CHAPTER 8

Toolbox

1. Categories

Definition 8.1.1. Let $\mathcal{C}$ consist of

1. a class $\mathrm{Ob}\,\mathcal{C}$ whose elements $A, B, C, \ldots \in \mathrm{Ob}\,\mathcal{C}$ are called *objects*,
2. a family $\{\mathrm{Mor}_{\mathcal{C}}(A,B) | A, B \in \mathrm{Ob}\,\mathcal{C}\}$ of mutually disjoint sets whose elements $f, g, \ldots \in \mathrm{Mor}_{\mathcal{C}}(A,B)$ are called *morphisms*, and
3. a family $\{\mathrm{Mor}_{\mathcal{C}}(A,B) \times \mathrm{Mor}_{\mathcal{C}}(B,C) \ni (f,g) \mapsto gf \in \mathrm{Mor}_{\mathcal{C}}(A,C) | A, B, C \in \mathrm{Ob}\,\mathcal{C}\}$ of maps called *compositions*.

$\mathcal{C}$ is called a *category* if the following axioms hold for $\mathcal{C}$

1. Associative Law:
$\forall A, B, C, D \in \mathrm{Ob}\,\mathcal{C}, f \in \mathrm{Mor}_{\mathcal{C}}(A,B), g \in \mathrm{Mor}_{\mathcal{C}}(B,C), h \in \mathrm{Mor}_{\mathcal{C}}(C,D):$
$$h(gf) = (hg)f;$$
2. Identity Law:
$\forall A \in \mathrm{Ob}\,\mathcal{C}\ \exists 1_A \in \mathrm{Mor}_{\mathcal{C}}(A,A)\ \forall B, C \in \mathrm{Ob}\,\mathcal{C},\ \forall f \in \mathrm{Mor}_{\mathcal{C}}(A,B),\ \forall g \in \mathrm{Mor}_{\mathcal{C}}(C,A):$
$$1_A g = g \qquad \text{and} \qquad f 1_A = f.$$

Examples 8.1.2. 1. The category of sets **Set**.

2. The categories of R-modules R-**Mod**, k-vector spaces k-**Vec** or k-**Mod**, groups **Gr**, abelian groups **Ab**, monoids **Mon**, commutative monoids **cMon**, rings **Ri**, fields **Fld**, topological spaces **Top**.

Since modules are highly important for all what follows, we recall the definition and some basic properties.

Definition and Remark 8.1.3. Let R be a ring (always associative with unit). A *left R-module* ${}_RM$ is an (additively written) abelian group M together with an operation $R \times M \ni (r,m) \mapsto rm \in M$ such that

1. $(rs)m = r(sm)$,
2. $(r+s)m = rm + sm$,
3. $r(m+m') = rm + rm'$,
4. $1m = m$

for all $r, s \in R$, $m, m' \in M$.

Each abelian group is a $\mathbb{Z}$-module in a unique way.

A *homomorphism of left R-modules* $f : {}_RM \longrightarrow {}_RN$ is a group homomorphism such that $f(rm) = rf(m)$.

Analogously we define right R-modules M_R and their homomorphisms.

We denote by $\mathrm{Hom}_R(.M, .N)$ the set of homomorphisms of left R-modules ${}_RM$ and ${}_RN$. Similarly $\mathrm{Hom}_R(M., N.)$ denotes the set of homomorphisms of right R-modules M_R and N_R. Both sets are abelian groups by $(f + g)(m) := f(m) + g(m)$.

For arbitrary categories we adopt many of the customary notations.

Notation 8.1.4. $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ will be written as $f : A \longrightarrow B$ or $A \xrightarrow{f} B$. A is called the *domain*, B the *range* of f.

The *composition* of two morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ is written as $gf : A \longrightarrow C$ or as $g \circ f : A \longrightarrow C$.

Definition and Remark 8.1.5. A morphism $f : A \longrightarrow B$ is called an *isomorphism* if there exists a morphism $g : B \longrightarrow A$ in $\mathcal{C}$ such that $fg = 1_B$ and $gf = 1_A$. The morphism g is uniquely determined by f since $g' = g'fg = g$. We write $f^{-1} := g$.

An object A is said to be *isomorphic* to an object B if there exists an isomorphism $f : A \longrightarrow B$. If f is an isomorphism the so is f^{-1}. If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are isomorphisms in $\mathcal{C}$ then so is $gf : A \longrightarrow C$. We have: $(f^{-1})^{-1} = f$ and $(gf)^{-1} = f^{-1}g^{-1}$. The relation of being isomorphic between objects is an equivalence relation.

Example 8.1.6. In the categories **Set**, R-**Mod**, k-**Vec**, **Gr**, **Ab**, **Mon**, **cMon**, **Ri**, **Fld** the isomorphisms are exactly those morphisms which are bijective as set maps.

In **Top** the set $M = \{a, b\}$ with $\mathfrak{T}_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and with $\mathfrak{T}_2 = \{\emptyset, M\}$ defines two different topological spaces. The map $f = \mathrm{id} : (M, \mathfrak{T}_1) \longrightarrow (M, \mathfrak{T}_2)$ is bijective and continuous. The inverse map, however, is not continuous, hence f is no isomorphism (homeomorphism).

Many well known concepts can be defined for arbitrary categories. We are going to apply some of them. Here are two examples.

Definition 8.1.7. 1. A morphism $f : A \longrightarrow B$ is called a *monomorphism* if $\forall C \in \mathrm{Ob}\,\mathcal{C},\ \forall g, h \in \mathrm{Mor}_{\mathcal{C}}(C, A)$:

$$fg = fh \Longrightarrow g = h \quad (f \text{ is left cancellable}).$$

2. A morphism $f : A \longrightarrow B$ is called an *epimorphism* if $\forall C \in \mathrm{Ob}\,\mathcal{C},\ \forall g, h \in \mathrm{Mor}_{\mathcal{C}}(B, C)$:

$$gf = hf \Longrightarrow g = h \quad (f \text{ is right cancellable}).$$

Definition 8.1.8. Given $A, B \in \mathcal{C}$. An object $A \times B$ in $\mathcal{C}$ together with morphisms $p_A : A \times B \longrightarrow A$ and $p_B : A \times B \longrightarrow B$ is called a (categorical) *product* of A and B if for every object $T \in \mathcal{C}$ and every pair of morphisms $f : T \longrightarrow A$ and

$g : T \longrightarrow B$ there exists a unique morphism $(f, g) : T \longrightarrow A \times B$ such that the diagram

$$\begin{tikzcd} & T \arrow[dl, "f"'] \arrow[d, "(f,g)"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B \end{tikzcd}$$

commutes.

An object $E \in \mathcal{C}$ is called a *final object* if for every object $T \in \mathcal{C}$ there exists a unique morphism $e : T \longrightarrow E$ (i.e. $\mathrm{Mor}_{\mathcal{C}}(T, E)$ consists of exactly one element).

A category $\mathcal{C}$ which has a product for any two objects A and B and which has a final object is called a category with finite products.

Remark 8.1.9. If the product $(A \times B, p_A, p_B)$ of two objects A and B in $\mathcal{C}$ exists then it is unique up to isomorphism.

If the final object E in $\mathcal{C}$ exists then it is unique up to isomorphism.

Problem 8.1.1. Let $\mathcal{C}$ be a category with finite products. Give a definition of a product of a family $A_1, \dots, A_n$ $(n \geq 0)$. Show that products of such families exist in $\mathcal{C}$.

Definition and Remark 8.1.10. Let $\mathcal{C}$ be a category. Then $\mathcal{C}^{op}$ with the following data $\mathrm{Ob}\,\mathcal{C}^{op} := \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}^{op}}(A, B) := \mathrm{Mor}_{\mathcal{C}}(B, A)$, and $f \circ_{op} g := g \circ f$ defines a new category, the *dual category* to $\mathcal{C}$.

Remark 8.1.11. Any notion expressed in categorical terms (with objects, morphisms, and their composition) has a *dual notion*, i.e. the given notion in the dual category.

Monomorphisms f in the dual category $\mathcal{C}^{op}$ are epimorphisms in the original category $\mathcal{C}$ and conversely. A final objects I in the dual category $\mathcal{C}^{op}$ is an *initial object* in the original category $\mathcal{C}$.

Definition 8.1.12. The *coproduct* of two objects in the category $\mathcal{C}$ is defined to be a product of the objects in the dual category $\mathcal{C}^{op}$.

Remark 8.1.13. Equivalent to the preceding definition is the following definition.

Given $A, B \in \mathcal{C}$. An object $A \amalg B$ in $\mathcal{C}$ together with morphisms $j_A : A \longrightarrow A \amalg B$ and $j_B : B \longrightarrow A \amalg B \longrightarrow B$ is a (categorical) coproduct of A and B if for every object $T \in \mathcal{C}$ and every pair of morphisms $f : A \longrightarrow T$ and $g : B \longrightarrow T$ there exists a unique morphism $[f, g] : A \amalg B \longrightarrow T$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "j_A"] \arrow[dr, "f"'] & A \amalg B \arrow[d, "{[f,g]}"] & B \arrow[l, "j_B"'] \arrow[dl, "g"] \\ & T & \end{tikzcd}$$

commutes.

The category $\mathcal{C}$ is said to have *finite coproducts* if $\mathcal{C}^{op}$ is a category with finite products. In particular coproducts are unique up to isomorphism.

2. Functors

Definition 8.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $\mathcal{F}$ consist of

1. a map $\operatorname{Ob}\mathcal{C} \ni A \mapsto \mathcal{F}(A) \in \operatorname{Ob}\mathcal{D}$,
2. a family of maps

$$\{\mathcal{F}_{A,B} : \operatorname{Mor}_{\mathcal{C}}(A,B) \ni f \mapsto \mathcal{F}_{A,B}(f) \in \operatorname{Mor}_{\mathcal{D}}(\mathcal{F}(A),\mathcal{F}(B)) | A,B \in \mathcal{C}\}$$

$$[\text{or } \{\mathcal{F}_{A,B} : \operatorname{Mor}_{\mathcal{C}}(A,B) \ni f \mapsto \mathcal{F}_{A,B}(f) \in \operatorname{Mor}_{\mathcal{D}}(\mathcal{F}(B),\mathcal{F}(A)) | A,B \in \mathcal{C}\}]$$

$\mathcal{F}$ is called a *covariant* [*contravariant*] *functor* if

1. $\mathcal{F}_{A,A}(1_A) = 1_{\mathcal{F}(A)}$ for all $A \in \operatorname{Ob}\mathcal{C}$,
2. $\mathcal{F}_{A,C}(gf) = \mathcal{F}_{B,C}(g)\mathcal{F}_{A,B}(f)$ for all $A,B,C \in \operatorname{Ob}\mathcal{C}$.
   $[\mathcal{F}_{A,C}(gf) = \mathcal{F}_{A,B}(f)\mathcal{F}_{B,C}(g)$ for all $A,B,C \in \operatorname{Ob}\mathcal{C}]$.

Notation: We write

$$\begin{array}{ccc} A \in \mathcal{C} & \text{instead of} & A \in \operatorname{Ob}\mathcal{C} \\ f \in \mathcal{C} & \text{instead of} & f \in \operatorname{Mor}_{\mathcal{C}}(A,B) \\ \mathcal{F}(f) & \text{instead of} & \mathcal{F}_{A,B}(f). \end{array}$$

Examples 8.2.2. 1. $\operatorname{Id} : \mathbf{Set} \to \mathbf{Set}$
2. Forget : R-$\mathbf{Mod} \to \mathbf{Set}$
3. Forget : $\mathbf{Ri} \to \mathbf{Ab}$
4. Forget : $\mathbf{Ab} \to \mathbf{Gr}$
5. $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}, \mathcal{P}(M) :=$ power set of M. $\mathcal{P}(f)(X) := f^{-1}(X)$ for $f : M \to N, X \subseteq N$ is a contravariant functor.
6. $\mathcal{Q} : \mathbf{Set} \to \mathbf{Set}, \mathcal{Q}(M) :=$ power set of M. $\mathcal{Q}(f)(X) := f(X)$ for $f : M \to N, X \subseteq M$ is a covariant functor.

Lemma 8.2.3. 1. *Let $X \in \mathcal{C}$. Then*

$$\operatorname{Ob}\mathcal{C} \ni A \mapsto \operatorname{Mor}_{\mathcal{C}}(X,A) \in \operatorname{Ob}\mathbf{Set}$$

$$\operatorname{Mor}_{\mathcal{C}}(A,B) \ni f \mapsto Mor_{\mathcal{C}}(X,f) \in \operatorname{Mor}_{\mathbf{Set}}(\operatorname{Mor}_{\mathcal{C}}(X,A),\operatorname{Mor}_{C}(X,B)),$$

with $\operatorname{Mor}_C(X,f) : \operatorname{Mor}_C(X,A) \ni g \mapsto fg \in \operatorname{Mor}_C(X,B)$ *or* $\operatorname{Mor}_C(X,f)(g) = fg$ *is a covariant functor* $\operatorname{Mor}_C(X,\text{-})$.

2. *Let $X \in C$. Then*

$$\operatorname{Ob}\mathcal{C} \ni A \mapsto \operatorname{Mor}_{\mathcal{C}}(A,X) \in \operatorname{Ob}\mathbf{Set}$$

$$\operatorname{Mor}_{\mathcal{C}}(A,B) \ni f \mapsto \operatorname{Mor}_{\mathcal{C}}(f,X) \in \operatorname{Mor}_{\mathbf{Set}}(\operatorname{Mor}_{\mathcal{C}}(B,X),\operatorname{Mor}_{\mathcal{C}}(A,X))$$

with $\operatorname{Mor}_{\mathcal{C}}(f,X) : \operatorname{Mor}_{\mathcal{C}}(B,X) \ni g \mapsto gf \in \operatorname{Mor}_{\mathcal{C}}(A,X)$ *or* $\operatorname{Mor}_{\mathcal{C}}(f,X)(g) = gf$ *is a contravariant functor* $\operatorname{Mor}_{\mathcal{C}}(\text{-},X)$.

Proof. 1. $\operatorname{Mor}_{\mathcal{C}}(X,1_A)(g) = 1_A g = g = \operatorname{id}(g)$, $\operatorname{Mor}_C(X,f)\operatorname{Mor}_C(X,g)(h) = fgh = \operatorname{Mor}_C(X,fg)(h)$.
2. analogously. $\square$

Remark 8.2.4. The preceding lemma shows that $\mathrm{Mor}_{\mathcal{C}}(\text{-},\text{-})$ is a functor in both arguments. A functor in two arguments is called a *bifunctor*. We can regards the bifunctor $\mathrm{Mor}_{\mathcal{C}}(\text{-},\text{-})$ as a covariant functor

$$\mathrm{Mor}_{\mathcal{C}}(\text{-},\text{-}) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathbf{Set}.$$

The use of the dual category removes the fact that the bifunctor $\mathrm{Mor}_{\mathcal{C}}(\text{-},\text{-})$ is contravariant in the first variable.

Obviously the composition of two functors is again a functor and this composition is associative. Furthermore for each category $\mathcal{C}$ there is an identity functor $\mathrm{Id}_{\mathcal{C}}$.

Functors of the form $\mathrm{Mor}_{\mathcal{C}}(X,\text{-})$ resp. $\mathrm{Mor}_{\mathcal{C}}(\text{-},X)$ are called *representable functors* (covariant resp. contravariant) and X is called the *representing object* (see also section 8.8).

3. Natural Transformations

Definition 8.3.1. Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$ be two functors. A *natural transformation* or a *functorial morphism* $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a family of morphisms $\{\varphi(A) : \mathcal{F}(A) \longrightarrow \mathcal{G}(A) | A \in \mathcal{C}\}$ such that the diagram

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\varphi(A)"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}(A) \arrow[d, "\mathcal{G}(f)"] \\ \mathcal{F}(B) \arrow[r, "\varphi(B)"'] & \mathcal{G}(B) \end{tikzcd}$$

commutes for all $f : A \longrightarrow B$ in $\mathcal{C}$, i.e. $\mathcal{G}(f)\varphi(A) = \varphi(B)\mathcal{F}(f)$.

Lemma 8.3.2. *Given covariant functors* $\mathcal{F} = \mathrm{Id}_{\mathbf{Set}} : \mathbf{Set} \longrightarrow \mathbf{Set}$ *and* $\mathcal{G} = \mathrm{Mor}_{\mathbf{Set}}(\mathrm{Mor}_{\mathbf{Set}}(-, A), A) : \mathbf{Set} \longrightarrow \mathbf{Set}$ *for a set* A. *Then* $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ *with*

$$\varphi(B) : B \ni b \mapsto (\mathrm{Mor}_{\mathbf{Set}}(B, A) \ni f \mapsto f(b) \in A) \in \mathcal{G}(B)$$

is a natural transformation.

Proof. Given $g : B \longrightarrow C$. Then the following diagram commutes

$$\begin{tikzcd} B \arrow[r, "\varphi(B)"] \arrow[d, "g"'] & \mathrm{Mor}_{\mathbf{Set}}(\mathrm{Mor}_{\mathbf{Set}}(B, A), A) \arrow[d, "\mathrm{Mor}_{\mathbf{Set}}(\mathrm{Mor}_{\mathbf{Set}}(g, A), A)"] \\ C \arrow[r, "\varphi(C)"] & \mathrm{Mor}_{\mathbf{Set}}(\mathrm{Mor}_{\mathbf{Set}}(C, A), A) \end{tikzcd}$$

since

$$\begin{aligned} &\varphi(C)\mathcal{F}(g)(b)(f) = \varphi(C)g(b)(f) = fg(b) = \varphi(B)(b)(fg) \\ &= [\varphi(B)(b)\mathrm{Mor}_{\mathbf{Set}}(g, A)](f) = [\mathrm{Mor}_{\mathbf{Set}}(\mathrm{Mor}_{\mathbf{Set}}(g, A), A)\varphi(A)(b)](f). \end{aligned}$$

□

Lemma 8.3.3. *Let* $f : A \longrightarrow B$ *be a morphism in* $\mathcal{C}$. *Then* $\mathrm{Mor}_{\mathcal{C}}(f, \text{-}) : \mathrm{Mor}_{\mathcal{C}}(B, \text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(A, \text{-})$ *given by* $\mathrm{Mor}_{\mathcal{C}}(f, C) : \mathrm{Mor}_{\mathcal{C}}(B, C) \ni g \mapsto gf \in \mathrm{Mor}_{\mathcal{C}}(A, C)$ *is a natural transformation of covariant functors.*

Let $f : A \longrightarrow B$ *be a morphism in* $\mathcal{C}$. *Then* $\mathrm{Mor}_{\mathcal{C}}(\text{-}, f) : \mathrm{Mor}_{\mathcal{C}}(\text{-}, A) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(\text{-}, B)$ *given by* $\mathrm{Mor}_{\mathcal{C}}(C, f) : \mathrm{Mor}_{\mathcal{C}}(C, A) \ni g \mapsto fg \in \mathrm{Mor}_{\mathcal{C}}(C, B)$ *is a natural transformation of contravariant functors.*

Proof. Let $h : C \longrightarrow C'$ be a morphism in $\mathcal{C}$. Then the diagrams

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(B,C) \arrow[r, "\mathrm{Mor}_{\mathcal{C}}(f,C)"] \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(B,h)"'] & \mathrm{Mor}_{\mathcal{C}}(A,C) \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(A,h)"] \\
\mathrm{Mor}_{\mathcal{C}}(B,C') \arrow[r, "\mathrm{Mor}_{\mathcal{C}}(f,C')"'] & \mathrm{Mor}_{\mathcal{C}}(A,C')
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(C',A) \arrow[r, "\mathrm{Mor}_{\mathcal{C}}(C',f)"] \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(h,A)"'] & \mathrm{Mor}_{\mathcal{C}}(C',B) \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(h,B)"] \\
\mathrm{Mor}_{\mathcal{C}}(C,A) \arrow[r, "\mathrm{Mor}_{\mathcal{C}}(C,f)"'] & \mathrm{Mor}_{\mathcal{C}}(C,B)
\end{tikzcd}$$

commute. $\square$

Remark 8.3.4. The composition of two natural transformations is again a natural transformation. The identity $\mathrm{id}_{\mathcal{F}}(A) := 1_{\mathcal{F}(A)}$ is also a natural transformation.

Definition 8.3.5. A natural transformation $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is called a *natural isomorphism* if there exists a natural transformation $\psi : \mathcal{G} \longrightarrow \mathcal{F}$ such that $\varphi \circ \psi = \mathrm{id}_{\mathcal{G}}$ and $\psi \circ \varphi = \mathrm{id}_{\mathcal{F}}$. The natural transformation ψ is uniquely determined by φ. We write $\varphi^{-1} := \psi$.

A functor $\mathcal{F}$ is said to be *isomorphic* to a functor $\mathcal{G}$ if there exists a natural isomorphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$.

Problem 8.3.2. 1. Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$ be functors. Show that a natural transformation $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a natural isomorphism if and only if $\varphi(A)$ is an isomorphism for all objects $A \in \mathcal{C}$.

2. Let $(A \times B, p_A, p_B)$ be the product of A and B in $\mathcal{C}$. Then there is a natural isomorphism

$$\mathrm{Mor}(\text{-}, A \times B) \cong \mathrm{Mor}_{\mathcal{C}}(\text{-}, A) \times \mathrm{Mor}_{\mathcal{C}}(\text{-}, B).$$

3. Let $\mathcal{C}$ be a category with finite products. For each object A in $\mathcal{C}$ show that there exists a morphism $\Delta_A : A \longrightarrow A \times A$ satisfying $p_1 \Delta_A = 1_A = p_2 \Delta_A$. Show that this defines a natural transformation. What are the functors?

4. Let $\mathcal{C}$ be a category with finite products. Show that there is a bi*functor* $\text{-} \times \text{-} : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ such that $(\text{-} \times \text{-})(A, B)$ is the object of a product of A and B. We denote elements in the image of this functor by $A \times B := (\text{-} \times \text{-})(A, B)$ and similarly $f \times g$.

5. With the notation of the preceding problem show that there is a natural transformation $\alpha(A, B, C) : (A \times B) \times C \cong A \times (B \times C)$. Show that the diagram

(*coherence or constraints*)

$$\begin{tikzcd}
((A \times B) \times C) \times D \arrow[r, "\alpha(A,B,C)\times 1"] \arrow[d, "\alpha(A\times B,C,D)"] & (A \times (B \times C)) \times D \arrow[r, "\alpha(A,B\times C,D)"] & A \times ((B \times C) \times D) \arrow[d, "1\times\alpha(B,C,D)"'] \\
(A \times B) \times (C \times D) \arrow[rr, "\alpha(A,B,C\times D)"] & & A \times (B \times (C \times D))
\end{tikzcd}$$

commutes.

6. With the notation of the preceding problem show that there are a natural transformations $\lambda(A) : E \times A \longrightarrow A$ and $\rho(A) : A \times E \longrightarrow A$ such that the diagram (*coherence or constraints*)

$$\begin{tikzcd}
(A \times E) \times B \arrow[rr, "\alpha(A,E,B)"] \arrow[dr, "\rho(A)\times 1"'] & & A \times (E \times B) \arrow[dl, "1\times\lambda(B)"] \\
& A \times B &
\end{tikzcd}$$

Definition 8.3.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is called an *equivalence of categories* if there exists a covariant functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ and natural isomorphisms $\varphi : \mathcal{GF} \cong \mathrm{Id}_{\mathcal{C}}$ and $\psi : \mathcal{FG} \cong \mathrm{Id}_{\mathcal{D}}$.

A contravariant functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is called a *duality of categories* if there exists a contravariant functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ and natural isomorphisms $\varphi : \mathcal{GF} \cong \mathrm{Id}_{\mathcal{C}}$ and $\psi : \mathcal{FG} \cong \mathrm{Id}_{\mathcal{D}}$.

A category $\mathcal{C}$ is said to be *equivalent* to a category $\mathcal{D}$ if there exists an equivalence $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$. A category $\mathcal{C}$ is said to be *dual* to a category $\mathcal{D}$ if there exists a duality $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$.

Problem 8.3.3. 1. Show that the dual category $\mathcal{C}^{op}$ is dual to the category $\mathcal{C}$.

2. Let $\mathcal{D}$ be a category dual to the category $\mathcal{C}$. Show that $\mathcal{D}$ is equivalent to the dual category $\mathcal{C}^{op}$.

3. Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be an equivalence with respect to $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$, $\varphi : \mathcal{GF} \cong \mathrm{Id}_{\mathcal{C}}$, and $\psi : \mathcal{FG} \cong \mathrm{Id}_{\mathcal{D}}$. Show that $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ is an equivalence. Show that $\mathcal{G}$ is uniquely determined by $\mathcal{F}$ up to a natural isomorphism.

4. Tensor Products

Definition and Remark 8.4.1. Let M_R and ${}_RN$ be R-modules, and let A be an abelian group. A map $f : M \times N \longrightarrow A$ is called *R-bilinear* if

1. $f(m + m', n) = f(m, n) + f(m', n)$,
2. $f(m, n + n') = f(m, n) + f(m, n')$,
3. $f(mr, n) = f(m, rn)$

for all $r \in R,\ m, m' \in M,\ n, n' \in N$.

Let $\mathrm{Bil}_R(M, N; A)$ denote the set of all R-bilinear maps $f : M \times N \longrightarrow A$.

$\mathrm{Bil}_R(M, N; A)$ is an abelian group with $(f + g)(m, n) := f(m, n) + g(m, n)$.

Definition 8.4.2. Let M_R and ${}_RN$ be R-modules. An abelian group $M \otimes_R N$ together with an R-bilinear map

$$\otimes : M \times N \ni (m, n) \mapsto m \otimes n \in M \otimes_R N$$

is called a *tensor product of M and N over R* if for each abelian group A and for each R-bilinear map $f : M \times N \longrightarrow A$ there exists a unique group homomorphism $g : M \otimes_R N \longrightarrow A$ such that the diagram

$$\begin{tikzcd}
M \times N \arrow[r, "\otimes"] \arrow[dr, "f"'] & M \otimes_R N \arrow[d, "g"] \\
 & A
\end{tikzcd}$$

commutes. The elements of $M \otimes_R N$ are called *tensors*, the elements of the form $m \otimes n$ are called *decomposable tensors*.

Warning: If you want to define a homomorphism $f : M \otimes_R N \longrightarrow A$ with a tensor product as domain you *must* define it by giving an R-bilinear map defined on $M \times N$.

Lemma 8.4.3. *A tensor product $(M \otimes_R N, \otimes)$ defined by M_R and ${}_RN$ is unique up to a unique isomorphism.*

Proof. Let $(M \otimes_R N, \otimes)$ and $(M \boxtimes_R N, \boxtimes)$ be tensor products. Then

$$\begin{tikzcd}
 & & M \times N \arrow[dll, "\otimes"'] \arrow[dl, "\boxtimes"] \arrow[dr, "\otimes"'] \arrow[drr, "\boxtimes"] & & \\
M \otimes_R N \arrow[r, "h"] & M \boxtimes_R N \arrow[rr, "k"] & & M \otimes_R N \arrow[r, "h"] & M \boxtimes_R N
\end{tikzcd}$$

implies $k = h^{-1}$. $\square$

Because of this fact we will henceforth talk about *the* tensor product of M and N over R.

Proposition 8.4.4. (Rules of computation in a tensor product) *Let $(M \otimes_R N, \otimes)$ be the tensor product. Then we have for all $r \in R,\ m, m' \in M,\ n, n' \in N$*

1. $M \otimes_R N = \{\sum_i m_i \otimes n_i \mid m_i \in M, n_i \in N\}$,
2. $(m + m') \otimes n = m \otimes n + m' \otimes n$,
3. $m \otimes (n + n') = m \otimes n + m \otimes n'$,
4. $mr \otimes n = m \otimes rn$ *(observe in particular, that* $\otimes : M \times N \longrightarrow M \otimes N$ *is not injective in general),*
5. *if* $f : M \times N \longrightarrow A$ *is an* R*-bilinear map and* $g : M \otimes_R N \longrightarrow A$ *is the induced homomorphism, then*
$$g(m \otimes n) = f(m, n).$$

Proof. 1. Let $B := \langle m \otimes n \rangle \subseteq M \otimes_R N$ denote the subgroup of $M \otimes_R N$ generated by the decomposable tensors $m \otimes n$. Let $j : B \longrightarrow M \otimes_R N$ be the embedding homomorphism. We get an induced map $\otimes' : M \times N \longrightarrow B$. In the following diagram

$$\begin{tikzcd}
M \times N \arrow[r, "\otimes'"] \arrow[dr, "\otimes'"'] & B \arrow[r, "j"] \arrow[d, "\mathrm{id}_B"'] & M \otimes_R N \arrow[dl, "p"] \arrow[d, "jp"] \\
 & B \arrow[r, "j"] & M \otimes_R N
\end{tikzcd}$$

we have $\mathrm{id}_B \circ \otimes' = \otimes'$, p with $p \circ j \circ \otimes' = p \circ \otimes = \otimes'$ exists since $\otimes'$ is R-bilinear. Because of $jp \circ \otimes = j \circ \otimes' = \otimes = \mathrm{id}_{M \otimes_R N} \circ \otimes$ we get $jp = \mathrm{id}_{M \otimes_R N}$, hence the embedding j is surjective and thus the identity.

2. $(m + m') \otimes n = \otimes(m + m', n) = \otimes(m, n) + \otimes(m', n) = m \otimes n + m' \otimes n$.

3. and 4. analogously.

5. is precisely the definition of the induced homomorphism. $\square$

Remark 8.4.5. To construct tensor products, we use the notion of a free module.

Let X be a set and R be a ring. An R-module RX together with a map $\iota : X \longrightarrow RX$ is called a *free R-module generated by X*, if for every R-module M and for every map $f : X \longrightarrow M$ there exists a unique homomorphism of R-modules $g : RX \longrightarrow M$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[dr, "f"'] & RX \arrow[d, "g"] \\
 & M
\end{tikzcd}$$

commutes.

Free R-modules exist and can be constructed as $RX := \{\alpha : X \longrightarrow R |$ for almost all $x \in X : \alpha(x) = 0\}$.

Proposition 8.4.6. *Given R-modules M_R and ${}_RN$. Then there exists a tensor product $(M \otimes_R N, \otimes)$.*

Proof. Define $M \otimes_R N := \mathbb{Z}\{M \times N\}/U$ where $\mathbb{Z}\{M \times N\}$ is a free $\mathbb{Z}$-module over $M \times N$ (the free abelian group) and U is generated by

$$\begin{aligned}
&\iota(m+m',n)-\iota(m,n)-\iota(m',n)\\
&\iota(m,m+n')-\iota(m,n)-\iota(m,n')\\
&\iota(mr,n)-\iota(m,rn)
\end{aligned}$$

for all $r \in R$, $m, m' \in M$, $n, n' \in N$. Consider

$$\begin{tikzcd}
M \times N \arrow[r, "\iota"] \arrow[drr, "\psi"'] & \mathbb{Z}\{M \times N\} \arrow[r, "\nu"] \arrow[dr, "\rho"] & M \otimes_R N = \mathbb{Z}\{M \times N\}/U \arrow[d, "g"] \\
 & & A
\end{tikzcd}$$

Let ψ be given. Then there is a unique $\rho \in \mathrm{Hom}(\mathbb{Z}\{M \times N\}, A)$ such that $\rho\iota = \psi$. Since ψ is R-bilinear we get $\rho(\iota(m+m',n) - \iota(m,n) - \iota(m'n)) = \psi(m+m',n) - \psi(m,n) - \psi(m',n) = 0$ and similarly $\rho(\iota(m,n+n') - \iota(m,n) - \iota(m,n')) = 0$ and $\rho(\iota(mr,n) - \iota(m,rn)) = 0$. So we get $\rho(U) = 0$. This implies that there is a unique $g \in \mathrm{Hom}(M \otimes_R N, A)$ such that $g\nu = \rho$ (homomorphism theorem). Let $\otimes := \nu \circ \iota$. Then $\otimes$ is bilinear since $(m+m') \otimes n = \nu \circ \iota(m+m',n) = \nu(\iota(m+m',n)) = \nu(\iota(m+m',n) - \iota(m,n) - \iota(m',n) + \iota(m,n) + \iota(m',n)) = \nu(\iota(m,n) + \iota(m',n)) = \nu \circ \iota(m,n) + \nu \circ \iota(m',n) = m \otimes n + m' \otimes n$. The other two properties are obtained in an analogous way.

We have to show that $(M \otimes_R N, \otimes)$ is a tensor product. The above diagram shows that for each abelian group A and for each R-bilinear map $\psi : M \times N \to A$ there is a $g \in \mathrm{Hom}(M \otimes_R N, A)$ such that $g \circ \otimes = \psi$. Given $h \in \mathrm{Hom}(M \otimes_R N, A)$ with $h \circ \otimes = \psi$. Then $h \circ \nu \circ \iota = \psi$. This implies $h \circ \nu = \rho = g \circ \nu$ hence $g = h$. □

Proposition and Definition 8.4.7. *Given two homomorphisms*

$$f \in \mathrm{Hom}_R(M., M'.) \text{ and } g \in \mathrm{Hom}_R(.N, .N').$$

Then there is a unique homomorphism

$$f \otimes_R g \in \mathrm{Hom}(M \otimes_R N, M' \otimes_R N')$$

such that $f \otimes_R g(m \otimes n) = f(m) \otimes g(n)$, i.e. the following diagram commutes

$$\begin{tikzcd}
M \times N \arrow[r, "\otimes"] \arrow[d, "f \times g"'] & M \otimes_R N \arrow[d, "f \otimes_R g"] \\
M' \times N' \arrow[r, "\otimes"'] & M' \otimes_R N'
\end{tikzcd}$$

Proof. $\otimes \circ (f \times g)$ is bilinear. □

Notation 8.4.8. We often write $f \otimes_R N := f \otimes_R 1_N$ and $M \otimes_R g := 1_M \otimes_R g$. We have the following rule of computation:

$$f \otimes_R g = (f \otimes_R N') \circ (M \otimes_R g) = (M' \otimes_R g) \circ (f \otimes_R N)$$

since $f \times g = (f \times N') \circ (M \times g) = (M' \times g) \circ (f \times N)$.

Proposition 8.4.9. *The following define covariant functors*

1. $- \otimes N : \mathbf{Mod}\text{-}R \longrightarrow \mathbf{Ab}$;
2. $M \otimes - : R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab}$;
3. $- \otimes - : \mathbf{Mod}\text{-}R \times R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab}$.

Proof. $(f \times g) \circ (f' \times g') = ff' \times gg'$ implies $(f \otimes_R g) \circ (f' \otimes_R g') = ff' \times gg'$. Furthermore $1_M \times 1_N = 1_{M \times N}$ implies $1_M \otimes_R 1_N = 1_{M \otimes_R N}$. □

Definition 8.4.10. Let R, S be rings and let M be a left R-module and a right S-module. M is called an *R-S-bimodule* if $(rm)s = r(ms)$. We define $\mathrm{Hom}_{R\text{-}S}(.M.,.N.) := \mathrm{Hom}_R(.M,.N) \cap \mathrm{Hom}_S(M.,N.)$.

Remark 8.4.11. Let M_S be a right S-module and let $R \times M \longrightarrow M$ a map. M is an R-S-bimodule if and only if

1. $\forall r \in R : (M \ni m \mapsto rm \in M) \in \mathrm{Hom}_S(M.,M.)$,
2. $\forall r, r' \in R, m \in M : (r + r')m = rm + r'm$,
3. $\forall r, r' \in R, m \in M : (rr')m = r(r'm)$,
4. $\forall m \in M : 1m = m$.

Lemma 8.4.12. *Let ${}_RM_S$ and ${}_SN_T$ be bimodules. Then ${}_R(M \otimes_S N)_T$ is a bimodule by $r(m \otimes n) := rm \otimes n$ and $(m \otimes n)t := m \otimes nt$.*

Proof. Obviously we have 2.-4. Furthermore $(r \otimes_S \mathrm{id})(m \otimes n) = rm \otimes n = r(m \otimes n)$ is a homomorphism. □

Corollary 8.4.13. *Given bimodules ${}_RM_S$, ${}_SN_T$, ${}_RM'_S$, ${}_SN'_T$ and homomorphisms $f \in \mathrm{Hom}_{R\text{-}S}(.M.,.M'.)$ and $g \in \mathrm{Hom}_{S\text{-}T}(.N.,.N'.)$. Then we have $f \otimes_S g \in \mathrm{Hom}_{R\text{-}T}(.M \otimes_S N.,.M' \otimes_S N'.)$.*

Proof. $f \otimes_S g(rm \otimes nt) = f(rm) \otimes g(nt) = r(f \otimes_S g)(m \otimes n)t$. □

Remark 8.4.14. Every module M over a commutative ring $\mathbb{K}$ and in particular every vector space over a field $\mathbb{K}$ is a $\mathbb{K}$-$\mathbb{K}$-bimodule by $\lambda m = m\lambda$. So there is an embedding functor $\iota : \mathbb{K}\text{-}\mathbf{Mod} \longrightarrow \mathbb{K}\text{-}\mathbf{Mod}\text{-}\mathbb{K}$. Observe that there are $\mathbb{K}$-$\mathbb{K}$-bimodules that do not satisfy $\lambda m = m\lambda$. Take for example an automorphism $\alpha : \mathbb{K} \longrightarrow \mathbb{K}$ and a left $\mathbb{K}$-module M and define $m\lambda := \alpha(\lambda)m$. Then M is such a $\mathbb{K}$-$\mathbb{K}$-bimodule.

The tensor product $M \otimes_{\mathbb{K}} N$ of two $\mathbb{K}$-$\mathbb{K}$-bimodules M and N is again a $\mathbb{K}$-$\mathbb{K}$-bimodule. If we have, however, $\mathbb{K}$-$\mathbb{K}$-bimodules M and N arising from $\mathbb{K}$-modules as above, i.e. satisfying $\lambda m = m\lambda$, then their tensor product $M \otimes_{\mathbb{K}} N$ also satisfies this equation, so $M \otimes_{\mathbb{K}} N$ comes from a module in $\mathbb{K}$-$\mathbf{Mod}$. Indeed we have $\lambda m \otimes n = m\lambda \otimes n = m \otimes \lambda n = m \otimes n\lambda$. Thus the following diagram of functors commutes:

$$\begin{tikzcd}
\mathbb{K}\text{-}\mathbf{Mod} \times \mathbb{K}\text{-}\mathbf{Mod} \arrow[r, "\iota \times \iota"] \arrow[d, "\otimes_{\mathbb{K}}"'] & \mathbb{K}\text{-}\mathbf{Mod}\text{-}\mathbb{K} \times \mathbb{K}\text{-}\mathbf{Mod}\text{-}\mathbb{K} \arrow[d, "\otimes_{\mathbb{K}}"] \\
\mathbb{K}\text{-}\mathbf{Mod} \arrow[r, "\iota"'] & \mathbb{K}\text{-}\mathbf{Mod}\text{-}\mathbb{K}.
\end{tikzcd}$$

So we can consider $\mathbb{K}$-**Mod** as a (proper) subcategory of $\mathbb{K}$-**Mod**-$\mathbb{K}$. The tensor product over $\mathbb{K}$ can be restricted to $\mathbb{K}$-**Mod**.

We write the tensor product of two vector spaces M and N as $M \otimes N$.

Theorem 8.4.15. *In the category* $\mathbb{K}$-**Mod** *there are natural isomorphisms*

1. Associativity Law: $\alpha : (M \otimes N) \otimes P \cong M \otimes (N \otimes P)$.
2. Law of the Left Unit: $\lambda : \mathbb{K} \otimes M \cong M$.
3. Law of the Right Unit: $\rho : M \otimes \mathbb{K} \cong M$.
4. Symmetry Law: $\tau : M \otimes N \cong N \otimes M$.
5. Existence of Inner Hom-Functors: $\mathrm{Hom}(P \otimes M, N) \cong \mathrm{Hom}(P, \mathrm{Hom}(M, N))$.

Proof. We only describe the corresponding homomorphisms.

1. Use (8.4.45.) to define $\alpha((m \otimes n) \otimes p) := m \otimes (n \otimes p)$.
2. Define $\lambda : \mathbb{K} \otimes M \longrightarrow M$ by $\lambda(r \otimes m) := rm$.
3. Define $\rho : M \otimes \mathbb{K} \longrightarrow M$ by $\rho(m \otimes r) := mr$.
4. Define $\tau(m \otimes n) := n \otimes m$.
5. For $f : P \otimes M \longrightarrow N$ define $\phi(f) : P \longrightarrow \mathrm{Hom}(M, N)$ by $\phi(f)(p)(m) := f(p \otimes m)$. $\square$

Usually one identifies threefold tensor products along the map α so that we use $M \otimes N \otimes P = (M \otimes N) \otimes P = M \otimes (N \otimes P)$. For the notion of a monoidal or tensor category, however, this natural transformation is of central importance.

Problem 8.4.4. 1. Give an explicit proof of $M \otimes (X \oplus Y) \cong M \otimes X \oplus M \otimes Y$.

2. Show that for every finite dimensional vector space V there is a unique element $\sum_{i=1}^n v_i \otimes v_i^* \in V \otimes V^*$ such that the following holds

$$\forall v \in V : \quad \sum_i v_i^*(v) v_i = v.$$

(Hint: Use an isomorphism $\mathrm{End}(V) \cong V \otimes V^*$ and dual bases $\{v_i\}$ of V and $\{v_i^*\}$ of V^*.)

3. Show that the following diagrams (coherence diagrams or constraints) commute in $\mathbb{K}$-**Mod**:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha(A,B,C)\otimes 1"] \arrow[d, "\alpha(A\otimes B,C,D)"] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha(A,B\otimes C,D)"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "1\otimes\alpha(B,C,D)"'] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha(A,B,C\otimes D)"] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \mathbb{K}) \otimes B \arrow[rr, "\alpha(A,\mathbb{K},B)"] \arrow[dr, "\rho(A)\otimes 1"'] & & A \otimes (\mathbb{K} \otimes B) \arrow[dl, "1\otimes\lambda(B)"] \\
& A \otimes B &
\end{tikzcd}$$

4. Write $\tau(A,B) : A \otimes B \longrightarrow B \otimes A$ for $\tau(A,B) : a \otimes b \mapsto b \otimes a$. Show that τ is a natural transformation (between which functors?). Show that

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\tau(X,B)\otimes 1"] \arrow[d, "\alpha"'] & (B \otimes A) \otimes C \arrow[r, "\alpha"] & B \otimes (A \otimes C) \arrow[d, "1\otimes\tau(A,C)"] \\
A \otimes (B \otimes C) \arrow[r, "\tau(A,B\otimes C)"] & (B \otimes C) \otimes A \arrow[r, "\alpha"] & B \otimes (C \otimes A)
\end{tikzcd}$$

commutes for all $A, B, C \in \mathbb{K}$-**Mod** and that

$$\tau(B,A)\tau(A,B) = \mathrm{id}_{A\otimes B}$$

for all A, B in $\mathbb{K}$-**Mod**.

5. Find an example of M, $N \in \mathbb{K}$-**Mod**-$\mathbb{K}$ such that $M \otimes_{\mathbb{K}} N \ncong N \otimes_{\mathbb{K}} M$.

5. Algebras

Let $\mathbb{K}$ be a commutative ring. In most of our applications $\mathbb{K}$ will be a field. Tensor products of $\mathbb{K}$-modules will be simply written as $M \otimes N := M \otimes_K N$. Every such tensor product is again a $\mathbb{K}$-bimodule since each $\mathbb{K}$-module M resp. N is a $\mathbb{K}$-bimodule (see 8.4.14).

Definition 8.5.1. A $\mathbb{K}$-*algebra* is a vector space A together with a *multiplication* $\nabla : A \otimes A \to A$ that is associative:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id} \otimes \nabla"] \arrow[d, "\nabla \otimes \mathrm{id}"'] & A \otimes A \arrow[d, "\nabla"] \\ A \otimes A \arrow[r, "\nabla"'] & A \end{tikzcd}$$

and a *unit* $\eta : \mathbb{K} \to A$:

$$\begin{tikzcd} \mathbb{K} \otimes A \cong A \cong A \otimes \mathbb{K} \arrow[r, "\mathrm{id} \otimes \eta"] \arrow[d, "\eta \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & A \otimes A \arrow[d, "\nabla"] \\ A \otimes A \arrow[r, "\nabla"'] & A. \end{tikzcd}$$

A $\mathbb{K}$-algebra A is *commutative* if the following diagram commutes

$$\begin{tikzcd} A \otimes A \arrow[rr, "\tau"] \arrow[dr, "\nabla"'] & & A \otimes A \arrow[dl, "\nabla"] \\ & A. & \end{tikzcd}$$

Let A and B be $\mathbb{K}$-algebras. A *homomorphism of algebras* $f : A \to B$ is a $\mathbb{K}$-linear map such that the following diagrams commute:

$$\begin{tikzcd} A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\nabla_A"'] & B \otimes B \arrow[d, "\nabla_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

and

$$\begin{tikzcd} & \mathbb{K} \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_B"] & \\ A \arrow[rr, "f"] & & B. \end{tikzcd}$$

Remark 8.5.2. Every $\mathbb{K}$-algebra A is a ring with the multiplication

$$A \times A \xrightarrow{\otimes} A \otimes A \xrightarrow{\nabla} A.$$

The unit element is $\eta(1)$, where 1 is the unit element of $\mathbb{K}$.

Obviously the composition of two homomorphisms of algebras is again a homomorphism of algebras. Furthermore the identity map is a homomorphism of algebras. Hence the $\mathbb{K}$-algebras form a category $\mathbb{K}$-**Alg**. The category of commutative $\mathbb{K}$-algebras will be denoted by $\mathbb{K}$-**cAlg**.

Problem 8.5.5. 1. Show that $\mathrm{End}_K(V)$ is a $\mathbb{K}$-algebra.

2. Show that $(A, \nabla : A \otimes A \to A, \eta : \mathbb{K} \to A)$ is a $\mathbb{K}$-algebra if and only if A with the multiplication $A \times A \xrightarrow{\otimes} A \otimes A \xrightarrow{\nabla} A$ and the unit $\eta(1)$ is a ring and $\eta : \mathbb{K} \to \mathrm{Cent}(A)$ is a ring homomorphism into the center of A.

3. Let V be a $\mathbb{K}$-module. Show that $D(V) := \mathbb{K} \times V$ with the multiplication $(r_1, v_1)(r_2, v_2) := (r_1 r_2, r_1 v_2 + r_2 v_1)$ is a commutative $\mathbb{K}$-algebra.

Lemma 8.5.3. *Let A and B be algebras. Then $A \otimes B$ is an algebra with the multiplication $(a_1 \otimes b_1)(a_2 \otimes b_2) := a_1 a_2 \otimes b_1 b_2$.*

Proof. Certainly the algebra properties can easily be checked by a simple calculation with elements. We prefer for later applications a diagrammatic proof.

Let $\nabla_A : A \otimes A \to A$ and $\nabla_B : B \otimes B \to B$ denote the multiplications of the two algebras. Then the new multiplication is $\nabla_{A \otimes B} := (\nabla_A \otimes \nabla_B)(1_A \otimes \tau \otimes 1_B) : A \otimes B \otimes A \otimes B \to A \otimes B$ where $\tau : B \otimes A \to A \otimes B$ is the symmetry map from Theorem 8.4.15. Now the following diagrams commute

$$\begin{tikzcd}
A \otimes B \otimes A \otimes B \otimes A \otimes B \arrow[rr, "1\otimes\tau\otimes 1^3"] \arrow[dd, "1^3\otimes\tau\otimes 1"'] & & A \otimes A \otimes B \otimes B \otimes A \otimes B \arrow[r, "\nabla\otimes\nabla\otimes 1^2"] \arrow[dl, "1^3\otimes\tau\otimes 1"'] \arrow[dd, "1\otimes\tau_{B\otimes B,A}\otimes 1"] & A \otimes B \otimes A \otimes B \arrow[dd, "1\otimes\tau\otimes 1"] \\
 & A \otimes A \otimes B \otimes A \otimes B \otimes B \arrow[dr, "1^2\otimes\tau\otimes 1^2"] & & \\
A \otimes B \otimes A \otimes A \otimes B \otimes B \arrow[ur, "1\otimes\tau\otimes 1^3"] \arrow[rr, "1\otimes\tau_{B,A\otimes A}\otimes 1^3"'] \arrow[d, "1\otimes\nabla\otimes\nabla"'] & & A \otimes A \otimes A \otimes B \otimes B \otimes B \arrow[r, "\nabla\otimes 1\otimes\nabla\otimes 1"] \arrow[d, "1\otimes\nabla\otimes 1\otimes\nabla"] & A \otimes A \otimes B \otimes B \arrow[d, "\nabla\otimes\nabla"] \\
A \otimes B \otimes A \otimes B \arrow[rr, "1\otimes\tau\otimes 1"] & & A \otimes A \otimes B \otimes B \arrow[r, "\nabla\otimes\nabla"] & A \otimes B
\end{tikzcd}$$

In the left upper rectangle of the diagram the quadrangle commutes by the properties of the tensor product and the two triangles commute by inner properties of τ. The right upper and left lower rectangles commute since τ is a natural transformation and the right lower rectangle commutes by the associativity of the algebras A and B.

Furthermore we use the homomorphism $\eta = \eta_{A\otimes B} : \mathbb{K} \longrightarrow \mathbb{K} \otimes K \longrightarrow A \otimes B$ in the following commutative diagram

$$\begin{tikzcd}
\mathbb{K}\otimes A\otimes B \arrow[r, phantom, "\cong"] \arrow[d] \arrow[dr] & A\otimes B \arrow[r, phantom, "\cong"] \arrow[ddrrr, "1"] & A\otimes B\otimes\mathbb{K} \arrow[r] \arrow[dr] & A\otimes B\otimes\mathbb{K}\otimes\mathbb{K} \arrow[r, "1^2\otimes\eta\otimes\eta"] \arrow[d, "1\otimes\tau\otimes1"] & A\otimes B\otimes A\otimes B \arrow[d, "1\otimes\tau\otimes1"] \\
\mathbb{K}\otimes\mathbb{K}\otimes A\otimes B \arrow[r, "1\otimes\tau\otimes1"] \arrow[d, "\eta\otimes\eta\otimes1^2"'] & \mathbb{K}\otimes A\otimes\mathbb{K}\otimes B \arrow[d, "\eta\otimes1\otimes\eta\otimes1"'] & & A\otimes\mathbb{K}\otimes B\otimes\mathbb{K} \arrow[r, "1\otimes\eta\otimes1\otimes\eta"'] & A\otimes A\otimes B\otimes B \arrow[d, "\nabla\otimes\nabla"] \\
A\otimes B\otimes A\otimes B \arrow[r, "1\otimes\tau\otimes1"] & A\otimes A\otimes B\otimes B \arrow[rrr, "\nabla\otimes\nabla"] & & & A\otimes B.
\end{tikzcd}$$

□

Definition 8.5.4. Let $\mathbb{K}$ be a commutative ring. Let V be a $\mathbb{K}$-module. A $\mathbb{K}$-algebra $T(V)$ together with a homomorphism of $\mathbb{K}$-modules $\iota : V \longrightarrow T(V)$ is called a *tensor algebra over* V if for each $\mathbb{K}$-algebra A and for each homomorphism of $\mathbb{K}$-modules $f : V \longrightarrow A$ there exists a unique homomorphism of $\mathbb{K}$-algebras $g : T(V) \longrightarrow A$ such that the diagram

$$\begin{tikzcd}
V \arrow[r, "\iota"] \arrow[dr, "f"'] & T(V) \arrow[d, "g"] \\
& A
\end{tikzcd}$$

commutes.

Note: If you want to define a homomorphism $g : T(V) \longrightarrow A$ with a tensor algebra as domain you should define it by giving a homomorphism of $\mathbb{K}$-modules defined on V.

Lemma 8.5.5. *A tensor algebra $(T(V), \iota)$ defined by V is unique up to a unique isomorphism.*

Proof. Let $(T(V), \iota)$ and $(T'(V), \iota')$ be tensor algebras over V. Then

$$\begin{tikzcd}
& V \arrow[dl, "\iota"'] \arrow[d, "\iota'"'] \arrow[dr, "\iota"] \arrow[drr, "\iota'"] & & \\
T(V) \arrow[r, "h"] & T'(V) \arrow[r, "k"] & T(V) \arrow[r, "h"] & T'(V)
\end{tikzcd}$$

implies $k = h^{-1}$. □

Proposition 8.5.6. (Rules of computation in a tensor algebra) *Let $(T(V), \iota)$ be the tensor algebra over V. Then we have*

1. *$\iota : V \longrightarrow T(V)$ is injective (so we may identify the elements $\iota(v)$ and v for all $v \in V$),*
2. *$T(V) = \{\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$,*
3. *if $f : V \longrightarrow A$ is a homomorphism of $\mathbb{K}$-modules, A is a $\mathbb{K}$-algebra, and $g : T(V) \longrightarrow A$ is the induced homomorphism of $\mathbb{K}$-algebras, then*

$$g(\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n}) = \sum_{n,\bar{i}} f(v_{i_1}) \cdot \ldots \cdot f(v_{i_n}).$$

Proof. 1. Use the embedding homomorphism $j : V \longrightarrow D(V)$, where $D(V)$ is defined as in 8.5.3. to construct $g : T(V) \longrightarrow D(V)$ such that $g \circ \iota = j$. Since j is injective so is ι.

2. Let $B := \{\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$. Obviously B is the subalgebra of $T(V)$ generated by the elements of V. Let $j : B \longrightarrow T(V)$ be the embedding homomorphism. Then $\iota : V \longrightarrow T(V)$ factors through a linear map $\iota' : V \longrightarrow B$. In the following diagram

$$\begin{tikzcd}
V \arrow[r, "\iota'"] \arrow[dr, "\iota'"'] & B \arrow[r, "j"] \arrow[d, "\mathrm{id}_B"] & T(V) \arrow[dl, "p"'] \arrow[d, "jp"] \\
 & B \arrow[r, "j"] & T(V)
\end{tikzcd}$$

we have $\mathrm{id}_B \circ \iota' = \iota'$. p with $p \circ j \circ \iota' = p \circ \iota = \iota'$ exists since ι' is a homomorphism of $\mathbb{K}$-modules. Because of $jp \circ \iota = j \circ \iota' = \iota = \mathrm{id}_{T(V)} \circ \iota$ we get $jp = \mathrm{id}_{T(V)}$, hence the embedding j is surjective and thus j is the identity.

3. is precisely the definition of the induced homomorphism. $\square$

Proposition 8.5.7. *Given a $\mathbb{K}$-module V. Then there exists a tensor algebra $(T(V), \iota)$.*

Proof. Define $T^n(V) := V \otimes \ldots \otimes V = V^{\otimes n}$ to be the n-fold tensor product of V. Define $T^0(V) := \mathbb{K}$ and $T^1(V) := V$. We define

$$T(V) := \bigoplus_{i \geq 0} T^i(V) = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \ldots.$$

The components $T^n(V)$ of $T(V)$ are called *homogeneous components.*

The canonical isomorphisms $T^m(V) \otimes T^n(V) \cong T^{m+n}(V)$ taken as multiplication

$$\nabla : T^m(V) \otimes T^n(V) \to T^{m+n}(V)$$
$$\nabla : T(V) \otimes T(V) \to T(V)$$

and the embedding $\eta : \mathbb{K} = T^0(V) \longrightarrow T(V)$ induce the structure of a $\mathbb{K}$-algebra on $T(V)$. Furthermore we have the embedding $\iota : V \longrightarrow T^1(V) \subseteq T(V)$.

We have to show that $(T(V), \iota)$ is a tensor algebra. Let $f : V \to A$ be a homomorphism of $\mathbb{K}$-modules. Each element in $T(V)$ is a sum of decomposable tensors $v_1 \otimes \ldots \otimes v_n$. Define $g : T(V) \to A$ by $g(v_1 \otimes \ldots \otimes v_n) := f(v_1)\ldots f(v_n)$ (and $(g : T^0(V) \to A) = (\eta : \mathbb{K} \to A)$). By induction one sees that g is a homomorphism of algebras. Since $(g : T^1(V) \to A) = (f : V \to A)$ we get $g \circ \iota = f$. If $h : T(V) \to A$ is a homomorphism of algebras with $h \circ \iota = f$ we get $h(v_1 \otimes \ldots \otimes v_n) = h(v_1)\ldots h(v_n) = f(v_1)\ldots f(v_n)$ hence $h = g$. $\square$

Proposition 8.5.8. *The construction of tensor algebras $T(V)$ defines a functor $T : \mathbb{K}\text{-}\mathbf{Mod} \to \mathbb{K}\text{-}\mathbf{Alg}$ that is left adjoint to the underlying functor $U : \mathbb{K}\text{-}\mathbf{Alg} \to \mathbb{K}\text{-}\mathbf{Mod}$.*

Proof. Follows from the universal property and 8.9.16. $\square$

Problem 8.5.6. 1. Let X be a set and $V := \mathbb{K}X$ be the free $\mathbb{K}$-module over X. Show that $X \to V \to T(V)$ defines a *free algebra* over X, i.e. for every $\mathbb{K}$-algebra A and every map $f : X \to A$ there is a unique homomorphism of $\mathbb{K}$-algebras $g : T(V) \to A$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & T(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.

We write $\mathbb{K}\langle X\rangle := T(\mathbb{K}X)$ and call it the *polynomial ring over $\mathbb{K}$ in the non-commuting variables X*.

2. Let $T(V)$ and $\iota : V \to T(V)$ be a tensor algebra. Regard V as a subset of $T(V)$ by ι. Show that there is a unique homomorphism $\Delta : T(V) \to T(V) \otimes T(V)$ with $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$.

3. Show that $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta : T(V) \to T(V) \otimes T(V) \otimes T(V)$.

4. Show that there is a unique homomorphism of algebras $\varepsilon : T(V) \to \mathbb{K}$ with $\varepsilon(v) = 0$ for all $v \in V$.

5. Show that $(\varepsilon \otimes 1)\Delta = (1 \otimes \varepsilon)\Delta = \mathrm{id}_{T(V)}$.

6. Show that there is a unique homomorphism of algebras $S : T(V) \to T(V)^{op}$ with $S(v) = -v$. ($T(V)^{op}$ is the *opposite algebra* of $T(V)$ with multiplication $s*t := ts$ for all $s, t \in T(V) = T(V)^{op}$ and where st denotes the product in $T(V)$.)

7. Show that the diagrams

$$\begin{tikzcd} T(V) \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & T(V) \\ T(V) \otimes T(V) \arrow[rr, "1\otimes S", "S\otimes 1"'] & & T(V) \otimes T(V) \arrow[u, "\nabla"'] \end{tikzcd}$$

commute.

Definition 8.5.9. Let $\mathbb{K}$ be a commutative ring. Let V be a $\mathbb{K}$-module. A $\mathbb{K}$-algebra $S(V)$ together with a homomorphism of $\mathbb{K}$-modules $\iota : V \longrightarrow S(V)$, such that $\iota(v) \cdot \iota(v') = \iota(v') \cdot \iota(v)$ for all $v, v' \in V$, is called a *symmetric algebra over* V if for each $\mathbb{K}$-algebra A and for each homomorphism of $\mathbb{K}$-modules $f : V \longrightarrow A$, such that $f(v) \cdot f(v') = f(v') \cdot f(v)$ for all $v, v' \in V$, there exists a unique homomorphism of $\mathbb{K}$-algebras $g : S(V) \longrightarrow A$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & S(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.

Note: If you want to define a homomorphism $g : S(V) \longrightarrow A$ with a symmetric algebra as domain you should define it by giving a homomorphism of $\mathbb{K}$-modules $f : V \longrightarrow A$ satisfying $f(v) \cdot f(v') = f(v') \cdot f(v)$ for all $v, v' \in V$.

Lemma 8.5.10. *A symmetric algebra $(S(V), \iota)$ defined by V is unique up to a unique isomorphism.*

Proof. Let $(S(V), \iota)$ and $(S'(V), \iota')$ be symmetric algebras over V. Then

$$\begin{tikzcd} & & V \arrow[dll, "\iota"'] \arrow[dl, "\iota'"] \arrow[dr, "\iota"] \arrow[drr, "\iota'"] & & \\ S(V) \arrow[r, "h"] & S'(V) \arrow[rr, "k"] & & S(V) \arrow[r, "h"] & S'(V) \end{tikzcd}$$

implies $k = h^{-1}$. $\square$

Proposition 8.5.11. (Rules of computation in a symmetric algebra) *Let $(S(V), \iota)$ be the symmetric algebra over V. Then we have*

1. *$\iota : V \longrightarrow S(V)$ is injective (we will identify the elements $\iota(v)$ and v for all $v \in V$),*
2. *$S(V) = \{\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$,*
3. *if $f : V \longrightarrow A$ is a homomorphism of $\mathbb{K}$-modules satisfying $f(v) \cdot f(v') = f(v') \cdot f(v)$ for all $v, v' \in V$, A is a $\mathbb{K}$-algebra, and $g : S(V) \longrightarrow A$ is the induced homomorphism $\mathbb{K}$-algebras, then*

$$g(\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n}) = \sum_{n,\bar{i}} f(v_{i_1}) \cdot \ldots \cdot f(v_{i_n}).$$

Proof. 1. Use the embedding homomorphism $j : V \longrightarrow D(V)$, where $D(V)$ is the commutative algebra defined in 8.5.3. to construct $g : S(V) \longrightarrow D(V)$ such that $g \circ \iota = j$. Since j is injective so is ι.

2. Let $B := \{\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$. Obviously B is the subalgebra of $S(V)$ generated by the elements of V. Let $j : B \longrightarrow S(V)$ be the embedding homomorphism. Then $\iota : V \longrightarrow S(V)$ factors through a linear map $\iota' : V \longrightarrow B$. In the following diagram

$$\begin{tikzcd}
V \arrow[r, "\iota'"] \arrow[dr, "\iota'"'] & B \arrow[r, "j"] \arrow[d, "\mathrm{id}_B"] & S(V) \arrow[dl, "p"'] \arrow[d, "jp"] \\
& B \arrow[r, "j"] & S(V)
\end{tikzcd}$$

we have $\mathrm{id}_B \circ \iota' = \iota'$, p with $p \circ j \circ \iota' = p \circ \iota = \iota'$ exists since ι' is a homomorphism of $\mathbb{K}$-modules satisfying $\iota'(v) \cdot \iota'(v') = \iota'(v') \cdot \iota'(v)$ for all $v, v' \in V$. Because of $jp \circ \iota = j \circ \iota' = \iota = \mathrm{id}_{S(V)} \circ \iota$ we get $jp = \mathrm{id}_{S(V)}$, hence the embedding j is surjective and thus the identity.

3. is precisely the definition of the induced homomorphism. $\square$

Proposition 8.5.12. *Let V be a $\mathbb{K}$-module. The symmetric algebra $(S(V), \iota)$ is commutative and satisfies the following universal property:*

for each commutative $\mathbb{K}$-algebra A and for each homomorphism of $\mathbb{K}$-modules $f : V \longrightarrow A$ there exists a unique homomorphism of $\mathbb{K}$-algebras $g : S(V) \longrightarrow A$ such that the diagram

$$\begin{tikzcd}
V \arrow[r, "\iota"] \arrow[dr, "f"'] & S(V) \arrow[d, "g"] \\
& A
\end{tikzcd}$$

commutes.

Proof. Commutativity follows from the commutativity of the generators: $vv' = v'v$ which carries over to the elements of the form $\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n}$. The universal property follows since the defining condition $f(v) \cdot f(v') = f(v') \cdot f(v)$ for all $v, v' \in V$ is automatically satisfied. $\square$

Proposition 8.5.13. *Given a $\mathbb{K}$-module V. Then there exists a symmetric algebra $(S(V), \iota)$.*

Proof. Define $S(V) := T(V)/I$ where $I = \langle vv' - v'v | v, v' \in V \rangle$ is the two-sided ideal generated by the elements $vv' - v'v$. Let ι be the canonical map $V \longrightarrow T(V) \longrightarrow S(V)$. Then the universal property is easily verified by the homomorphism theorem for algebras. $\square$

Proposition 8.5.14. *The construction of symmetric algebras $S(V)$ defines a functor $S : \mathbb{K}$-**Mod** $\longrightarrow \mathbb{K}$-**cAlg** that is left adjoint to the underlying functor $U : \mathbb{K}$-**cAlg** $\longrightarrow \mathbb{K}$-**Mod**.*

Proof. Follows from the universal property and 8.9.16. $\square$

Problem 8.5.7. Let X be a set and $V := \mathbb{K}X$ be the free $\mathbb{K}$-module over X. Show that $X \to V \to S(V)$ defines a *free commutative algebra* over X, i.e. for every commutative $\mathbb{K}$-algebra A and every map $f : X \to A$ there is a unique homomorphism of $\mathbb{K}$-algebras $g : S(V) \to A$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & S(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.

The algebra $\mathbb{K}[X] := S(\mathbb{K}X)$ is called the *polynomial ring over* $\mathbb{K}$ *in the (commuting) variables* X.

2. Let $S(V)$ and $\iota : V \to S(V)$ be a symmetric algebra. Show that there is a unique homomorphism $\Delta : S(V) \to S(V) \otimes S(V)$ with $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$.

3. Show that $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta : S(V) \to S(V) \otimes S(V) \otimes S(V)$.

4. Show that there is a unique homomorphism of algebras $\varepsilon : S(V) \to \mathbb{K}$ with $\varepsilon(v) = 0$ for all $v \in V$.

5. Show that $(\varepsilon \otimes 1)\Delta = (1 \otimes \varepsilon)\Delta = \mathrm{id}_{S(V)}$.

6. Show that there is a unique homomorphism of algebras $S : S(V) \to S(V)$ with $S(v) = -v$.

7. Show that the diagrams

$$\begin{tikzcd} S(V) \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & S(V) \\ S(V) \otimes S(V) \arrow[rr, "1\otimes S", "S\otimes 1"'] & & S(V) \otimes S(V) \arrow[u, "\nabla"'] \end{tikzcd}$$

commute.

Definition 8.5.15. Let $\mathbb{K}$ be a commutative ring. Let V be a $\mathbb{K}$-module. A $\mathbb{K}$-algebra $E(V)$ together with a homomorphism of $\mathbb{K}$-modules $\iota : V \to E(V)$, such that $\iota(v)^2 = 0$ for all $v \in V$, is called an *exterior algebra or Grassmann algebra over* V if for each $\mathbb{K}$-algebra A and for each homomorphism of $\mathbb{K}$-modules $f : V \to A$, such that $f(v)^2 = 0$ for all $v \in V$, there exists a unique homomorphism of $\mathbb{K}$-algebras $g : E(V) \to A$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & E(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.

The multiplication in $E(V)$ is usually denoted by $u \wedge v$.

Note: If you want to define a homomorphism $g : E(V) \longrightarrow A$ with an exterior algebra as domain you should define it by giving a homomorphism of $\mathbb{K}$-modules defined on V satisfying $f(v)^2 = 0$ for all $v, v' \in V$.

Problem 8.5.8. 1. Let $f : V \longrightarrow A$ be a linear map satisfying $f(v)^2 = 0$ for all $v \in V$. Then $f(v)f(v') = -f(v')f(v)$ for all $v, v' \in V$.

2. Let 2 be invertible in $\mathbb{K}$ (e.g. $\mathbb{K}$ a field of characteristic $\neq 2$). Let $f : V \longrightarrow A$ be a linear map satisfying $f(v)f(v') = -f(v')f(v)$ for all $v, v' \in V$. Then $f(v)^2 = 0$ for all $v \in V$.

Lemma 8.5.16. *An exterior algebra $(E(V), \iota)$ defined by V is unique up to a unique isomorphism.*

PROOF. Let $(E(V), \iota)$ and $(E'(V), \iota')$ be exterior algebras over V. Then

$$\begin{tikzcd}
& & V \arrow[dll, "\iota"'] \arrow[dl, "\iota'"] \arrow[dr, "\iota"] \arrow[drr, "\iota'"] & & \\
E(V) \arrow[r, "h"] & E'(V) \arrow[rr, "k"] & & E(V) \arrow[r, "h"] & E'(V)
\end{tikzcd}$$

implies $k = h^{-1}$. □

Proposition 8.5.17. (Rules of computation in an exterior algebra) *Let $(E(V), \iota)$ be the exterior algebra over V. Then we have*

1. *$\iota : V \longrightarrow E(V)$ is injective (we will identify the elements $\iota(v)$ and v for all $v \in V$),*
2. *$E(V) = \{\sum_{n,\bar{i}} v_{i_1} \wedge \ldots \wedge v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$,*
3. *if $f : V \longrightarrow A$ is a homomorphism of $\mathbb{K}$-modules satisfying $f(v) \cdot f(v') = -f(v') \cdot f(v)$ for all $v, v' \in V$, A is a $\mathbb{K}$-algebra, and $g : E(V) \longrightarrow A$ is the induced homomorphism $\mathbb{K}$-algebras, then*

$$g(\sum_{n,\bar{i}} v_{i_1} \wedge \ldots \wedge v_{i_n}) = \sum_{n,\bar{i}} f(v_{i_1}) \cdot \ldots \cdot f(v_{i_n}).$$

PROOF. 1. Use the embedding homomorphism $j : V \longrightarrow D(V)$, where $D(V)$ is the algebra defined in 8.5.3. to construct $g : E(V) \longrightarrow D(V)$ such that $g \circ \iota = j$. Since j is injective so is ι.

2. Let $B := \{\sum_{n,\bar{i}} v_{i_1} \wedge \ldots \wedge v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$. Obviously B is the subalgebra of $E(V)$ generated by the elements of V. Let $j : B \longrightarrow E(V)$ be the embedding homomorphism. Then $\iota : V \longrightarrow E(V)$ factors through a linear map

$\iota' : V \to B$. In the following diagram

$$\begin{tikzcd}
V \arrow[r, "\iota'"] \arrow[dr, "\iota'"'] & B \arrow[r, "j"] \arrow[d, "\mathrm{id}_B"'] & E(V) \arrow[dl, "p"'] \arrow[d, "jp"] \\
& B \arrow[r, "j"] & E(V)
\end{tikzcd}$$

we have $\mathrm{id}_B \circ \iota' = \iota'$, p with $p \circ j \circ \iota' = p \circ \iota = \iota'$ exists since ι' is a homomorphism of $\mathbb{K}$-modules satisfying $\iota'(v) \cdot \iota'(v') = -\iota'(v') \cdot \iota'(v)$ for all $v, v' \in V$. Because of $jp \circ \iota = j \circ \iota' = \iota = \mathrm{id}_{E(V)} \circ \iota$ we get $jp = \mathrm{id}_{E(V)}$, hence the embedding j is surjective and thus j is the identity.

3. is precisely the definition of the induced homomorphism. □

Proposition 8.5.18. *Given a $\mathbb{K}$-module V. Then there exists an exterior algebra $(E(V), \iota)$.*

Proof. Define $E(V) := T(V)/I$ where $I = \langle v^2 | v \in V \rangle$ is the two-sided ideal generated by the elements v^2. Let ι be the canonical map $V \to T(V) \to E(V)$. Then the universal property is easily verified by the homomorphism theorem for algebras. □

Problem 8.5.9. 1. Let V be a finite dimensional vector space of dimension n. Show that $E(V)$ is finite dimensional of dimension 2^n. (Hint: The homogeneous components $E^i(V)$ have dimension $\binom{n}{i}$.

2. Show that the symmetric group S_n operates (from the left) on $T^n(V)$ by $\sigma(v_1 \otimes \ldots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}$ with $\sigma \in S_n$ and $v_i \in V$.

3. A tensor $a \in T^n(V)$ is called *a symmetric tensor* if $\sigma(a) = a$ for all $\sigma \in S_n$. Let $\hat{S}^n(V)$ be the subspace of symmetric tensors in $T^n(V)$.

a) Show that $\mathcal{S} : T^n(V) \ni a \mapsto \sum_{\sigma \in S_n} \sigma(a) \in T^n(V)$ is a linear map.

b) Show that $\mathcal{S}$ has its image in $\hat{S}^n(V)$.

c) Show that $\mathrm{Im}(\mathcal{S}) = \hat{S}^n(V)$ if $n!$ is invertible in $\mathbb{K}$.

d) Show that $\hat{S}^n(V) \hookrightarrow T^n(V) \xrightarrow{\nu} S^n(V)$ is an isomorphism if $n!$ is invertible in $\mathbb{K}$ and $\nu : T^n(V) \to S^n(V)$ is the restriction of $\nu : T(V) \to S(V)$, the symmetric algebra.

4. A tensor $a \in T^n(V)$ is called *an antisymmetric tensor* if $\sigma(a) = \varepsilon(\sigma)a$ for all $\sigma \in S_n$ where $\varepsilon(\sigma)$ is the sign of the permutation σ. Let $\hat{E}^n(V)$ be the subspace of antisymmetric tensors in $T^n(V)$.

a) Show that $\mathcal{E} : T^n(V) \ni a \mapsto \sum_{\sigma \in S_n} \varepsilon(\sigma)\sigma(a) \in T^n(V)$ is a linear map.

b) Show that $\mathcal{E}$ has its image in $\hat{E}^n(V)$.

c) Show that $\mathrm{Im}(\mathcal{E}) = \hat{E}^n(V)$ if $n!$ is invertible in $\mathbb{K}$.

d) Show that $\hat{E}^n(V) \hookrightarrow T^n(V) \xrightarrow{\nu} E^n(V)$ is an isomorphism if $n!$ is invertible in $\mathbb{K}$ and $\nu : T^n(V) \to E^n(V)$ is the restriction of $\nu : T(V) \to E(V)$, the exterior algebra.

Definition 8.5.19. Let A be a $\mathbb{K}$-algebra. A *left A-module* is a $\mathbb{K}$-module M together with a homomorphism $\mu_M : A \otimes M \to M$, such that the diagrams

$$\begin{tikzcd}
A \otimes A \otimes M \arrow[r, "\mathrm{id} \otimes \mu"] \arrow[d, "\nabla \otimes \mathrm{id}"'] & A \otimes M \arrow[d, "\mu"] \\
A \otimes M \arrow[r, "\mu"'] & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \cong \mathbb{K} \otimes M \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[dr, "\mathrm{id}"] & A \otimes M \arrow[d, "\mu"] \\
 & M
\end{tikzcd}$$

commute.

Let ${}_AM$ and ${}_AN$ be A-modules and let $f : M \to N$ be a $\mathbb{K}$-linear map. The map f is called a *homomorphism of modules* if the diagram

$$\begin{tikzcd}
A \otimes M \arrow[r, "\mu_M"] \arrow[d, "1 \otimes f"'] & M \arrow[d, "f"] \\
A \otimes N \arrow[r, "\mu_N"'] & N
\end{tikzcd}$$

commutes.

The left A-modules and their homomorphisms form the *category ${}_A\mathcal{M}$ of A-modules*.

Problem 8.5.10. Show that an abelian group M is a left module over the ring A if and only if M is a $\mathbb{K}$-module and an A-module in the sense of Definition 8.5.19.

6. Coalgebras

Definition 8.6.1. A $\mathbb{K}$-*coalgebra* is a $\mathbb{K}$-module C together with a *comultiplication* or *diagonal* $\Delta : C \longrightarrow C \otimes C$ that is coassociative:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta\otimes\mathrm{id}"] \\ C \otimes C \arrow[r, "\mathrm{id}\otimes\Delta"'] & C \otimes C \otimes C \end{tikzcd}$$

and a *counit* or *augmentation* $\epsilon : C \longrightarrow \mathbb{K}$:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "\mathrm{id}"] & C \otimes C \arrow[d, "\mathrm{id}\otimes\epsilon"] \\ C \otimes C \arrow[r, "\epsilon\otimes\mathrm{id}"'] & \mathbb{K} \otimes C \cong C \cong C \otimes \mathbb{K}. \end{tikzcd}$$

A $\mathbb{K}$-coalgebra C is *cocommutative* if the following diagram commutes

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[rr, "\tau"] & & C \otimes C \end{tikzcd}$$

Let C and D be $\mathbb{K}$-coalgebras. A *homomorphism of coalgebras* $f : C \longrightarrow D$ is a $\mathbb{K}$-linear map such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f\otimes f"'] & D \otimes D \end{tikzcd}$$

and

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\epsilon_C"'] & & D. \arrow[dl, "\epsilon_D"] \\ & \mathbb{K} & \end{tikzcd}$$

Remark 8.6.2. Obviously the composition of two homomorphisms of coalgebras is again a homomorphism of coalgebras. Furthermore the identity map is a homomorphism of coalgebras. Hence the $\mathbb{K}$-coalgebras form a category $\mathbb{K}$-**Coalg**. The category of cocommutative $\mathbb{K}$-coalgebras will be denoted by $\mathbb{K}$-**cCoalg**.

Problem 8.6.11. 1. Show that $V \otimes V^*$ is a coalgebra for every finite dimensional vector space V over a field $\mathbb{K}$ if the comultiplication is defined by $\Delta(v \otimes v^*) := \sum_{i=1}^{n} v \otimes v_i^* \otimes v_i \otimes v^*$ where $\{v_i\}$ and $\{v_i^*\}$ are dual bases of V resp. V^*.

2. Show that the free $\mathbb{K}$-modules $\mathbb{K}X$ with the basis X and the comultiplication $\Delta(x) = x \otimes x$ is a coalgebra. What is the counit? Is the counit unique?

3. Show that $\mathbb{K} \oplus V$ with $\Delta(1) = 1 \otimes 1$, $\Delta(v) = v \otimes 1 + 1 \otimes v$ defines a coalgebra.

4. Let C and D be coalgebras. Then $C \otimes D$ is a coalgebra with the comultiplication $\Delta_{C\otimes D} := (1_C \otimes \tau \otimes 1_D)(\Delta_C \otimes \Delta_D) : C \otimes D \otimes C \otimes D \to C \otimes D$ and counit $\varepsilon = \varepsilon_{C\otimes D} : C \otimes D \to \mathbb{K} \otimes K \to \mathbb{K}$. (The proof is analogous to the proof of Lemma 8.5.3.)

To describe the comultiplication of a $\mathbb{K}$-coalgebra in terms of elements we introduce a notation first introduced by Sweedler similar to the notation $\nabla(a \otimes b) = ab$ used for algebras. Instead of $\Delta(c) = \sum c_i \otimes c_i'$ we write

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}.$$

Observe that only the complete expression on the right hand side makes sense, not the components $c_{(1)}$ or $c_{(2)}$ which are *not* considered as families of elements of C. This notation alone does not help much in the calculations we have to perform later on. So we introduce a more general notation.

Definition 8.6.3. (Sweedler Notation) Let M be an arbitrary $\mathbb{K}$-module and C be a $\mathbb{K}$-coalgebra. Then there is a bijection between all multilinear maps

$$f : C \times \ldots \times C \to M$$

and all linear maps

$$f' : C \otimes \ldots \otimes C \to M.$$

These maps are associated to each other by the formula

$$f(c_1, \ldots, c_n) = f'(c_1 \otimes \ldots \otimes c_n).$$

For $c \in C$ we define

$$\sum f(c_{(1)}, \ldots, c_{(n)}) := f'(\Delta^{n-1}(c)),$$

where Δ^{n-1} denotes the $n-1$-fold application of Δ, for example $\Delta^{n-1} = (\Delta \otimes 1 \otimes \ldots \otimes 1) \circ (\Delta \otimes 1) \circ \Delta$.

In particular we obtain for the bilinear map $\otimes : C \times C \ni (c, d) \mapsto c \otimes d \in C \otimes C$

$$\sum c_{(1)} \otimes c_{(2)} = \Delta(c),$$

and for the multilinear map $\otimes^2 : C \times C \times C \to C \otimes C \otimes C$

$$\sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)} = (\Delta \otimes 1)\Delta(c) = (1 \otimes \Delta)\Delta(c).$$

With this notation one verifies easily

$$\sum c_{(1)} \otimes \ldots \otimes \Delta(c_{(i)}) \otimes \ldots \otimes c_{(n)} = \sum c_{(1)} \otimes \ldots \otimes c_{(n+1)}$$

and
$$\begin{aligned}\sum c_{(1)} \otimes \ldots \otimes \epsilon(c_{(i)}) \otimes \ldots \otimes c_{(n)} &= \sum c_{(1)} \otimes \ldots \otimes 1 \otimes \ldots \otimes c_{(n-1)} \\ &= \sum c_{(1)} \otimes \ldots \otimes c_{(n-1)}\end{aligned}$$

This notation and its application to multilinear maps will also be used in more general contexts like comodules.

Proposition 8.6.4. *Let C be a coalgebra and A an algebra. Then the composition $f * g := \nabla_A(f \otimes g)\Delta_C$ defines a multiplication*

$$\mathrm{Hom}(C, A) \otimes \mathrm{Hom}(C, A) \ni f \otimes g \mapsto f * g \in \mathrm{Hom}(C, A),$$

such that $\mathrm{Hom}(C, A)$ becomes an algebra. The unit element is given by $\mathbb{K} \ni \alpha \mapsto (c \mapsto \eta(\alpha\epsilon(c))) \in \mathrm{Hom}(C, A)$.

Proof. The multiplication of $\mathrm{Hom}(C, A)$ obviously is a bilinear map. The multiplication is associative since $(f * g) * h = \nabla_A((\nabla_A(f \otimes g)\Delta_C) \otimes h)\Delta_C = \nabla_A(\nabla_A \otimes 1)((f \otimes g) \otimes h)(\Delta_C \otimes 1)\Delta_C = \nabla_A(1 \otimes \nabla_A)(f \otimes (g \otimes h))(1 \otimes \Delta_C)\Delta_C = \nabla_A(f \otimes (\nabla_A(g \otimes h)\Delta_C))\Delta_C = f * (g * h)$. Furthermore it is unitary with unit $1_{\mathrm{Hom}(C,A)} = \eta_A\epsilon_C$ since $\eta_A\epsilon_C * f = \nabla_A(\eta_A\epsilon_C \otimes f)\Delta_C = \nabla_A(\eta_A \otimes 1_A)(1_K \otimes f)(\epsilon_C \otimes 1_C)\Delta_C = f$ and similarly $f * \eta_A\epsilon_C = f$. □

Definition 8.6.5. The multiplication $* : \mathrm{Hom}(C, A) \otimes \mathrm{Hom}(C, A) \longrightarrow \mathrm{Hom}(C, A)$ is called *convolution*.

Corollary 8.6.6. *Let C be a $\mathbb{K}$-coalgebra. Then $C^* = \mathrm{Hom}_K(C, \mathbb{K})$ is an $\mathbb{K}$-algebra.*

Proof. Use that $\mathbb{K}$ itself is a $\mathbb{K}$-algebra. □

Remark 8.6.7. If we write the evaluation as $C^* \otimes C \ni a \otimes c \mapsto \langle a, c\rangle \in \mathbb{K}$ then an element $a \in C^*$ is completely determined by the values of $\langle a, c\rangle$ for all $c \in C$. So the product of a and b in C^* is uniquely determined by the formula

$$\langle a * b, c\rangle = \langle a \otimes b, \Delta(c)\rangle = \sum a(c_{(1)})b(c_{(2)}).$$

The unit element of C^* is $\epsilon \in C^*$.

Lemma 8.6.8. *Let $\mathbb{K}$ be a field and A be a finite dimensional $\mathbb{K}$-algebra. Then $A^* = \mathrm{Hom}_K(A, \mathbb{K})$ is a $\mathbb{K}$-coalgebra.*

Proof. Define the comultiplication on C^* by

$$\Delta : A^* \xrightarrow{\nabla^*} (A \otimes A)^* \xrightarrow{\mathrm{can}^{-1}} A^* \otimes A^*.$$

The canonical map $\mathrm{can} : A^* \otimes A^* \longrightarrow (A \otimes A)^*$ is invertible, since A is finite dimensional. By a diagrammatic proof or by calculation with elements it is easy to show that A^* becomes a $\mathbb{K}$-coalgebra. □

Remark 8.6.9. If $\mathbb{K}$ is an arbitrary commutative ring, then $A^* = \mathrm{Hom}_K(A, \mathbb{K})$ is a $\mathbb{K}$-coalgebra if A is a finitely generated projective $\mathbb{K}$-module.

Problem 8.6.12. Find sufficient conditions for an algebra A resp. a coalgebra C such that $\mathrm{Hom}(A, C)$ becomes a coalgebra with co-convolution as comultiplication.

Definition 8.6.10. Let C be a $\mathbb{K}$-coalgebra. A *left C-comodule* is a $\mathbb{K}$-module M together with a homomorphism $\delta_M : M \to C \otimes M$, such that the diagrams

$$\begin{tikzcd} M \arrow[r, "\delta"] \arrow[d, "\delta"'] & C \otimes M \arrow[d, "\Delta\otimes\mathrm{id}"] \\ C \otimes M \arrow[r, "\mathrm{id}\otimes\delta"'] & C \otimes C \otimes M \end{tikzcd}$$

and

$$\begin{tikzcd} M \arrow[d, "\delta"'] \arrow[dr, "\mathrm{id}"] & \\ C \otimes M \arrow[r, "\epsilon\otimes\mathrm{id}"'] & \mathbb{K} \otimes M \cong M. \end{tikzcd}$$

commute.

Let ${}^CM$ and ${}^CN$ be C-comodules and let $f : M \to N$ be a $\mathbb{K}$-linear map. The map f is called a *homomorphism of comodules* if the diagram

$$\begin{tikzcd} M \arrow[r, "\delta_M"] \arrow[d, "f"'] & C \otimes M \arrow[d, "1\otimes f"] \\ N \arrow[r, "\delta_N"'] & C \otimes N \end{tikzcd}$$

commutes.

The left C-comodules and their homomorphisms form the *category ${}^C\mathcal{M}$ of comodules.*

Let N be an arbitrary $\mathbb{K}$-module and M be a C-comodule. Then there is a bijection between all multilinear maps

$$f : C \times \ldots \times M \to N$$

and all linear maps

$$f' : C \otimes \ldots \otimes M \to N.$$

These maps are associated to each other by the formula

$$f(c_1, \ldots, c_n, m) = f'(c_1 \otimes \ldots \otimes c_n \otimes m).$$

For $m \in M$ we define

$$\sum f(m_{(1)}, \ldots, m_{(n)}, m_{(M)}) := f'(\delta^n(m)),$$

where δ^n denotes the n-fold application of δ, i.e. $\delta^n = (1 \otimes \ldots \otimes 1 \otimes \delta) \circ (1 \otimes \delta) \circ \delta$.

In particular we obtain for the bilinear map $\otimes : C \times M \to C \otimes M$

$$\sum m_{(1)} \otimes m_{(M)} = \delta(m),$$

and for the multilinear map $\otimes^2 : C \times C \times M \to C \otimes C \otimes M$

$$\sum m_{(1)} \otimes m_{(2)} \otimes m_{(M)} = (1 \otimes \delta)\delta(c) = (\Delta \otimes 1)\delta(m).$$

Problem 8.6.13. Show that a finite dimensional vector space V is a comodule over the coalgebra $V \otimes V^*$ as defined in problem 8.11.1 with the coaction $\delta(v) := \sum v \otimes v_i^* \otimes v_i \in (V \otimes V^*) \otimes V$ where $\sum v_i^* \otimes v_i$ is the dual basis of V in $V^* \otimes V$.

Theorem 8.6.11. *(Fundamental Theorem for Comodules) Let $\mathbb{K}$ be a field. Let M be a left C-comodule and let $m \in M$ be given. Then there exists a finite dimensional subcoalgebra $C' \subseteq C$ and a finite dimensional C'-comodule M' with $m \in M' \subseteq M$ where $M' \subseteq M$ is a $\mathbb{K}$-submodule, such that the diagram*

$$\begin{tikzcd} M' \arrow[r, "\delta'"] \arrow[d] & C' \otimes M' \arrow[d] \\ M \arrow[r, "\delta"'] & C \otimes M \end{tikzcd}$$

commutes.

Corollary 8.6.12. *1. Each element $c \in C$ of a coalgebra is contained in a finite dimensional subcoalgebra of C.*

2. Each element $m \in M$ of a comodule is contained in a finite dimensional subcomodule of M.

Corollary 8.6.13. *1. Each finite dimensional subspace V of a coalgebra C is contained in a finite dimensional subcoalgebra C' of C.*

2. Each finite dimensional subspace V of a comodule M is contained in a finite dimensional subcomodule M' of M.

Corollary 8.6.14. *1. Each coalgebra is a union of finite dimensional subcoalgebras.*

2. Each comodule is a union of finite dimensional subcomodules.

Proof. (*of the Theorem*) We can assume that $m \neq 0$ for else we can use $M' = 0$ and $C' = 0$.

Under the representations of $\delta(m) \in C \otimes M$ as finite sums of decomposable tensors pick one

$$\delta(m) = \sum_{i=1}^{s} c_i \otimes m_i$$

of shortest length s. Then the families $(c_i|i = 1,\dots,s)$ and $(m_i|i = 1,\dots,s)$ are linearly independent. Choose coefficients $c_{ij} \in C$ such that

$$\Delta(c_j) = \sum_{i=1}^{t} c_i \otimes c_{ij}, \quad \forall j = 1,\dots,s,$$

by suitably extending the linearly independent family $(c_i|i = 1,\dots,s)$ to a linearly independent family $(c_i|i = 1,\dots,t)$ and $t \geq s$.

We first show that we can choose $t = s$. By coassociativity we have $\sum_{i=1}^{s} c_i \otimes \delta(m_i) = \sum_{j=1}^{s} \Delta(c_j) \otimes m_j = \sum_{j=1}^{s} \sum_{i=1}^{t} c_i \otimes c_{ij} \otimes m_j$. Since the c_i and the m_j are linearly independent we can compare coefficients and get

$$\delta(m_i) = \sum_{j=1}^{s} c_{ij} \otimes m_j, \quad \forall i = 1,\dots,s \tag{1}$$

and $0 = \sum_{j=1}^{s} c_{ij} \otimes m_j$ for $i > s$. The last statement implies

$$c_{ij} = 0, \quad \forall i > s, j = 1,\dots,s.$$

Hence we get $t = s$ and

$$\Delta(c_j) = \sum_{i=1}^{s} c_i \otimes c_{ij}, \quad \forall j = 1,\dots,s.$$

Define finite dimensional subspaces $C' = \langle c_{ij}|i,j = 1,\dots,s\rangle \subseteq C$ and $M' = \langle m_i|i = 1,\dots,s\rangle \subseteq M$. Then by (1) we get $\delta : M' \longrightarrow C' \otimes M'$. We show that $m \in M'$ and that the restriction of Δ to C' gives a linear map $\Delta : C' \longrightarrow C' \otimes C'$ so that the required properties of the theorem are satisfied. First observe that $m = \sum \varepsilon(c_i)m_i \in M'$ and $c_j = \sum \varepsilon(c_i)c_{ij} \in C'$. Using coassociativity we get

$$\begin{aligned}\textstyle\sum_{i,j=1}^{n} c_i \otimes \Delta(c_{ij}) \otimes m_j &= \textstyle\sum_{k,j=1}^{s} \Delta(c_k) \otimes c_{kj} \otimes m_j \\ &= \textstyle\sum_{i,j,k=1}^{s} c_i \otimes c_{ik} \otimes c_{kj} \otimes m_j\end{aligned}$$

hence

$$\Delta(c_{ij}) = \sum_{k=1}^{s} c_{ik} \otimes c_{kj}. \tag{2}$$

□

Remark 8.6.15. We give a sketch of a second proof which is somewhat more technical. Since C is a $\mathbb{K}$-coalgebra, the dual C^* is an algebra. The comodule structure $\delta : M \longrightarrow C \otimes M$ leads to a module structure by $\rho = (\mathrm{ev} \otimes 1)(1 \otimes \delta) : C^* \otimes M \longrightarrow C^* \otimes C \otimes M \longrightarrow M$. Consider the submodule $N := C^*m$. Then N is finite dimensional, since $c^*m = \sum_{i=1}^{n} \langle c^*, c_i\rangle m_i$ for all $c^* \in C^*$ where $\sum_{i=1}^{n} c_i \otimes m_i = \delta(m)$. Observe that C^*m is a subspace of the space generated by the m_i. But it does not depend on the choice of the m_i. Furthermore if we take $\delta(m) = \sum c_i \otimes m_i$ with a shortest representation then the m_i are in C^*m since $c^*m = \sum\langle c^*, c_i\rangle m_i = m_i$ for c^* an element of a dual basis of the c_i.

N is a C-comodule since $\delta(c^*m) = \sum\langle c^*, c_i\rangle\delta(m_i) = \sum\langle c^*, c_{i(1)}\rangle c_{i(2)} \otimes m_i \in C \otimes C^*m$.

Now we construct a subcoalgebra D of C such that N is a D-comodule with the induced coaction. Let $D := N \otimes N^*$. By 8.13 N is a comodule over the coalgebra $N \otimes N^*$. Construct a linear map $\phi : D \longrightarrow C$ by $n \otimes n^* \mapsto \sum n_{(1)}\langle n^*, n_{(N)}\rangle$. By definition of the dual basis we have $n = \sum n_i\langle n_i^*, n\rangle$. Thus we get

$$\begin{aligned}(\phi \otimes \phi)\Delta_D(n \otimes n^*) &= (\phi \otimes \phi)(\textstyle\sum n \otimes n_i^* \otimes n_i \otimes n^*)\\ &= \textstyle\sum n_{(1)}\langle n_i^*, n_{(N)}\rangle \otimes n_{i(1)}\langle n^*, n_{i(N)}\rangle\\ &= \textstyle\sum n_{(1)} \otimes n_{i(1)}\langle n^*, n_{i(N)}\rangle\langle n_i^*, n_{(N)}\rangle\\ &= \textstyle\sum n_{(1)} \otimes n_{(2)}\langle n^*, n_{(N)}\rangle = \sum \Delta_C(n_{(1)})\langle n^*, n_{(N)}\rangle\\ &= \Delta_C\phi(n \otimes n^*).\end{aligned}$$

Furthermore $\varepsilon_C\phi(n \otimes n^*) = \varepsilon(\sum n_{(1)}\langle n^*, n_{(N)}\rangle = \langle n^*, \sum \varepsilon(n_{(1)})n_{(N)}\rangle = \langle n^*, n\rangle = \varepsilon(n \otimes n^*)$. Hence $\phi : D \longrightarrow C$ is a homomorphism of coalgebras, D is finite dimensional and the image $C' := \phi(D)$ is a finite dimensional subcoalgebra of C. Clearly N is also a C'-comodule, since it is a D-comodule.

Finally we show that the D-comodule structure on N if lifted to the C-comodule structure coincides with the one defined on M. We have

$$\begin{aligned}\delta_C(c^*m) &= \delta_C(\textstyle\sum\langle c^*, m_{(1)}\rangle m_{(M)}) = \sum\langle c^*, m_{(1)}\rangle m_{(2)} \otimes m_{(M)}\\ &= \textstyle\sum\langle c^*, m_{(1)}\rangle m_{(2)} \otimes m_i\langle m_i^*, m_{(M)}\rangle = \sum\langle c^*, m_{(1)}\rangle m_{(2)}\langle m_i^*, m_{(M)}\rangle \otimes m_i\\ &= (\phi \otimes 1)(\textstyle\sum\langle c^*, m_{(1)}\rangle m_{(M)} \otimes m_i^* \otimes m_i) = (\phi \otimes 1)(\sum c^*m \otimes m_i^* \otimes m_i)\\ &= (\phi \otimes 1)\delta_D(c^*m).\end{aligned}$$

7. Bialgebras

Definition 8.7.1. 1. A *bialgebra* $(B, \nabla, \eta, \Delta, \epsilon)$ consists of an algebra (B, ∇, η) and a coalgebra (B, Δ, ϵ) such that the diagrams

$$\begin{tikzcd}
B \otimes B \arrow[r, "\Delta\otimes\Delta"] \arrow[dd, "\nabla"'] & B \otimes B \otimes B \otimes B \arrow[dr, "1\otimes\tau\otimes 1"] & \\
 & & B \otimes B \otimes B \otimes B \arrow[d, "\nabla\otimes\nabla"] \\
B \arrow[rr, "\Delta"] & & B \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \mathbb{K} \arrow[dl, "\eta"'] \arrow[dr, "\eta\otimes\eta"] & \\
B \arrow[rr, "\Delta"] & & B \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \otimes B \arrow[rr, "\nabla"] \arrow[dr, "\epsilon\otimes\epsilon"'] & & B \arrow[dl, "\epsilon"] \\
 & \mathbb{K} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K} \arrow[rr, "\mathrm{id}"] \arrow[dr, "\eta"'] & & \mathbb{K} \\
 & B \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

commute, i.e. Δ and ϵ are homomorphisms of algebras resp. ∇ and η are homomorphisms of coalgebras.

2. Given bialgebras A and B. A map $f : A \longrightarrow B$ is called a *homomorphism of bialgebras* if it is a homomorphism of algebras and a homomorphism of coalgebras.

3. The category of bialgebras is denoted by $\mathbb{K}$-**Bialg**.

Problem 8.7.14. 1. Let (B, ∇, η) be an algebra and (B, Δ, ε) be a coalgebra. The following are equivalent:

a) $(B, \nabla, \eta, \Delta, \varepsilon)$ is a bialgebra.
b) $\Delta : B \longrightarrow B \otimes B$ and $\varepsilon : B \longrightarrow \mathbb{K}$ are homomorphisms of $\mathbb{K}$-algebras.
c) $\nabla : B \otimes B \longrightarrow B$ and $\eta : \mathbb{K} \longrightarrow B$ are homomorphisms of $\mathbb{K}$-coalgebras.

2. Let B be a finite dimensional bialgebra over field $\mathbb{K}$. Show that the dual space B^* is a bialgebra.

One of the most important properties of bialgebras B is that the tensor product over $\mathbb{K}$ of two B-modules or two B-comodules is again a B-module.

Proposition 8.7.2. 1. *Let B be a bialgebra. Let M and N be left B-modules. Then $M \otimes_{\mathbb{K}} N$ is a B-module by the map*

$$B \otimes M \otimes N \stackrel{\Delta\otimes 1}{\longrightarrow} B \otimes B \otimes M \otimes N \stackrel{1\otimes\tau\otimes 1}{\longrightarrow} B \otimes M \otimes B \otimes N \stackrel{\mu\otimes\mu}{\longrightarrow} M \otimes N.$$

2. *Let B be a bialgebra. Let M and N be left B-comodules. Then $M \otimes_{\mathbb{K}} N$ is a B-comodule by the map*

$$M \otimes N \stackrel{\delta\otimes\delta}{\longrightarrow} B \otimes M \otimes B \otimes N \stackrel{1\otimes\tau\otimes 1}{\longrightarrow} B \otimes B \otimes M \otimes M \otimes N \stackrel{\nabla\otimes 1}{\longrightarrow} B \otimes M \otimes N.$$

3. $\mathbb{K}$ *is a B-module by the map* $B \otimes \mathbb{K} \cong B \xrightarrow{\varepsilon} \mathbb{K}$.
4. $\mathbb{K}$ *is a B-comodule by the map* $\mathbb{K} \xrightarrow{\eta} B \cong B \otimes \mathbb{K}$.

Proof. We give a diagrammatic proof for 1. The associativity law is given by

$$\begin{tikzcd}
B \otimes B \otimes M \otimes N \arrow[r, "1\otimes\Delta\otimes 1\otimes 1"] \arrow[ddd, "\nabla\otimes 1\otimes 1"'] & B \otimes B \otimes B \otimes M \otimes N \arrow[r, "1\otimes 1\otimes\tau\otimes 1"] \arrow[d, "\Delta\otimes 1\otimes 1\otimes 1\otimes 1"] & B \otimes B \otimes M \otimes B \otimes N \arrow[r, "1\otimes\mu\otimes\mu"] \arrow[d, "\Delta\otimes 1\otimes 1\otimes 1\otimes 1"] & B \otimes M \otimes N \arrow[d, "\Delta\otimes 1\otimes 1"] \\
 & B \otimes B \otimes B \otimes B \otimes M \otimes N \arrow[r, "1\otimes 1\otimes 1\otimes\tau\otimes 1"] \arrow[d, "1\otimes\tau\otimes 1\otimes 1\otimes 1"] & B \otimes B \otimes B \otimes M \otimes B \otimes N \arrow[r, "1\otimes 1\otimes\mu\otimes\mu"] \arrow[d, "1\otimes\tau(B,B\otimes M)\otimes 1\otimes 1"] & B \otimes B \otimes M \otimes N \arrow[d, "1\otimes\tau\otimes 1"] \\
 & B \otimes B \otimes B \otimes B \otimes M \otimes N \arrow[r, "1\otimes 1\otimes\tau(B\otimes B,M)\otimes 1"] \arrow[d, "\nabla\otimes\nabla\otimes 1\otimes 1"] & B \otimes B \otimes M \otimes B \otimes B \otimes N \arrow[r, "1\otimes\mu\otimes 1\otimes\mu"] \arrow[d, "\nabla\otimes 1\otimes\nabla\otimes 1"] & B \otimes M \otimes B \otimes N \arrow[d, "\mu\otimes\mu"] \\
B \otimes M \otimes N \arrow[r, "\Delta\otimes 1\otimes 1"] & B \otimes B \otimes M \otimes N \arrow[r, "1\otimes\tau\otimes 1"] & B \otimes M \otimes B \otimes N \arrow[r, "\mu\otimes\mu"] & M \otimes N
\end{tikzcd}$$

The unit law is the commutativity of

$$\begin{tikzcd}
M \otimes N \arrow[r, phantom, "\cong"] \arrow[dd, "="'] & \mathbb{K} \otimes M \otimes N \arrow[r, "\eta\otimes 1\otimes 1"] \arrow[d, "\cong"] & B \otimes M \otimes N \arrow[d, "\Delta\otimes 1\otimes 1"] \\
 & \mathbb{K} \otimes \mathbb{K} \otimes M \otimes N \arrow[r, "\eta\otimes\eta\otimes 1\otimes 1"] \arrow[d, "1\otimes\tau\otimes 1"] & B \otimes B \otimes M \otimes N \arrow[d, "1\otimes\tau\otimes 1"] \\
M \otimes N \arrow[r, phantom, "\cong"] & \mathbb{K} \otimes M \otimes \mathbb{K} \otimes N \arrow[r, "\eta\otimes 1\otimes\eta\otimes 1"] \arrow[dr, "1"'] & B \otimes M \otimes B \otimes N \arrow[d, "\mu\otimes\mu"] \\
 & & M \otimes N
\end{tikzcd}$$

The corresponding properties for comodules follows from the dualized diagrams. The module and comodule properties of $\mathbb{K}$ are easily checked. $\square$

Definition 8.7.3. 1. Let $(B, \nabla, \eta, \Delta, \epsilon)$ be a bialgebra. Let A be a left B-module with structure map $\mu : B \otimes A \to A$. Let furthermore (A, ∇_A, η_A) be an algebra such that ∇_A and η_A are homomorphisms of B-modules. Then $(A, \nabla_A, \eta_A, \mu)$ is called a *B-module algebra.*

2. Let $(B, \nabla, \eta, \Delta, \epsilon)$ be a bialgebra. Let C be a left B-module with structure map $\mu : B \otimes C \to C$. Let furthermore $(C, \Delta_C, \varepsilon_C)$ be a coalgebra such that Δ_C and ε_C are homomorphisms of B- modules. Then $(C, \Delta_C, \varepsilon_C, \mu)$ is called a *B-module coalgebra.*

3. Let $(B, \nabla, \eta, \Delta, \epsilon)$ be a bialgebra. Let A be a left B-comodule with structure map $\delta : A \to B \otimes A$. Let furthermore (A, ∇_A, η_A) be an algebra such that ∇_A and η_A are homomorphisms of B-comodules. Then $(A, \nabla_A, \eta_A, \delta)$ is called a *B-comodule algebra.*

4. Let $(B, \nabla, \eta, \Delta, \epsilon)$ be a bialgebra. Let C be a left B-comodule with structure map $\delta : C \to B \otimes C$. Let furthermore $(C, \Delta_C, \varepsilon_C)$ be a coalgebra such that Δ_C and ε_C are homomorphisms of B-comodules. Then $(C, \Delta_C, \varepsilon_C, \delta)$ is called a *B-comodule coalgebra*.

Remark 8.7.4. If $(C, \Delta_C, \varepsilon_C)$ is a $\mathbb{K}$-coalgebra and (C, μ) is a B-module, then $(C, \Delta_C, \varepsilon_C, \mu)$ is a B-module coalgebra iff μ is a homomorphism of $\mathbb{K}$-coalgebras.

If (A, ∇_A, η_A) is a $\mathbb{K}$-algebra and (A, δ) is a B-comodule, then $(A, \nabla_A, \eta_A, \delta)$ is a B-comodule algebra iff δ is a homomorphism of $\mathbb{K}$-algebras.

Similar statement for module algebras or comodule coalgebras do *not* hold.

8. Representable Functors

Definition 8.8.1. Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathbf{Set}$ be a covariant functor. A pair (A, x) with $A \in \mathcal{C}, x \in \mathcal{F}(A)$ is called a *representing (generic, universal) object* for $\mathcal{F}$ and $\mathcal{F}$ is called a *representable functor*, if for each $B \in \mathcal{C}$ and $y \in \mathcal{F}(B)$ there exists a unique $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ such that $\mathcal{F}(f)(x) = y$:

$$\begin{tikzcd}
A \arrow[d, "f"'] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(f)"'] & [-2em] \ni x \arrow[d, mapsto] \\
B & \mathcal{F}(B) & [-2em] \ni y
\end{tikzcd}$$

Proposition 8.8.2. *Let (A, x) and (B, y) be representing objects for $\mathcal{F}$. Then there exists a unique isomorphism $f : A \longrightarrow B$ such that $\mathcal{F}(f)(x) = y$.*

$$\begin{tikzcd}
A \arrow[d, "h"] \arrow[dd, bend right=60, "1_A"'] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(h)"'] \arrow[dd, bend left=60, "1_{\mathcal{F}(A)}"] & x \arrow[d, mapsto] \\
B \arrow[d, "k"] \arrow[dd, bend right=60, "1_B"'] & \mathcal{F}(B) \arrow[d, "\mathcal{F}(k)"'] \arrow[dd, bend left=60, "1_{\mathcal{F}(B)}"] & y \arrow[d, mapsto] \\
A \arrow[d, "h"] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(h)"'] & x \arrow[d, mapsto] \\
B & \mathcal{F}(B) & y
\end{tikzcd}$$

Examples 8.8.3. 1. Let $X \in \mathbf{Set}$ and let R be a ring. $\mathcal{F} : R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Set}$, $\mathcal{F}(M) := \mathrm{Map}(X, M)$ is a covariant functor. A representing object for $\mathcal{F}$ is given by $(RX, x : X \longrightarrow RX)$ with the property, that for all $(M, y : X \longrightarrow M)$ there exists a unique $f \in \mathrm{Hom}_R(RX, M)$ such that $\mathcal{F}(f)(x) = \mathrm{Map}(X, f)(x) = fx = y$

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[dr, "y"'] & RX \arrow[d, "f"] \\
& M.
\end{tikzcd}$$

2. Given modules M_R and ${}_RN$. Define $\mathcal{F} : \mathbf{Ab} \longrightarrow \mathbf{Set}$ by $\mathcal{F}(A) := \mathrm{Bil}_R(M, N; A)$. Then $\mathcal{F}$ is a covariant functor. A representing object for $\mathcal{F}$ is given by $(M \otimes_R N, \otimes : M \times N \longrightarrow M \otimes_R N)$ with the property that for all $(A, f : M \times N \longrightarrow A)$ there exists

a unique $g \in \mathrm{Hom}(M \otimes_R N, A)$ such that $\mathcal{F}(g)(\otimes) = \mathrm{Bil}_R(M, N; g)(\otimes) = g\otimes = f$

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[dr, "f"'] & M \otimes_R N \arrow[d, "g"] \\ & A. \end{tikzcd}$$

3. Given a $\mathbb{K}$-module V. Define $\mathcal{F} : \mathbf{Alg} \to \mathbf{Set}$ by $\mathcal{F}(A) := \mathrm{Hom}(V, A)$. Then $\mathcal{F}$ is a covariant functor. A representing object for $\mathcal{F}$ is given by $(T(V), \iota : V \to T(V))$ with the property that for all $(A, f : V \to A)$ the exists a unique $g \in \mathrm{Mor}_{\mathbf{Alg}}(T(V), A)$ such that $\mathcal{F}(g)(\iota) = \mathrm{Hom}(V, g)(\iota) = g\iota = f$

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & T(V) \arrow[d, "g"] \\ & A. \end{tikzcd}$$

4. Given a $\mathbb{K}$-module V. Define $\mathcal{F} : \mathbf{cAlg} \to \mathbf{Set}$ by $\mathcal{F}(A) := \mathrm{Hom}(V, A)$. Then $\mathcal{F}$ is a covariant functor. A representing object for $\mathcal{F}$ is given by $(S(V), \iota : V \to S(V))$ with the property that for all $(A, f : V \to A)$ the exists a unique $g \in \mathrm{Mor}_{\mathbf{Alg}}(S(V), A)$ such that $\mathcal{F}(g)(\iota) = \mathrm{Hom}(V, g)(\iota) = g\iota = f$

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & S(V) \arrow[d, "g"] \\ & A. \end{tikzcd}$$

Proposition 8.8.4. *$\mathcal{F}$ has a representing object (A, a) if and only if there is a natural isomorphism $\varphi : \mathcal{F} \cong \mathrm{Mor}_{\mathcal{C}}(A, -)$ (with $a = \varphi(A)^{-1}(1_A)$).*

PROOF. $\Longrightarrow$: The map

$$\varphi(B) : \mathcal{F}(B) \ni y \mapsto f \in \mathrm{Mor}_{\mathcal{C}}(A, B) \text{ with } \mathcal{F}(f)(a) = y$$

is bijective with the inverse map

$$\psi(B) : \mathrm{Mor}_{\mathcal{C}}(A, B) \ni f \mapsto \mathcal{F}(f)(a) \in \mathcal{F}(B).$$

In fact we have $y \mapsto f \mapsto \mathcal{F}(f)(a) = y$ and $f \mapsto y := \mathcal{F}(f)(a) \mapsto g : \mathcal{F}(g)(a) = y = \mathcal{F}(f)(a)$. By uniqueness we get $f = g$. Hence all $\varphi(B)$ are bijective with inverse map $\psi(B)$. It is sufficient to show that ψ is a natural transformation.

Given $g : B \to C$. Then the following diagram commutes

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(A,B) \arrow[r, "\psi(B)"] \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(A,g)"'] & \mathcal{F}(B) \arrow[d, "\mathcal{F}(g)"] \\
\mathrm{Mor}_{\mathcal{C}}(A,C) \arrow[r, "\psi(C)"'] & \mathcal{F}(C)
\end{tikzcd}$$

since $\psi(C)\mathrm{Mor}_{\mathcal{C}}(A,g)(f) = \psi(C)(gf) = \mathcal{F}(gf)(a) = \mathcal{F}(g)\mathcal{F}(f)(a) = \mathcal{F}(g)\psi(B)(f)$.

$\Leftarrow$: Let A be given. Let $a := \varphi(A)^{-1}(1_A)$. For $y \in \mathcal{F}(B)$ we get $y = \varphi(B)^{-1}(f) = \varphi(B)^{-1}(f1_A) = \varphi(B)^{-1}\mathrm{Mor}_{\mathcal{C}}(A,f)(1_A) = \mathcal{F}(f)\varphi(A)^{-1}(1_A) = \mathcal{F}(f)(a)$ for a uniquely determined $f \in \mathrm{Mor}_{\mathcal{C}}(A,B)$. □

Proposition 8.8.5. *Given a representable functor $\mathcal{F}_X : \mathcal{C} \to$ **Set** for each $X \in \mathcal{D}$. Given a natural transformation $\mathcal{F}_g : \mathcal{F}_Y \to \mathcal{F}_X$ for each $g : X \to Y$ (contravariant!) such that $\mathcal{F}$ depends functorially on X, i.e. $\mathcal{F}_{1_X} = 1_{\mathcal{F}_X}, \mathcal{F}_{hg} = \mathcal{F}_g\mathcal{F}_h$. Then the representing objects (A_X, a_X) for $\mathcal{F}_X$ depend functorially on X, i.e. for each $g : X \to Y$ there is a unique homomorphism $A_g : A_X \to A_Y$ (with $\mathcal{F}_X(A_g)(a_X) = \mathcal{F}_g(A_Y)(a_Y)$) and the following identities hold $A_{1_X} = 1_{A_X}, A_{hg} = A_hA_g$.*

Proof. Choose a representing object (A_X, a_X) for $\mathcal{F}_X$ for each $X \in \mathcal{C}$ (by the axiom of choice). Then there is a unique homomorphism $A_g : A_X \to A_Y$ with

$$\mathcal{F}_X(A_g)(a_X) = \mathcal{F}_g(A_Y)(a_Y) \in \mathcal{F}_X(A_Y),$$

for each $g : X \to Y$ because $\mathcal{F}_g(A_Y) : \mathcal{F}_Y(A_Y) \to \mathcal{F}_X(A_Y)$ is given. We have $\mathcal{F}_X(A_1)(a_X) = \mathcal{F}_1(A_X)(a_X) = a_X = \mathcal{F}_X(1)(a_X)$ hence $A_1 = 1$, and $\mathcal{F}_X(A_{hg})(a_X) = \mathcal{F}_{hg}(A_Z)(a_Z) = \mathcal{F}_g(A_Z)\mathcal{F}_h(A_Z)(a_Z) = \mathcal{F}_g(A_Z)\mathcal{F}_Y(A_h)(a_Y) = \mathcal{F}_X(A_h)\mathcal{F}_g(A_Y)(a_Y) = \mathcal{F}_X(A_h)\mathcal{F}_X(A_g)(a_X) = \mathcal{F}_X(A_hA_g)(a_X)$ hence $A_hA_g = A_{hg}$ for $g : X \to Y$ and $h : Y \to Z$ in $\mathcal{D}$. □

Corollary 8.8.6. 1. $\mathrm{Map}(X,M) \cong \mathrm{Hom}_R(RX,M)$ *is a natural transformation in M (and in X!). In particular* **Set** $\ni X \mapsto RX \in R$-**Mod** *is a functor.*

2. $\mathrm{Bil}_R(M,N;A) \cong \mathrm{Hom}(M \otimes_R N, A)$ *is a natural transformation in A (and in $(M,N) \in$* **Mod**-$R \times R$-**Mod**). *In particular* **Mod**-$R \times R$-**Mod** $\ni M,N \mapsto M \otimes_r N \in$ **Ab** *is a functor.*

3. R-**Mod**-$S \times S$-**Mod**-$T \ni (M,N) \mapsto M \otimes_S N \in R$-**Mod**-$T$ *is a functor.*

9. Adjoint Functors and the Yoneda Lemma

Theorem 8.9.1. *(Yoneda Lemma) Let $\mathcal{C}$ be a category. Given a covariant functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathbf{Set}$ and an object $A \in \mathcal{C}$. Then the map*

$$\pi : \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A, \text{-}), \mathcal{F}) \ni \phi \mapsto \phi(A)(1_A) \in \mathcal{F}(A)$$

is bijective with the inverse map

$$\pi^{-1} : \mathcal{F}(A) \ni a \mapsto h^a \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A, \text{-}), \mathcal{F}),$$

where $h^a(B)(f) = \mathcal{F}(f)(a)$.

Proof. For $\phi \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A, \text{-}), \mathcal{F})$ we have a map $\phi(A) : \mathrm{Mor}_{\mathcal{C}}(A, A) \longrightarrow \mathcal{F}(A)$, hence π with $\pi(\phi) := \phi(A)(1_A)$ is a well defined map. For π^{-1} we have to check that h^a is a natural transformation. Given $f : B \longrightarrow C$ in $\mathcal{C}$. Then the diagram

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{C}}(A, B) \arrow[r, "\mathrm{Mor}(A,f)"] \arrow[d, "h^a(B)"'] & \mathrm{Mor}_{\mathcal{C}}(A, C) \arrow[d, "h^a(C)"] \\ \mathcal{F}(B) \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(C) \end{tikzcd}$$

is commutative. In fact if $g \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ then $h^a(C)\mathrm{Mor}_{\mathcal{C}}(A, f)(g) = h^a(C)(fg) = \mathcal{F}(fg)(a) = \mathcal{F}(f)\mathcal{F}(g)(a) = \mathcal{F}(f)h^a(B)(a)$. Thus π^{-1} is well defined.

Let $\pi^{-1}(a) = h^a$. Then $\pi\pi^{-1}(a) = h^a(A)(1_A) = \mathcal{F}(1_A)(a) = a$. Let $\pi(\phi) = \phi(A)(1_A) = a$. Then $\pi^{-1}\pi(\phi) = h^a$ and $h^a(B)(f) = \mathcal{F}(f)(a) = \mathcal{F}(f)(\phi(A)(1_A)) = \phi(B)\mathrm{Mor}_{\mathcal{C}}(A, f)(1_A) = \phi(B)(f)$, also $h^a = \phi$. $\square$

Corollary 8.9.2. *Given $A, B \in \mathcal{C}$. Then the following hold*

1. $\mathrm{Mor}_{\mathcal{C}}(A, B) \ni f \mapsto \mathrm{Mor}_{\mathcal{C}}(f, \text{-}) \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(B, \text{-}), \mathrm{Mor}_{\mathcal{C}}(A, \text{-}))$ *is a bijective map.*
2. *With the bijective map from 1. the isomorphisms from* $\mathrm{Mor}_{\mathcal{C}}(A, B)$ *correspond to natural isomorphisms from* $\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(B, \text{-}), \mathrm{Mor}_{\mathcal{C}}(A, \text{-}))$.
3. *For contravariant functors $\mathcal{F} : \mathcal{C} \longrightarrow \mathbf{Set}$ we have* $\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(\text{-}, A), \mathcal{F}) \cong \mathcal{F}(A)$.
4. $\mathrm{Mor}_{\mathcal{C}}(A, B) \ni f \mapsto \mathrm{Mor}_{\mathcal{C}}(\text{-}, f) \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(\text{-}, A), \mathrm{Mor}_{\mathcal{C}}(\text{-}, B))$ *is a bijective map that defines a one-to-one correspondence between the isomorphisms from* $\mathrm{Mor}_{\mathcal{C}}(A, B)$ *and the natural isomorphisms from* $\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(\text{-}, A), \mathrm{Mor}_{\mathcal{C}}(\text{-}, B))$.

Proof. 1. follows from the Yoneda Lemma with $\mathcal{F} = \mathrm{Mor}_{\mathcal{C}}(A, \text{-})$.

2. Observe that $h^f(C)(g) = \mathrm{Mor}_{\mathcal{C}}(A, g)(f) = gf = \mathrm{Mor}_{\mathcal{C}}(f, C)(g)$ hence $h^f = \mathrm{Mor}_{\mathcal{C}}(f, \text{-})$. Since we have $\mathrm{Mor}_{\mathcal{C}}(f, \text{-})\mathrm{Mor}_{\mathcal{C}}(g, \text{-}) = \mathrm{Mor}_{\mathcal{C}}(gf, \text{-})$ and $\mathrm{Mor}_{\mathcal{C}}(f, \text{-}) = \mathrm{id}_{\mathrm{Mor}_{\mathcal{C}}(A, \text{-})}$ if and only if $f = 1_A$ we get the one-to-one correspondence between the isomorphisms from 1.

3. and 4. follow by dualizing. $\square$

Remark 8.9.3. The map π is a natural transformation in the arguments A and $\mathcal{F}$. More precisely: if $f : A \longrightarrow B$ and $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ are given then the following diagrams commute

$$\begin{tikzcd} \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A,\text{-}),\mathcal{F}) \arrow[r, "\pi"] \arrow[d, "\mathrm{Nat}(\mathrm{Mor}(A,\text{-}),\phi)"'] & \mathcal{F}(A) \arrow[d, "\phi(A)"] \\ \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A,\text{-}),\mathcal{G}) \arrow[r, "\pi"'] & \mathcal{G}(A) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A,\text{-}),\mathcal{F}) \arrow[r, "\pi"] \arrow[d, "\mathrm{Nat}(\mathrm{Mor}(f,\text{-}),\mathcal{F})"'] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(f)"] \\ \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(B,\text{-}),\mathcal{F}) \arrow[r, "\pi"'] & \mathcal{F}(B). \end{tikzcd}$$

This can be easily checked. Furthermore we have for $\psi : \mathrm{Mor}_{\mathcal{C}}(A,\text{-}) \longrightarrow \mathcal{F}$

$$\pi\,\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A,\text{-}),\phi)(\psi) = \pi(\phi\psi) = (\phi\psi)(A)(1_A) = \phi(A)\psi(A)(1_A) = \phi(A)\pi(\psi)$$

and

$$\begin{aligned} &\pi\,\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(f,\text{-}),\mathcal{F})(\psi) = \pi(\psi\mathrm{Mor}_{\mathcal{C}}(f,\text{-})) = (\psi\mathrm{Mor}_{\mathcal{C}}(f,\text{-}))(B)(1_B) = \psi(B)(f) \\ &= \psi(B)\mathrm{Mor}_{\mathcal{C}}(A,f)(1_A) = \mathcal{F}(f)\psi(A)(1_A) = \mathcal{F}(f)\pi(\psi). \end{aligned}$$

Remark 8.9.4. By the previous corollary the representing object A is uniquely determined up to isomorphism by the isomorphism class of the functor $\mathrm{Mor}_{\mathcal{C}}(A,\text{-})$.

Problem 8.9.15. 1. Determine explicitly all natural endomorphisms from $\mathbb{G}_a$ to $\mathbb{G}_a$ (as defined in Lemma 2.3.5).
2. Determine all additive natural endomorphisms of $\mathbb{G}_a$.
3. Determine all natural transformations from $\mathbb{G}_a$ to $\mathbb{G}_m$ (see Lemma 2.3.7).
4. Determine all natural automorphisms of $\mathbb{G}_m$.

Proposition 8.9.5. *Let $\mathcal{G} : \mathcal{C} \times \mathcal{D} \longrightarrow$ **Set** be a covariant bifunctor such that the functor $\mathcal{G}(C,\text{-}) : \mathcal{D} \longrightarrow$ **Set** is representable for all $C \in \mathcal{C}$. Then there exists a contravariant functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ such that $\mathcal{G} \cong \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-},\text{-})$ holds. Furthermore $\mathcal{F}$ is uniquely determined by $\mathcal{G}$ up to isomorphism.*

PROOF. For each $C \in \mathcal{C}$ choose an object $\mathcal{F}(C) \in \mathcal{D}$ and an isomorphism $\xi_C : \mathcal{G}(C,\text{-}) \cong \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C),\text{-})$. Given $f : C \to C'$ in $\mathcal{C}$ then let $\mathcal{F}(f) : \mathcal{F}(C') \to \mathcal{F}(C)$ be the uniquely determined morphism (by the Yoneda Lemma) in $\mathcal{D}$ such that the diagram

$$\begin{tikzcd} \mathcal{G}(C,\text{-}) \arrow[r, "\xi_C"] \arrow[d, "\mathcal{G}(f,\text{-})"'] & \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C),\text{-}) \arrow[d, "\mathrm{Mor}(\mathcal{F}(f),\text{-})"] \\ \mathcal{G}(C',\text{-}) \arrow[r, "\xi_{C'}"'] & \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C'),\text{-}) \end{tikzcd}$$

commutes. Because of the uniqueness $\mathcal{F}(f)$ and because of the functoriality of $\mathcal{G}$ it is easy to see that $\mathcal{F}(fg) = \mathcal{F}(g)\mathcal{F}(f)$ and $\mathcal{F}(1_C) = 1_{\mathcal{F}(C)}$ hold and that $\mathcal{F}$ is a contravariant functor.

If $\mathcal{F}' : \mathcal{C} \longrightarrow \mathcal{D}$ is given with $\mathcal{G} \cong \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'\text{-}, \text{-})$ then $\phi : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}) \cong \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'\text{-}, \text{-})$. Hence by the Yoneda Lemma $\psi(C) : \mathcal{F}(C) \cong \mathcal{F}'(C)$ is an isomorphism for all $C \in \mathcal{C}$. With these isomorphisms induced by ϕ the diagram

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'(C), \text{-}) \arrow[r, "\mathrm{Mor}(\psi(C), \text{-})"] \arrow[d, "\mathrm{Mor}(\mathcal{F}'(f), \text{-})"'] & \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), \text{-}) \arrow[d, "\mathrm{Mor}(\mathcal{F}(f), \text{-})"] \\
\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'(C'), \text{-}) \arrow[r, "\mathrm{Mor}(\psi(C'), \text{-})"'] & \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C'), \text{-})
\end{tikzcd}$$

commutes. Hence the diagram

$$\begin{tikzcd}
\mathcal{F}(C') \arrow[r, "\psi(C')"] \arrow[d, "\mathcal{F}'(f)"'] & \mathcal{F}'(C') \arrow[d, "\mathcal{F}(f)"] \\
\mathcal{F}(C) \arrow[r, "\psi(C)"'] & \mathcal{F}'(C)
\end{tikzcd}$$

commutes. Thus $\psi : \mathcal{F} \longrightarrow \mathcal{F}'$ is a natural isomorphism. $\square$

Definition 8.9.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ be covariant functors. $\mathcal{F}$ is called *leftadjoint* to $\mathcal{G}$ and $\mathcal{G}$ *rightadjoint* to $\mathcal{F}$ if there is a natural isomorphism of bifunctors $\phi : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-})$ from $\mathcal{C}^{op} \times \mathcal{D}$ to **Set**.

Lemma 8.9.7. *If $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is leftadjoint to $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ then $\mathcal{F}$ is uniquely determined by $\mathcal{G}$ up to isomorphism. Similarly $\mathcal{G}$ is uniquely determined by $\mathcal{F}$ up to isomorphism.*

Proof. Now we prove the first claim. Assume that also $\mathcal{F}'$ is leftadjoint to $\mathcal{G}$ with $\phi' : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'\text{-}, \text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-})$. Then we have a natural isomorphism ${\phi'}^{-1}\phi : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'\text{-}, \text{-})$. By Proposition 8.9.5 we get $\mathcal{F} \cong \mathcal{F}'$. $\square$

Lemma 8.9.8. *A functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ has a leftadjoint functor iff all functors $\mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}\text{-})$ are representable.*

Proof. follows from 8.9.5. $\square$

Lemma 8.9.9. *Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ be covariant functors. Then*

$$\mathrm{Nat}(\mathrm{Id}_{\mathcal{C}}, \mathcal{G}\mathcal{F}) \ni \Phi \mapsto \mathcal{G}\text{-}\Phi\text{-} \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}), \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}))$$

is a bijective map with inverse map

$$\mathrm{Nat}(\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}), \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-})) \ni \phi \mapsto \phi(\text{-}, \mathcal{F}\text{-})(1_{\mathcal{F}\text{-}}) \in \mathrm{Nat}(\mathrm{Id}_{\mathcal{C}}, \mathcal{G}\mathcal{F}).$$

Furthermore

$$\mathrm{Nat}(\mathcal{F}\mathcal{G}, \mathrm{Id}_{\mathcal{C}}) \ni \Psi \mapsto \Psi\text{-}\mathcal{F}\text{-} \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}), \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}))$$

is a bijective map with inverse map

$$\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}), \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-})) \ni \psi \mapsto \psi(\mathcal{G}\text{-}, \text{-})(1_{\mathcal{G}\text{-}}) \in \mathrm{Nat}(\mathcal{F}\mathcal{G}, \mathrm{Id}_{\mathcal{C}}).$$

PROOF. The natural transformation $\mathcal{G}$-Φ- is defined as follows. Given $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $f \in \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), D)$ then let $(\mathcal{G}\text{-}\Phi\text{-})(C, D)(f) := \mathcal{G}(f)\Phi(C) : C \longrightarrow \mathcal{G}\mathcal{F}(C) \longrightarrow \mathcal{G}(D)$. It is easy to check the properties of a natural transformation.

Given Φ then one obtains by composition of the two maps $\mathcal{G}(1_{\mathcal{F}(C)})\Phi(C) = \mathcal{G}\mathcal{F}(1_C)\Phi(C) = \Phi(C)$. Given ϕ one obtains

$$\begin{aligned}&\mathcal{G}(f)(\phi(C, \mathcal{F}(C))(1_{\mathcal{F}(C)}) = \mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(f))\phi(C, \mathcal{F}(C))(1_{\mathcal{F}(C)})\\ &= \phi(C, D)\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), f)(1_{\mathcal{F}(C)}) = \phi(C, D)(f).\end{aligned}$$

The second part of the lemma is proved similarly. $\square$

Proposition 8.9.10. *Let*

$$\phi : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}) \quad \textit{and} \quad \psi : \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-})$$

be natural transformations with associated natural transformations (by Lemma 8.9.9) $\Phi : \mathrm{Id}_{\mathcal{C}} \longrightarrow \mathcal{G}\mathcal{F}$ *resp.* $\Psi : \mathcal{F}\mathcal{G} \longrightarrow \mathrm{Id}_{\mathcal{D}}$.

1) Then we have $\phi\psi = \mathrm{id}_{\mathrm{Mor}(\text{-},\mathcal{G}\text{-})}$ *if and only if* $(\mathcal{G} \xrightarrow{\Phi\mathcal{G}} \mathcal{G}\mathcal{F}\mathcal{G} \xrightarrow{\mathcal{G}\Psi} \mathcal{G}) = \mathrm{id}_{\mathcal{G}}$.

2) We also have $\psi\phi = \mathrm{id}_{\mathrm{Mor}(\mathcal{F}\text{-},\text{-})}$ *if and only if* $(\mathcal{F} \xrightarrow{\mathcal{F}\Phi} \mathcal{F}\mathcal{G}\mathcal{F} \xrightarrow{\Psi\mathcal{F}} \mathcal{F}) = \mathrm{id}_{\mathcal{F}}$.

PROOF. We get

$$\begin{aligned}&\mathcal{G}\Psi(D)\Phi\mathcal{G}(D) = \mathcal{G}\Psi(D)\phi(\mathcal{G}(D), \mathcal{F}\mathcal{G}(D))(1_{\mathcal{F}\mathcal{G}(D)})\\ &= \mathrm{Mor}_{\mathcal{C}}(\mathcal{G}(D), \mathcal{G}\Psi(D))\phi(\mathcal{G}(D), \mathcal{F}\mathcal{G}(D))(1_{\mathcal{F}\mathcal{G}(D)})\\ &= \phi(\mathcal{G}(D), D)\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\mathcal{G}(D), \Psi(D))(1_{\mathcal{F}\mathcal{G}(D)})\\ &= \phi(\mathcal{G}(D), D)(\Psi(D))\\ &= \phi(\mathcal{G}(D), D)\psi(\mathcal{G}(D), D)(1_{\mathcal{G}(D)})\\ &= \phi\psi(\mathcal{G}(D), D)(1_{\mathcal{G}(D)}).\end{aligned}$$

Similarly we get

$$\begin{aligned}&\phi\psi(C, D)(f) = \phi(C, D)\psi(C, D)(f) = \mathcal{G}(\Psi(D)\mathcal{F}(f))\Phi(C)\\ &= \mathcal{G}\Psi(D)\mathcal{G}\mathcal{F}(f)\Phi(C) = \mathcal{G}\Psi(D)\Phi\mathcal{G}(D)f.\end{aligned}$$ $\square$

Corollary 8.9.11. *Let* $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ *and* $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ *be functors.* $\mathcal{F}$ *is leftadjoint to* $\mathcal{G}$ *if and only if there are natural transformations* $\Phi : \mathrm{Id}_{\mathcal{C}} \longrightarrow \mathcal{G}\mathcal{F}$ *and* $\Psi : \mathcal{F}\mathcal{G} \longrightarrow \mathrm{Id}_{\mathcal{D}}$ *such that* $(\mathcal{G}\Psi)(\Phi\mathcal{G}) = \mathrm{id}_{\mathcal{G}}$ *and* $(\Psi\mathcal{F})(\mathcal{F}\Phi) = \mathrm{id}_{\mathcal{F}}$.

Definition 8.9.12. The natural transformations $\Phi : \mathrm{Id}_{\mathcal{C}} \longrightarrow \mathcal{G}\mathcal{F}$ and $\Psi : \mathcal{F}\mathcal{G} \longrightarrow \mathrm{Id}_{\mathcal{D}}$ given in 8.9.11 are called *unit* and *counit* resp. for the adjoint functors $\mathcal{F}$ and $\mathcal{G}$.

Problem 8.9.16. 1. Let ${}_RM_S$ be a bimodule. Show that the functor $M \otimes_S$ - : ${}_S\mathcal{M} \longrightarrow {}_R\mathcal{M}$ is leftadjoint to $\mathrm{Hom}_R(M,\text{-}) : {}_R\mathcal{M} \longrightarrow {}_S\mathcal{M}$. Determine the associated unit and counit.

b) Show that there is a natural isomorphism $\mathrm{Map}(A \times B, C) \cong \mathrm{Map}(B, \mathrm{Map}(A, C))$. Determine the associated unit and counit.

c) Show that there is a natural isomorphism $\mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}G, A) \cong \mathbf{Gr}(G, U(A))$. Determine the associated unit and counit.

d) Show that there is a natural isomorphism $\mathbb{K}\text{-}\mathbf{Alg}(U(\mathfrak{g}), A) \cong \text{Lie-Alg}(\mathfrak{g}, A^L)$. Determine the corresponding leftadjoint functor and the associated unit and counit.

Definition 8.9.13. Let $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ be a covariant functor. $\mathcal{G}$ generates a *(co-)universal problem* a follows:

Given $C \in \mathcal{C}$. Find an object $\mathcal{F}(C) \in \mathcal{D}$ and a morphism $\iota : C \longrightarrow \mathcal{G}(\mathcal{F}(C))$ in $\mathcal{C}$ such that there is a unique morphism $g : \mathcal{F}(C) \longrightarrow D$ in $\mathcal{D}$ for each object $D \in \mathcal{D}$ and for each morphism $f : C \longrightarrow \mathcal{G}(D)$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathcal{G}(\mathcal{F}(C)) \arrow[d, "\mathcal{G}(g)"] \\ & \mathcal{G}(D) \end{tikzcd}$$

commutes.

A pair $(\mathcal{F}(C), \iota)$ that satisfies the above conditions is called a *universal solution* of the (co-)universal problem defined by $\mathcal{G}$ and C.

Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be a covariant functor. $\mathcal{F}$ generates a *universal problem* a follows:

Given $D \in \mathcal{D}$. Find an object $\mathcal{G}(D) \in \mathcal{C}$ and a morphism $\nu : \mathcal{F}(\mathcal{G}(D)) \longrightarrow D$ in $\mathcal{D}$ such that there is a unique morphism $g : C \longrightarrow \mathcal{G}(D)$ in $\mathcal{C}$ for each object $C \in \mathcal{C}$ and for each morphism $f : \mathcal{F}(C) \longrightarrow D$ in $\mathcal{D}$ such that the diagram

$$\begin{tikzcd} \mathcal{F}(C) \arrow[d, "\mathcal{F}(g)"'] \arrow[dr, "f"] & \\ \mathcal{F}\mathcal{G}(D) \arrow[r, "\nu"'] & D \end{tikzcd}$$

commutes.

A pair $(\mathcal{G}(D), \nu)$ that satisfies the above conditions is called a *universal solution* of the (co-)universal problem defined by $\mathcal{F}$ and D.

Proposition 8.9.14. *Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be leftadjoint to $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$. Then $\mathcal{F}(C)$ and the unit $\iota = \Phi(C) : C \longrightarrow \mathcal{G}\mathcal{F}(C)$ form a (co-)universal solution for the (co-)universal problem defined by $\mathcal{G}$ and C.*

Furthermore $\mathcal{G}(D)$ and the counit $\nu = \Psi(D) : \mathcal{F}\mathcal{G}(D) \longrightarrow D$ form a universal solution for the universal problem defined by $\mathcal{F}$ and D.

PROOF. By Theorem 8.9.10 the morphisms $\phi : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-},\text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(\text{-},\mathcal{G}\text{-})$ and $\psi : \mathrm{Mor}_{\mathcal{C}}(\text{-},\mathcal{G}\text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-},\text{-})$ are inverses of each other. They are defined with unit and counit as $\phi(C,D)(g) = \mathcal{G}(g)\Phi(C)$ resp. $\psi(C,D)(f) = \Psi(D)\mathcal{F}(f)$. Hence for each $f : C \longrightarrow \mathcal{G}(D)$ there is a unique $g : \mathcal{F}(C) \longrightarrow D$ such that $f = \phi(C,D)(g) = \mathcal{G}(g)\Phi(C) = \mathcal{G}(g)\iota$.

The second statement follows analogously. □

Remark 8.9.15. If $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ and $C \in \mathcal{C}$ are given then the (co-)universal solution $(\mathcal{F}(C), \iota : C \longrightarrow \mathcal{G}(D))$ can be considered as the best (co-)approximation of the object C in $\mathcal{C}$ by an object D in $\mathcal{D}$ with the help of a functor $\mathcal{G}$. The object $D \in \mathcal{D}$ turns out to be $\mathcal{F}(C)$.

If $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $D \in \mathcal{D}$ are given then the universal solution $(\mathcal{G}(D), \nu : \mathcal{F}\mathcal{G}(D) \longrightarrow D)$ can be considered as the best approximation of the object D in $\mathcal{D}$ by an object C in $\mathcal{C}$ with the help of a functor $\mathcal{F}$. The object $C \in \mathcal{C}$ turns out to be $\mathcal{G}(D)$.

Proposition 8.9.16. *Given $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$. Assume that for each $C \in \mathcal{C}$ the universal problem defined by $\mathcal{G}$ and C is solvable. Then there is a leftadjoint functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ to $\mathcal{G}$.*

Given $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$. Assume that for each $D \in \mathcal{D}$ the universal problem defined by $\mathcal{F}$ and D is solvable. Then there is a leftadjoint functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ to $\mathcal{F}$.

PROOF. Assume that the (co-)universal problem defined by $\mathcal{G}$ and C is solved by $\iota : C \longrightarrow \mathcal{F}(C)$. Then the map $\mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(D)) \ni f \mapsto g \in \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), D)$ with $\mathcal{G}(g)\iota = f$ is bijective. The inverse map is given by $g \mapsto \mathcal{G}(g)\iota$. This is a natural transformation since the diagram

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), D) \arrow[r, "\mathcal{G}(\text{-})\iota"] \arrow[d, "\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C),h)"'] & \mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(D)) \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(C,\mathcal{G}(h))"] \\
\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), D') \arrow[r, "\mathcal{G}(\text{-})\iota"'] & \mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(D'))
\end{tikzcd}$$

commutes for each $h \in \mathrm{Mor}_D(D, D')$. In fact we have

$$\mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(h))(\mathcal{G}(g)\iota) = \mathcal{G}(h)\mathcal{G}(g)\iota = \mathcal{G}(hg)\iota = \mathcal{G}(\mathrm{Mor}_{\mathcal{C}}(\mathcal{F}(C), h)(g))\iota.$$

Hence for all $C \in \mathcal{C}$ the functor $\mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(\text{-})) : \mathcal{D} \longrightarrow \mathbf{Set}$ induced by the bifunctor $\mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}(\text{-})) : \mathcal{C}^{op} \times \mathcal{D} \longrightarrow \mathbf{Set}$ is representable. By Theorem 8.9.5 there is a functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ such that $\mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}(\text{-})) \cong \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(\text{-}), \text{-})$.

The second statement follows analogously. □

Remark 8.9.17. One can characterize the properties that $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ (resp. $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$) must have in order to possess a left-(right-)adjoint functor. One of the essential properties for this is that $\mathcal{G}$ preserves limits (hence direct products and difference kernels).