}^{op} := \mathrm{Ob}\, \mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}^{op}}(A, B) := \mathrm{Mor}_{\mathcal{C}}(B, A)$, and $f \circ_{op} g := g \circ f$ defines a new category, the *dual category* to $\mathcal{C}$.

Remark 8.1.11. Any notion expressed in categorical terms (with objects, morphisms, and their composition) has a *dual notion*, i.e. the given notion in the dual category.

Monomorphisms f in the dual category $\mathcal{C}^{op}$ are epimorphisms in the original category $\mathcal{C}$ and conversely. A final objects I in the dual category $\mathcal{C}^{op}$ is an *initial object* in the original category $\mathcal{C}$.

Definition 8.1.12. The *coproduct* of two objects in the category $\mathcal{C}$ is defined to be a product of the objects in the dual category $\mathcal{C}^{op}$.

Remark 8.1.13. Equivalent to the preceding definition is the following definition.

Given $A, B \in \mathcal{C}$. An object $A \amalg B$ in $\mathcal{C}$ together with morphisms $j_A : A \to A \amalg B$ and $j_B : B \to A \amalg B \to B$ is a (categorical) coproduct of A and B if for every object $T \in \mathcal{C}$ and every pair of morphisms $f : A \to T$ and $g : B \to T$ there exists a unique morphism $[f, g] : A \amalg B \to T$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "j_A"] \arrow[dr, "f"'] & A \amalg B \arrow[d, "{[f,g]}"] & B \arrow[l, "j_B"'] \arrow[dl, "g"] \\ & T & \end{tikzcd}$$

commutes.

The category $\mathcal{C}$ is said to have *finite coproducts* if $\mathcal{C}^{op}$ is a category with finite products. In particular coproducts are unique up to isomorphism.