CHAPTER 4

The Infinitesimal Theory

2. Derivations

Definition 4.2.1. Let A be a -algebra and ${}_AM_A$ be an A-A-bimodule (with identical -action on both sides). A linear map $D : A \longrightarrow M$ is called a *derivation* if

$$D(ab) = aD(b) + D(a)b.$$

The set of derivations Der $(A, {}_AM_A)$ is a -module and a functor in ${}_AM_A$.

By induction one sees that D satisfies

$$D(a_1 \ldots a_n) = \sum_{i=1}^{n} a_1 \ldots a_{i-1} D(a_i) a_{i+1} \ldots a_n.$$

Let A be a commutative -algebra and ${}_AM$ be an A-module. Consider M as an A-A-bimodule by $ma := am$. We denote the set of derivations from A to M by Der $(A, M)_c$.

Proposition 4.2.2. 1. *Let A be a -algebra. Then the functor* Der $(A, \text{-})$: A-**Mod**-$A \longrightarrow$ **Vec** *is representable by the* module of differentials Ω_A.

2. *Let A be a commutative -algebra. Then the functor* Der $(A, \text{-})_c$: A-**Mod** $\longrightarrow$ **Vec** *is representable by the* module of commutative differentials Ω_A^c.

Proof. 1. Represent A as a quotient of a free -algebra $A :=\ \langle X_i | i \in J\rangle / I$ where $B =\ \langle X_i | i \in J\rangle$ is the free algebra with generators X_i. We first prove the theorem for free algebras.

a) A representing module for Der $(B, \text{-})$ is $(\Omega_B, d : B \longrightarrow \Omega_B)$ with

$$\Omega_B := B \otimes F(dX_i | i \in J) \otimes B$$

where $F(dX_i | i \in J)$ is the free -module on the set of formal symbols $\{dX_i | i \in J\}$ as a basis.

We have to show that for every derivation $D : B \longrightarrow M$ there exists a unique homomorphisms $\varphi : \Omega_B \longrightarrow M$ of B-B-bimodules such that the diagram

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "D"'] & \Omega_B \arrow[d, "\varphi"] \\ & M \end{tikzcd}$$

commutes. The module Ω_B is a B-B-bimodule in the canonical way. The products $X_1 \ldots X_n$ of the generators X_i of B form a basis for B. For any product $X_1 \ldots X_n$ we define $d(X_1 \ldots X_n) := \sum_{i=1}^n X_1 \ldots X_{i-1} \otimes dX_i \otimes X_{i+1} \ldots X_n$ in particular $d(X_i) = 1 \otimes dX_i \otimes 1$. To see that d is a derivation it suffices to show this on the basis elements:

$$\begin{aligned} & d(X_1 \ldots X_k X_{k+1} \ldots X_n) \\ &\quad = \textstyle\sum_{j=1}^k X_1 \ldots X_{j-1} \otimes dX_j \otimes X_{j+1} \ldots X_k X_{k+1} \ldots X_n \\ &\quad\ + \textstyle\sum_{j=k+1}^n X_1 \ldots X_k X_{k+1} \ldots X_{j-1} \otimes dX_j \otimes X_{j+1} \ldots X_n \\ &\quad = d(X_1 \ldots X_k) X_{k+1} \ldots X_n + X_1 \ldots X_k d(X_{k+1} \ldots X_n) \end{aligned}$$

Now let $D : B \longrightarrow M$ be a derivation. Define φ by $\varphi(1 \otimes dX_i \otimes 1) := D(X_i)$. This map obviously extends to a homomorphism of B-B-bimodules. Furthermore we have

$$\begin{aligned}\varphi d(X_1 \dots X_n) &= \varphi(\textstyle\sum_j X_1 \dots X_{j-1} \otimes dX_j \otimes X_{j+1} \dots X_n) \\ &= \textstyle\sum_j X_1 \dots X_{j-1} \varphi(1 \otimes dX_j \otimes 1) X_{j+1} \dots X_n = D(X_1 \dots X_n)\end{aligned}$$

hence $\varphi d = D$.

To show the uniqueness of φ let $\psi : \Omega_B \longrightarrow M$ be a bimodule homomorphism such that $\psi d = D$. Then $\psi(1 \otimes dX_i \otimes 1) = \psi d(X_i) = D(X_i) = \varphi(1 \otimes dX_i \otimes 1)$. Since ψ and φ are B-B-bimodules homomorphisms this extends to $\psi = \varphi$.

b) Now let $A := \langle X_i | i \in J\rangle / I$ be an arbitrary algebra with $B = \langle X_i | i \in J\rangle$ free. Define

$$\Omega_A := \Omega_B / (I\Omega_B + \Omega_B I + Bd_B(I) + d_B(I)B).$$

We first show that $I\Omega_B + \Omega_B I + Bd_B(I) + d_B(I)B$ is a B-B-subbimodule. Since Ω_B and I are B-B-bimodules the terms $I\Omega_B$ and $\Omega_B I$ are bimodules. Furthermore we have $bd_B(i)b' = bd_B(ib') - bid_B(b') \in Bd_B(I) + I\Omega_B$ hence $I\Omega_B + \Omega_B I + Bd_B(I) + d_B(I)B$ is a bimodule.

Now $I\Omega_B$ and $\Omega_B I$ are subbimodules of $I\Omega_B + \Omega_B I + Bd_B(I) + d_B(I)B$. Hence $A = B/I$ acts on both sides on Ω_A so that Ω_A becomes an A-A-bimodule.

Let $\nu : \Omega_B \longrightarrow \Omega_A$ and also $\nu : B \longrightarrow A$ be the residue homomorphisms. Since $\nu d_B(i) \in \nu d_B(I) = 0 \subseteq \Omega_A$ we get a unique factorization map $d_A : A \longrightarrow \Omega_A$ such that

$$\begin{tikzcd} B \arrow[r, "d_B"] \arrow[d, "\nu"'] & \Omega_B \arrow[d, "\nu"] \\ A \arrow[r, "d_A"'] & \Omega_A \end{tikzcd}$$

commutes. Since $d_A(\overline{b}) = \overline{d_B(b)}$ it is clear that d_A is a derivation.

Let $D : A \longrightarrow M$ be a derivation. The A-A-bimodule M is also a B-B-bimodule by $bm = \overline{b}m$. Furthermore $D\nu : B \longrightarrow A \longrightarrow M$ is again a derivation. Let $\varphi_B : \Omega_B \longrightarrow M$ be the unique factorization map for the B-derivation $D\nu$. Consider the following diagram

$$\begin{tikzcd} B \arrow[r, "d_B"] \arrow[d] & \Omega_B \arrow[d] \arrow[dd, bend left, "\varphi"] \\ A \arrow[r, "d_A"] \arrow[dr, "D"'] & \Omega_A \arrow[d, "\psi"'] \\ & M \end{tikzcd}$$

We want to construct ψ such that the diagram commutes. Let $i\omega \in I\Omega_B$. Then $\varphi(i\omega) = \overline{i}\varphi(\omega) = 0$ and similarly $\varphi(\omega i) = 0$. Let $bd_B(i) \in Bd_B(I)$ then $\varphi(bd_B(i)) = \overline{b}\varphi d_B(i) = \overline{b}D(\overline{i}) = 0$ and similarly $\varphi(d_B(i)b) = 0$. Hence φ vanishes on $I\Omega_B + \Omega_B I +$

$Bd_B(I) + d_B(I)B$ and thus factorizes through a unique map $\psi : \Omega_A \longrightarrow M$. Obviously ψ is a homomorphism of A-A-bimodules. Furthermore we have $D\nu = \varphi d_B = \psi\nu d_B = \psi d_A \nu$ and, since ν is surjective, $D = \psi d_A$. It is clear that ψ is uniquely determined by this condition.

2. If A is commutative then we can write $A = \ [X_i | i \in J]/I$ and $\Omega^c_B = B \otimes F(dX_i)$. With $\Omega^c_A = \Omega^c_B/(I\Omega^c_B + Bd_B(I))$ the proof is analogous to the proof in the noncommutative situation. $\square$

Remark 4.2.3. 1. Ω_A is generated by $d(A)$ as a bimodule, hence all elements are of the form $\sum_i a_i d(a'_i)a''_i$. These elements are called *differentials*.

2. If $A = \ \langle X_i \rangle / I$, then Ω_A is generated as a bimodule by the elements $\{\overline{d(X_i)}\}$.

3. Let $f \in B = \ \langle X_i \rangle$. Let B^{op} be the algebra opposite to B (with opposite multiplication). Then $\Omega_B = B \otimes F(dX_i) \otimes B$ is the free $B \otimes B^{op}$ left module over the free generating set $\{d(X_i)\}$. Hence $d(f)$ has a unique representation

$$d(f) = \sum_i \frac{\partial f}{\partial X_i} d(X_i)$$

with uniquely defined coefficients

$$\frac{\partial f}{\partial X_i} \in B \otimes B^{op}.$$

In the commutative situation we have unique coefficients

$$\frac{\partial f}{\partial X_i} \in \ [X_i].$$

4. We give the following examples for part 3:

$$\frac{\partial X_i}{\partial X_j} = \delta_{ij},$$
$$\frac{\partial X_1 X_2}{\partial X_1} = 1 \otimes X_2,$$
$$\frac{\partial X_1 X_2}{\partial X_2} = X_1 \otimes 1,$$
$$\frac{\partial X_1 X_2 X_3}{\partial X_2} = X_1 \otimes X_3,$$
$$\frac{\partial X_1 X_3 X_2}{\partial X_2} = X_1 X_3 \otimes 1.$$

This is obtained by direct calculation or by the *product rule*

$$\frac{\partial fg}{\partial X_i} = (1 \otimes g)\frac{\partial f}{\partial X_i} + (f \otimes 1)\frac{\partial g}{\partial X_i}.$$

The product rule follows from

$$d(fg) = d(f)g + fd(g) = \sum((1 \otimes g)\frac{\partial f}{\partial X_i} + (f \otimes 1)\frac{\partial g}{\partial X_i})d(X_i).$$

Let $A = \quad \langle X_i \rangle / I$. If $f \in I$ then $\overline{d(f)} = d_A(\overline{f}) = 0$ hence

$$\sum \frac{\partial f}{\partial X_i} d_A(\overline{X_i}) = 0.$$

These are the defining relations for the A-A-bimodule Ω_A with the generators $d_A(\overline{X_i})$.

For motivation of the quantum group case we consider an affine algebraic group G with representing commutative Hopf algebra A. Recall that $\mathrm{Hom}(A, R)$ is an algebra with the convolution multiplication for every $R \in \quad$-**cAlg** and that $G(R) = \quad$-**cAlg**$(A, R) \subseteq \mathrm{Hom}(A, R)$ is a subgroup of the group of units of the algebra $\mathrm{Hom}(A, R)$.

Definition and Remark 4.2.4. A linear map $T : A \longrightarrow A$ is called *left translation invariant*, if the following diagram functorial in $R \in \quad$-**cAlg** commutes:

$$\begin{tikzcd}
G(R) \times \mathrm{Hom}(A, R) \arrow[r, "*"] \arrow[d, "1 \otimes \mathrm{Hom}(T,R)"'] & \mathrm{Hom}(A, R) \arrow[d, "\mathrm{Hom}(T,R)"] \\
G(R) \times \mathrm{Hom}(A, R) \arrow[r, "*"'] & \mathrm{Hom}(A, R)
\end{tikzcd}$$

i. e. if we have

$$\forall g \in G(R), \forall x \in \mathrm{Hom}(A, R) : g * (x \circ T) = (g * x) \circ T.$$

This condition is equivalent to

$$\Delta_A \circ T = (1_A \otimes T) \circ \Delta_A. \tag{1}$$

In fact if (1) holds then $g * (x \circ T) = \nabla_R(g \otimes x)(1_A \otimes T)\Delta_A = \nabla_R(g \otimes x)\Delta_A T = (g * x) \circ T$.

Conversely if the diagram commutes, then take $R = A$, $g = 1_A$ and we get $\nabla_A(1_A \otimes x)(1_A \otimes T)\Delta_A = 1_A * (x \circ T) = (1_A * x) \circ T = \nabla_A(1_A \otimes x)\Delta_A T$ for all $x \in \mathrm{Hom}(A, A)$. To get (1) it suffices to show that the terms $\nabla_A(1_A \otimes x)$ can be cancelled in this equation. Let $\sum_{i=1}^n a_i \otimes b_i \in A \otimes A$ be given such that $\nabla_A(1_A \otimes x)(\sum a_i \otimes b_i) = 0$ for all $x \in \mathrm{Hom}(A, A)$ and choose such an element with a shortest representation (n minimal). Then $\sum a_i x(b_i) = 0$ for all x. Since the b_i are linearly independent in such a shortest representation, there are x_i with $x_j(b_i) = \delta_{ij}$. Hence $a_j = \sum a_i x_j(b_i) = 0$ and thus $\sum a_i \otimes b_i = 0$. From this follows (1).

Definition 4.2.5. Let H be an arbitrary Hopf algebra. An element $T \in \mathrm{Hom}(H, H)$ is called *left translation invariant* if it satisfies

$$\Delta_H T = (1_H \otimes T)\Delta_H.$$

Proposition 4.2.6. *Let H be an arbitrary Hopf algebra. Then $\Phi : H^* \longrightarrow \mathrm{End}(H)$ with $\Phi(f) := \mathrm{id} * u_H f$ is an algebra monomorphism satisfying*

$$\Phi(f * g) = \Phi(f) \circ \Phi(g).$$

The image of Φ is precisely the set of left translation invariant elements $T \in \mathrm{End}(H)$.

Proof. For $f \in \mathrm{Hom}(H, \ )$ we have $u_H f \in \mathrm{End}(H)$ hence $\mathrm{id} * u_H f \in \mathrm{End}(H)$. Thus Φ is a well defined homomorphism. Observe that

$$\Phi(f)(a) = (\mathrm{id}_H * u_H f)(a) = \sum a_{(1)} f(a_{(2)}).$$

Φ is injective since it has a retraction $\mathrm{End}(H) \ni g \mapsto \varepsilon_H \circ g \in \mathrm{Hom}(H, \ )$. In fact we have $(\varepsilon\Phi(f))(a) = \varepsilon(\sum a_{(1)} f(a_{(2)})) = \sum \varepsilon(a_{(1)}) f(a_{(2)}) = f(\sum \varepsilon(a_{(1)}) a_{(2)}) = f(a)$ hence $\varepsilon\Phi(f) = f$.

The map Φ preserves the algebra unit since $\Phi(1_{H^*}) = \Phi(\varepsilon_H) = \mathrm{id}_H * u_H \varepsilon_H = \mathrm{id}_H$.

The map Φ is compatible with the multiplication: $\Phi(f * g)(a) = \sum a_{(1)}(f * g)(a_{(2)}) = \sum a_{(1)} f(a_{(2)}) g(a_{(3)}) = \sum (\mathrm{id} * u_H f)(a_{(1)}) g(a_{(2)}) = \Phi(f)(\sum a_{(1)} g(a_{(2)})) = \Phi(f)\Phi(g)(a)$ so that $\Phi(f * g) = \Phi(f) \circ \Phi(g)$.

For each $f \in H^*$ the element $\Phi(f)$ is left translation invariant since $\Delta\Phi(f)(a) = \Delta(\sum a_{(1)} f(a_{(2)})) = \sum a_{(1)} \otimes a_{(2)} f(a_{(3)}) = (1 \otimes \Phi(f))\Delta(a)$.

Let $T \in \mathrm{End}(H)$ be left translation invariant then $S * T = \nabla_H (S \otimes 1)(1 \otimes T)\Delta_H = \nabla_H (S \otimes 1)\Delta_H T = u_H \varepsilon_H T$. Thus $\Phi(\varepsilon T) = \mathrm{id} * u_H \varepsilon_H T = \mathrm{id} * S * T = T$, so that T is in the image of Φ. $\square$

Proposition 4.2.7. *Let $d \in \mathrm{Hom}(H, \ )$ and $\Phi(d) = D \in \mathrm{Hom}(H, H)$ be given. The following are equivalent:*

1. *$d : H \longrightarrow {}_\varepsilon \ {}_\varepsilon$ is a derivation.*
2. *$D : H \longrightarrow {}_H H_H$ is a (left translation invariant) derivation.*

In particular Φ induces an isomorphism between the set of derivations $d : H \longrightarrow {}_\varepsilon \ {}_\varepsilon$ and the set of left translation invariant derivations $D : H \longrightarrow {}_H H_H$.

Proof. Assume that 1. holds so that d satisfies $d(ab) = \varepsilon(a)d(b) + d(a)\varepsilon(b)$. Then we get $D(ab) = \Phi(d)(ab) = \sum a_{(1)} b_{(1)} d(a_{(2)} b_{(2)}) = \sum a_{(1)} b_{(1)} \varepsilon(a_{(2)}) d(b_{(2)}) + \sum a_{(1)} b_{(1)} d(a_{(2)}) \varepsilon(b_{(2)}) = aD(b) + D(a)b$. Conversely assume that $D(ab) = aD(b) + D(a)b$. Then $d(ab) = \varepsilon D(ab) = \varepsilon(a)\varepsilon D(b) + \varepsilon D(a)\varepsilon(b) = \varepsilon(a)d(b) + d(a)\varepsilon(b)$. $\square$