CHAPTER 4

The Infinitesimal Theory

4. Derivations and Lie Algebras of Affine Algebraic Groups

Lemma and Definition 7.4.1. *Let* $\mathcal{G}:$ **-cAlg** $\to$ **Set** *be a group valued functor. The kernel* $\mathcal{L}ie(\mathcal{G})(R)$ *of the sequence*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}ie(\mathcal{G})(R) \arrow[r] & \mathcal{G}(R(\delta)) \arrow[r, shift left, "\mathcal{G}(p)"] & \mathcal{G}(R) \arrow[l, shift left, "\mathcal{G}(j)"] \arrow[r] & 0 \end{tikzcd}$$

is called the Lie algebra *of* $\mathcal{G}$ *and is a group valued functor in* R.

PROOF. For every algebra homomorphism $f: R \to S$ the following diagram of groups commutes

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}ie(\mathcal{G})(R) \arrow[r] \arrow[d] & \mathcal{G}(R(\delta)) \arrow[r, shift left, "\mathcal{G}(p)"] \arrow[d, "\mathcal{G}(f(\delta))"] & \mathcal{G}(R) \arrow[l, shift left, "\mathcal{G}(j)"] \arrow[r] \arrow[d, "\mathcal{G}(f)"] & 0 \\ 0 \arrow[r] & \mathcal{L}ie(\mathcal{G})(S) \arrow[r] & \mathcal{G}(S(\delta)) \arrow[r, shift left, "\mathcal{G}(p)"] & \mathcal{G}(S) \arrow[l, shift left, "\mathcal{G}(j)"] \arrow[r] & 0 \end{tikzcd}$$

$\square$

Proposition 4.4.2. *Let* $\mathcal{G}:$ **-cAlg** $\to$ **Set** *be a group valued functor with multiplication* $*$. *Then there are functorial operations*

$$\mathcal{G}(R) \times \mathcal{L}ie(\mathcal{G})(R) \ni (g, x) \mapsto g \cdot x \in \mathcal{L}ie(\mathcal{G})(R)$$

$$R \times \mathcal{L}ie(\mathcal{G})(R) \ni (a, x) \mapsto ax \in \mathcal{L}ie(\mathcal{G})(R)$$

such that

$$\begin{aligned} g \cdot (x + y) &= g \cdot x + g \cdot y, \\ h \cdot (g \cdot x) &= (h * g) \cdot x, \\ a(x + y) &= ax + ay, \\ (ab)x &= a(bx), \\ g \cdot (ax) &= a(g \cdot x). \end{aligned}$$

PROOF. First observe that the composition $+$ on $\mathcal{L}ie(\mathcal{G})(R)$ is induced by the multiplication $*$ of $\mathcal{G}(R(\delta))$ so it is not necessarily commutative.

We define $g \cdot x := \mathcal{G}(j)(g) * x * \mathcal{G}(j)(g)^{-1}$. Then $\mathcal{G}(p)(g \cdot x) = \mathcal{G}(p)\mathcal{G}(j)(g) * \mathcal{G}(p)(x) * \mathcal{G}(p)\mathcal{G}(j)(g)^{-1} = g * 1 * g^{-1} = 1$ hence $g \cdot x \in \mathcal{L}ie(\mathcal{G})(R)$.

Now let $a \in R$. To define $a : \mathcal{L}ie(\mathcal{G})(R) \to \mathcal{L}ie(\mathcal{G})(R)$ we use $u_a : R(\delta) \to R(\delta)$ defined by $u_a(\delta) := a\delta$ and thus $u_a(b + c\delta) := b + ac\delta$. Obviously u_a is a homomorphism of R-algebras. Furthermore we have $pu_a = p$ and $u_a j = j$. Thus we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}ie(\mathcal{G})(R) \arrow[r] \arrow[d, "a"] & \mathcal{G}(R(\delta)) \arrow[r, shift left, "\mathcal{G}(p)"] \arrow[d, "\mathcal{G}(u_a)"] & \mathcal{G}(R) \arrow[l, shift left, "\mathcal{G}(j)"] \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mathcal{L}ie(\mathcal{G})(R) \arrow[r] & \mathcal{G}(R(\delta)) \arrow[r, shift left, "\mathcal{G}(p)"] & \mathcal{G}(R) \arrow[l, shift left, "\mathcal{G}(j)"] \arrow[r] & 0 \end{tikzcd}$$

that defines a group homomorphism $a : \mathcal{L}ie(\mathcal{G})(R) \longrightarrow \mathcal{L}ie(\mathcal{G})(R)$ on the kernel of the exact sequences. In particular we have then $a(x+y) = ax + ay$.

Furthermore we have $u_{ab} = u_a u_b$ hence $(ab)x = a(bx)$.

The next formula follows from $g \cdot (x+y) = \mathcal{G}(j)(g) * x * y * \mathcal{G}(j)(g)^{-1} = \mathcal{G}(j)(g) * x * \mathcal{G}(j)(g)^{-1} * \mathcal{G}(j)(g) * y * \mathcal{G}(j)(g)^{-1} = g \cdot x + g \cdot y$.

We also see $(h*g) \cdot x = \mathcal{G}(j)(h*g)*x*\mathcal{G}(j)(h*g)^{-1} = \mathcal{G}(j)(h)*\mathcal{G}(j)(g)*x*\mathcal{G}(j)(g)^{-1}*\mathcal{G}(j)(h)^{-1} = h \cdot (g \cdot x)$. Finally we have $g \cdot (ax) = \mathcal{G}(j)(g) * \mathcal{G}(u_a)(x) * \mathcal{G}(j)(g^{-1}) = \mathcal{G}(u_a)(\mathcal{G}(j)(g) * x * \mathcal{G}(j)(g^{-1})) = a(g \cdot x)$. □

Proposition 4.4.3. *Let $\mathcal{G} = \ $-$\mathbf{cAlg}(H,\text{-})$ be an affine algebraic group. Then $\mathcal{L}ie(\mathcal{G})(\ ) \cong \mathbf{Lie}(H^o)$ as additive groups. The isomorphism is compatible with the operations given in 4.4.2 and 4.3.6.*

PROOF. We consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}ie(\mathcal{G})(\ ) \arrow[r] & \text{-}\mathbf{cAlg}(H,\ (\delta)) \arrow[r, shift left, "p"] \arrow[d, "\omega"] & \text{-}\mathbf{cAlg}(H,\ ) \arrow[l, shift left] \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathbf{Lie}(H^o) \arrow[u, "e"] \arrow[r, "e^{\delta -}"] & G_{\ (\delta)}(H^o \otimes\ (\delta)) \arrow[r, "p"] & G(H^o) \arrow[r] & 0
\end{tikzcd}$$

We know by definition that the top sequence is exact. The bottom sequence is exact by Corollary 4.3.9.

Let $f \in \ $-$\mathbf{cAlg}(H,\ )$. Since $\mathrm{Ker}(f)$ is an ideal of codimension 1 we get $f \in H^o$. The map f is an algebra homomorphism iff $\langle f, ab \rangle = \langle f \otimes f, a \otimes b \rangle$ and $\langle f, 1 \rangle = 1$ iff $\Delta_{H^o}(f) = f \otimes f$ and $\varepsilon_{H^o}(f) = 1$ iff $f \in G(H^o)$. Hence we get the right hand vertical isomorphism $\ $-$\mathbf{cAlg}(H,\ ) \cong G(H^o)$.

Consider an element $f \in \ $-$\mathbf{cAlg}(H,\ (\delta)) \subseteq \mathrm{Hom}(H,\ (\delta))$. It can be written as $f = f_0 + f_1\delta$ with $f_0, f_1 \in \mathrm{Hom}(H,\ )$. The linear map f is an algebra homomorphism iff $f_0 : H \longrightarrow\ $ is an algebra homomorphism and f_1 satisfies $f_1(1) = 0$ and $f_1(ab) = f_0(a)f_1(b) + f_1(a)f_0(b)$. In fact we have $f(1) = f_0(1) + f_1(1)\delta = 1$ iff $f_0(1) = 1$ and $f_1(1) = 0$ (by comparing coefficients). Furthermore we have $f(ab) = f(a)f(b)$ iff $f_0(ab) + f_1(ab)\delta = (f_0(a) + f_1(a)\delta)(f_0(b) + f_1(b)\delta) = f_0(a)f_0(b) + f_0(a)f_1(b)\delta + f_1(a)f_0(b)\delta$ iff $f_0(ab) = f_0(a)f_0(b)$ and $f_1(ab) = f_0(a)f_1(b) + f_1(a)f_0(b)$.

Since f_0 is an algebra homomorphism we have as above $f_0 \in H^o$. For f_1 we have $(b \rightharpoonup f_1)(a) = f_1(ab) = f_0(a)f_1(b) + f_1(a)f_0(b) = (f_1(b)f_0 + f_0(b)f_1)(a)$ hence $(b \rightharpoonup f_1) = f_1(b)f_0 + f_0(b)f_1 \in\ f_0 +\ f_1$, a two dimensional subspace. Thus $f_1 \in H^o$.

In the following computations we will identify $(H^o \otimes\ (\delta)) \otimes_{\ (\delta)} (H^o \otimes\ (\delta))$ with $H^o \otimes H^o \otimes |K(\delta)$.

Let $f = f_0 + f_1\delta = f_0 \otimes 1 + f_1 \otimes \delta \in H^o \oplus H^o\delta = H^o \otimes\ (\delta)$. Then f is a homomorphism of algebras iff $f(ab) = f(a)f(b)$ and $f(1) = 1$ iff $f_0(ab) = f_0(a)f_0(b)$ and $f_1(ab) = f_0(a)f_1(b) + f_1(a)f_0(b)$ and $f_0(1) = 1$ and $f_1(1) = 0$ iff $\Delta_{H^o}(f_0) = f_0 \otimes f_0$ and $\Delta_{H^o}(f_1) = f_0 \otimes f_1 + f_1 \otimes f_0$ and $\varepsilon_{H^o}(f_0) = 1$ and $\varepsilon_{H^o}(f_1) = 0$ iff $(\Delta_{H^o} \otimes \mathrm{id}_{\ (\delta)})(f_0 \otimes 1 + f_1 \otimes \delta) = f_0 \otimes f_0 \otimes 1 + f_0 \otimes f_1 \otimes \delta + f_1 \otimes f_0 \otimes \delta = (f_0 \otimes 1 + f_1 \otimes \delta) \otimes_{\ (\delta)} (f_0 \otimes 1 + f_1 \otimes \delta)$

and $(\varepsilon_{H^o} \otimes \mathrm{id}_{\ (\delta)})(f_0 \otimes 1 + f_1 \otimes \delta) = 1 \otimes 1$ iff $(\Delta_{H^o} \otimes \mathrm{id}_{\ (\delta)})(f) = f \otimes_{\ (\delta)} f$ and $(\varepsilon_{H^o} \otimes \mathrm{id}_{\ (\delta)})(f) = 1$ iff $f \in G_{\ (\delta)}(H^o \otimes \ (\delta))$.

Hence we have a bijective map $\omega : \ \text{-}\mathbf{cAlg}(H, \ (\delta)) \ni f = f_0 + f_1\delta \mapsto f_0 \otimes 1 + f_1 \otimes \delta \in G_{\ (\delta)}(H^o \otimes \ (\delta))$. Since the group multiplication in $\ \text{-}\mathbf{cAlg}(H, \ (\delta)) \subseteq \mathrm{Hom}(H, \ (\delta))$ is the convolution $*$ and the group multiplication in $G_{\ (\delta)}(H^o \otimes \ (\delta)) \subseteq H^o \otimes \ (\delta)$ is the ordinary algebra multiplication, where the multiplication of H^o again is the convolution, it is clear that ω is a group homomorphism. Furthermore the right hand square of the above diagram commutes. Thus we get an isomorphism $e : \mathbf{Lie}(H^o) \longrightarrow \mathcal{L}ie(\mathcal{G})(\ )$ on the kernels. This map is defined by $e(d) = 1 + d\delta \in \ \text{-}\mathbf{cAlg}(H, \ (\delta))$.

To show that this isomorphism is compatible with the actions of resp. $G(H^o)$ let $\alpha \in \ $, $a \in H$, and $d \in \mathbf{Lie}(H^o)$. We have $e(\alpha d)(a) = \varepsilon(a) + \alpha d(a)\delta = u_\alpha(\varepsilon(a) + d(a)\delta) = (u_\alpha \circ (1 + d\delta))(a) = (u_\alpha \circ e(d))(a) = (\alpha e(d))(a)$ hence $e(\alpha d) = \alpha e(d)$.

Furthermore let $g \in G(H^o) = \ \text{-}\mathbf{cAlg}(H, \ )$, $a \in H$, and $d \in \mathbf{Lie}(H^o)$. Then we have $e(g \cdot d)(a) = e(gdg^{-1})(a) = (1 + gdg^{-1}\delta)(a) = \varepsilon(a) + gdg^{-1}(a)\delta = \sum g(a_{(1)})\varepsilon(a_{(2)})gS(a_{(3)}) + \sum g(a_{(1)})d(a_{(2)})gS(a_{(3)})\delta = \sum g(a_{(1)})e(d)(a_{(2)})gS(a_{(3)}) = (j \circ g * e(d) * j \circ g^{-1})(a) = (g \cdot e(d))(a)$ hence $e(g \cdot d) = g \cdot e(d)$. $\square$

Proposition 4.4.4. *Let H be a Hopf algebra and let $I := \mathrm{Ker}(\varepsilon)$. Then $\mathrm{Der}_\varepsilon(H, \text{-}) : \mathbf{Vec} \longrightarrow \mathbf{Vec}$ is representable by I/I^2 and $d : H \xrightarrow{1-\varepsilon} I \xrightarrow{\nu} I/I^2$, in particular*

$$\mathrm{Der}_\varepsilon(H, \text{-}) \cong \mathrm{Hom}(I/I^2, \text{-}) \qquad \textit{and} \qquad \mathbf{Lie}(H^o) \cong \mathrm{Hom}(I/I^2, \ ).$$

Proof. Because of $\varepsilon(\mathrm{id} - u\varepsilon)(a) = \varepsilon(a) - \varepsilon u \varepsilon(a) = 0$ we have $\mathrm{Im}(\mathrm{id} - \varepsilon) \subseteq I$. Let $i \in I$. Then we have $i = i - \varepsilon(i) = (\mathrm{id} - \varepsilon)(i)$ hence $\mathrm{Im}(\mathrm{id} - \varepsilon) = \mathrm{Ker}(\varepsilon)$. We have $I^2 \ni (\mathrm{id} - \varepsilon)(a)(\mathrm{id} - \varepsilon)(b) = ab - \varepsilon(a)b - a\varepsilon(b) + \varepsilon(a)\varepsilon(b) = (\mathrm{id} - \varepsilon)(ab) - \varepsilon(a)(\mathrm{id} - \varepsilon)(b) - (\mathrm{id} - \varepsilon)(b)$. Hence we have in I/I^2 the equation $(\mathrm{id} - \varepsilon)(ab) = \varepsilon(a)(\mathrm{id} - \varepsilon)(b) + (\mathrm{id} - \varepsilon)(a)\varepsilon(b)$ so that $\nu(\mathrm{id} - \varepsilon) : H \longrightarrow I \longrightarrow I/I^2$ is an ε-derivation.

Now let $D : H \longrightarrow M$ be an ε-derivation. Then $D(1) = D(1\,1) = 1D(1) + D(1)1$ hence $D(1) = 0$. It follows $D(a) = D(\mathrm{id} - \varepsilon)(a)$. From $\varepsilon(I) = 0$ we get $D(I^2) \subseteq \varepsilon(I)D(I) + D(I)\varepsilon(I) = 0$ hence there is a unique factorization

$$\begin{tikzcd}
H \arrow[r, "\mathrm{id} - \varepsilon"] \arrow[drr, "D"'] & I \arrow[r, "\nu"] \arrow[dr, "D"'] & I/I^2 \arrow[d, "f"] \\
& & M.
\end{tikzcd}$$

$\square$

Corollary 4.4.5. *Let H be a Hopf algebra that is finitely generated a s an algebra. Then $\mathbf{Lie}(H^o)$ is finite dimensional.*

Proof. Let $H = \ \langle a_1, \ldots, a_n \rangle$. Since $H = \ \oplus I$ we can choose $a_1 = 1$ and $a_2, \ldots, a_n \in I$. Thus any element in $i \in I$ can be written as $\sum \alpha_J a_{j_1} \ldots a_{j_k}$ so that $I/I^2 = \ \overline{a_2} + \ldots + \overline{a_n}$. This gives the result. $\square$

Proposition 4.4.6. *Let H be a commutative Hopf algebra and ${}_HM$ be an H-module. Then we have $\Omega_H \cong H \otimes I/I^2$ and $d : H \longrightarrow H \otimes I/I^2$ is given by $d(a) = \sum a_{(1)} \otimes \overline{(\mathrm{id} - \varepsilon)(a_{(2)})}$.*

Proof. Consider the algebra $B := H \oplus M$ with $(a, m)(a', m') = (aa', am' + a'm)$. Let $\mathcal{G} = \ \text{-}\mathbf{cAlg}(H, \text{-})$. Then we have $\mathcal{G}(B) \subseteq \mathrm{Hom}(H, B) \cong \mathrm{Hom}(H, H) \oplus \mathrm{Hom}(H, M)$. An element $(\varphi, D) \in \mathrm{Hom}(H, B)$ is in $\mathcal{G}(B)$ iff $(\varphi, D)(1) = (\varphi(1), D(1)) = (1, 0)$, hence $\varphi(1) = 1$ and $D(1) = 0$, and $(\varphi(ab), D(ab)) = (\varphi, D)(ab) = (\varphi, D)(a)(\varphi, D)(b) = (\varphi(a), D(a))(\varphi(b), D(b)) = (\varphi(a)\varphi(b), \varphi(a)D(b) + D(a)\varphi(b)$, hence $\varphi(ab) = \varphi(a)\varphi(b)$ and $D(ab) = \varphi(a)D(b) + D(a)\varphi(b)$. So (φ, D) is in $\mathcal{G}(B)$ iff $\varphi \in \mathcal{G}(H)$ and D is a φ-derivation. The $*$-multiplication in $\mathrm{Hom}(H, B)$ is given by $(\varphi, D) * (\varphi', D') = (\varphi * \varphi', \varphi * D' + D * \varphi')$ by applying this to an element $a \in H$. Since $(\varphi, 0) \in \mathcal{G}(B)$ and $(u\varepsilon, D) \in \mathcal{G}(B)$ for every ε-derivation D, there is a bijection $\mathrm{Der}_\varepsilon(H, M) \cong \{(u\varepsilon, D_\varepsilon) \in \mathcal{G}_\varepsilon(B)\} \cong \{(1_H, D_1) \in \mathcal{G}_1(B)\} \cong \mathrm{Der}\ (H, M)$ by $(u\varepsilon, D_\varepsilon) \mapsto (1, 0) * (u\varepsilon, D_\varepsilon) = (1, 1 * D_\varepsilon) \in \mathcal{G}_1(B)$ with inverse map $(1, D_1) \mapsto (S, 0) * (1, D_1) = (u\varepsilon, S * D_1) \in \mathcal{G}_\varepsilon(B)$. Hence we have isomorphisms $\mathrm{Der}\ (H, M) \cong \mathrm{Der}_\varepsilon(H, M) \cong \mathrm{Hom}(I/I^2, M) \cong \mathrm{Hom}_H(H \otimes I/I^2, M)$.

The universal ε-derivation for vector spaces is $\overline{\mathrm{id} - \varepsilon} : A \longrightarrow I/I^2$. The universal ε-derivation for H-modules is $D_\varepsilon(a) = 1 \otimes \overline{(\mathrm{id} - \varepsilon)(a)} \in A \otimes I/I^2$. The universal 1-derivation for H-modules is $1 * D_\varepsilon$ with $(1 * D_\varepsilon)(a) = \sum a_{(1)} \otimes \overline{(\mathrm{id} - \varepsilon)(a_{(2)})} \in A \otimes I/I^2$. $\square$