CHAPTER 4

# The Infinitesimal Theory

## 1. Integrals and Fourier Transforms

Assume for this chapter that   is a field.

**Lemma 4.1.1.** *Let $C$ be a finite dimensional coalgebra. Every right $C$-comodule $M$ is a left $C^*$-module by $c^*m = \sum m_{(M)}\langle c^*, m_{(1)}\rangle$ and conversely by $\delta(m) = \sum_i c_i^* m \otimes c_i$ where $\sum c_i^* \otimes c_i$ is the dual basis.*

Proof. We check that $M$ becomes a left $C^*$-module

$$\begin{aligned}(c^*c'^*)m &= \sum m_{(M)}\langle c^*c'^*, m_{(1)}\rangle = \sum m_{(M)}\langle c^*, m_{(1)}\rangle\langle c'^*, m_{(2)}\rangle \\ &= c^* \sum m_{(M)}\langle c'^*, m_{(1)}\rangle = c^*(c'^*m).\end{aligned}$$

It is easy to check that the two constructions are inverses of each other. In particular assume that $M$ is a right $C$-comodule. Choose $m_i$ such that $\delta(m) = \sum m_i \otimes c_i$. Then $c_j^*m = \sum m_i\langle c_j^*, c_i\rangle = m_j$ and $\sum c_i^*m \otimes c_i = \sum m_i \otimes c_i = \delta(m)$. $\square$

**Definition 4.1.2.** 1. Let $A$ be an algebra with augmentation $\varepsilon : A \longrightarrow$  , an algebra homomorphism. Let $M$ be a left $A$-module. Then ${}^AM = \{m \in M | am = \varepsilon(a)m\}$ is called the *space of left invariants* of $M$.

This defines a functor ${}^A$- : $A - \mathbf{Mod} \longrightarrow \mathbf{Vec}$.

2. Let $C$ be a coalgebra with a group-like element $1 \in C$. Let $M$ be a right $C$-comodule. Then $M^{coC} := \{m \in M | \delta(m) = m \otimes 1\}$ is called the *space of right coinvariants* of $M$.

This defines a functor -${}^{coC}$ : $\mathbf{Comod}$-$C \longrightarrow \mathbf{Vec}$.

**Lemma 4.1.3.** *Let $C$ be a finite dimensional coalgebra with a group like element $1 \in C$. Then $A := C^*$ is an augmented algebra with augmentation $\varepsilon : C^* \ni a \mapsto \langle a, 1\rangle \in$  . Let $M$ be a right $C$-comodule. Then $M$ is a left $C^*$-module and we have*

$${}^{C^*}M = M^{coC}.$$

Proof. Since $1 \in C$ is group-like we have $\varepsilon_A(ab) = \langle ab, 1\rangle = \langle a, 1\rangle\langle b, 1\rangle = \varepsilon_A(a)\varepsilon_A(b)$ and $\varepsilon_A(1_A) = \langle 1_A, 1_C\rangle = \varepsilon_C(1_C) = 1$.

We have $m \in M^{coH}$ iff $\delta(m) = \sum m_{(M)} \otimes m_{(1)} = m \otimes 1$ iff $\sum m_{(M)}\langle a, m_{(1)}\rangle = m\langle a, 1\rangle$ for all $a \in A = C^*$ and by identifying $C^* \otimes C = \mathrm{Hom}(C^*, C^*)$ iff $am = \varepsilon_A(a)m$ iff $m \in {}^AM$. $\square$

**Remark 4.1.4.** The theory of Fourier transforms contains the following statements. Let $H$ be the (Schwartz) space of infinitely differentiable functions $f :\ \ \longrightarrow\ \ $, such that $f$ and all derivatives rapidly decrease at infinity. ($f$ decreases rapidly at infinity if $|x|^m f(x)$ is bounded for all $m$.) This space is an algebra (without unit) under the multiplication of values. There is a second multiplication on $H$, the convolution

$$(f * g)(x) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} f(t)g(x-t)dt.$$

The Fourier transform is a homomorphism $\hat{}: H \longrightarrow H$ defined by

$$\hat{f}(x) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} f(t)e^{-itx}dt.$$

It satisfies the identity $(f * g)\hat{} = \hat{f}\hat{g}$ hence it is an algebra homomorphism. We want to find an analogue of this theory for finite quantum groups.

A similar example is the following. Let $G$ be a locally compact topological group. Let $\mu$ be the (left) *Haar measure* on $G$ and $\int f := \int_G f(x)d\mu(x)$ be the *Haar integral.*

The Haar measure is left invariant in the sense that $\mu(E) = \mu(gE)$ for all $g \in G$ and all compact subsets $E$ of $G$. The Haar measure exists and is unique up to a positive factor. The Haar integral is translation invariant i.e. for all $y \in G$ we have $\int f(yx)d\mu(x) = \int f(x)d\mu(x)$.

If $\mu$ is a left-invariant Haar measure then there is a continuous homomorphism $\mathrm{mod} : G \longrightarrow (\quad^+, \cdot)$ such that $\int f(xy^{-1})d\mu(x) = \mathrm{mod}\,(y) \int (f(x)d\mu(x)$. The homomorphism $\mu$ does not depend on $f$ and is called the *modulus* of $G$. The group $G$ is called *unimodular* if the homomorphism $\mathrm{mod}$ is the identity.

If $G$ is a compact, or discrete, or Abelian group, or a connected semisimple or nilpotent Lie group, then $G$ is unimodular.

Let $G$ be a quantum group (or a quantum monoid) with function algebra $H$ an arbitrary Hopf algebra. We also use the algebra of linear functionals $H^* = \mathrm{Hom}(H, \quad)$ (called the bialgebra of $G$ in the French literature). The operation $H^* \otimes H \ni a \otimes f \mapsto \langle a, f\rangle \in \quad$ is nondegenerate on both sides. We denote the elements of $H$ by $f, g, h \in H$, the elements of $H^*$ by $a, b, c \in H^*$, the (non existing) elements of the quantum group $G$ by $x, y, z \in G$.

**Remark 4.1.5.** In 2.4.8 we have seen that the dual vector space $H^*$ of a finite dimensional Hopf algebra $H$ is again a Hopf algebra. The Hopf algebra structures are connected by the evaluation bilinear form

$$\mathrm{ev} : H^* \otimes H \ni a \otimes f \mapsto \langle a, f\rangle \in$$

as follows:

$$\begin{array}{cc} \langle a \otimes b, \sum f_{(1)} \otimes f_{(2)}\rangle = \langle ab, f\rangle, & \langle \sum a_{(1)} \otimes a_{(2)}, f \otimes g\rangle = \langle a, fg\rangle, \\ \langle a, 1\rangle = \varepsilon(a), & \langle 1, f\rangle = \varepsilon(f), \\ \multicolumn{2}{c}{\langle a, S(f)\rangle = \langle S(a), f\rangle.} \end{array}$$

**Definition 4.1.6.** 1. The linear functionals $a \in H^*$ are called *generalized integrals on $H$* ([Riesz-Nagy] S.123).

2. An element $\int \in H^*$ is called a *left (invariant) integral on* $H$ if

$$a\int = \langle a, 1_H \rangle \int$$

or $a\int = \varepsilon_{H^*}(a)\int$ for all $a \in H^*$.

3. An element $\delta \in H$ is called a *left integral in* $H$ if

$$f\delta = \varepsilon(f)\delta$$

for all $f \in H$.

4. The set of left integrals in $H$ is denoted by $\mathrm{Int}_l(H)$, the set of right integrals by $\mathrm{Int}_r(H)$. The set of left (right) integrals on $H$ is $\mathrm{Int}_l(H^*)$ ($\mathrm{Int}_r(H^*)$).

5. A Hopf algebra $H$ is called *unimodular* if $\mathrm{Int}_l(H) = \mathrm{Int}_r(H)$.

**Lemma 4.1.7.** *The left integrals* $\mathrm{Int}_l(H^*)$ *form a two sided ideal of* $H^*$. *If the antipode* $S$ *is bijective then* $S$ *induces an isomorphism* $S : \mathrm{Int}_l(H^*) \longrightarrow \mathrm{Int}_r(H^*)$.

PROOF. For $\int$ in $\mathrm{Int}_l(H^*)$ we have $a\int = \varepsilon(a)\int \in \mathrm{Int}_l(H^*)$ and $a\int b = \varepsilon(a)\int b$ hence $\int b \in \mathrm{Int}_l(H^*)$. If $S$ is bijective then the induced map $S : H^* \longrightarrow H^*$ is also bijective and satisfies $S(\int)b = S(\int)S(S^{-1}(b)) = S(S^{-1}(b)\int) = S(\int)\varepsilon(b)$ hence $S(\int) \in \mathrm{Int}_r(H^*)$. $\square$

**Remark 4.1.8.** Maschke's Theorem has an extension to finite dimensional Hopf algebras: $\varepsilon(\int) \neq 0$ iff $H^*$ is semisimple.

**Corollary 4.1.9.** *Let* $H$ *be a finite dimensional Hopf algebra. Then* $H^*$ *is a left* $H^*$*-module by the usual multiplication, hence a right* $H$*-comodule. We have*

$$(H^*)^{coH} = \mathrm{Int}_l(H^*).$$

PROOF. By definition we have $\mathrm{Int}_l(H^*) = {}^{H^*}H^*$. $\square$

**Example 4.1.10.** Let $G$ be a finite group. Let $H := \mathrm{Map}(G, \ )$ be the Hopf algebra defined by the following isomorphism

$$^G = \mathrm{Map}(G, \ ) \cong \mathrm{Hom}(\ G, \ ) = (\ G)^*.$$

This isomorphism between the vector space $^G$ of all set maps from the group $G$ to the base ring and the dual vector space $(\ G)^*$ of the group algebra $G$ defines the structure of a Hopf algebra on $^G$.

We regard $H := {}^G$ as the function algebra on the set $G$. In the sense of algebraic geometry this is not quite true. The algebra $^G$ represents a functor from -**cAlg** to **Set** that has $G$ as value for all connected algebras $A$ in particular for all field extensions of .

As before we use the map $\mathrm{ev} : \ G \otimes {}^G \longrightarrow$ . The multiplication of $^G$ is given by pointwise multiplication of maps since $\langle x, ff' \rangle = \langle \sum x_{(1)} \otimes x_{(2)}, f \otimes f' \rangle = \langle x \otimes x, f \otimes f' \rangle = \langle x, f \rangle \langle x, f' \rangle$ for all $f, f' \in {}^G$ and all $x \in G$. The unit element $1_{\ G}$ of $^G$ is the map $\varepsilon : \ G \longrightarrow$ restricted to $G$, hence $\varepsilon(x) = 1 = \langle x, 1_{\ G} \rangle$ for all $x \in G$. The antipode of $f \in {}^G$ is given by $S(f)(x) = \langle x, S(f) \rangle = f(x^{-1})$.

The elements of the dual basis $(x^*|x \in G)$ with $\langle x, y^* \rangle = \delta_{x,y}$ considered as maps from $G$ to form a basis of $^G$. They satisfy the conditions

$$x^* y^* = \delta_{x,y} x^* \text{ and } \sum_{x \in G} x^* = 1_{\ G}$$

since $\langle z, x^* y^* \rangle = \langle z, x^* \rangle \langle z, y^* \rangle = \delta_{z,x} \delta_{z,y} = \delta_{x,y} \langle z, x^* \rangle$ and $\langle z, \sum_{x \in G} x^* \rangle = 1 = \langle z, 1_{\ G} \rangle$.

Hence the dual basis $(x^*|x \in G)$ is a decomposition of the unit into a set of minimal orthogonal idempotents and the algebra of $^G$ has the structure

$$^G = \oplus_{x \in G} \quad x^* \cong \quad \times \ldots \times \quad .$$

In particular $^G$ is commutative and semisimple.

The diagonal of $^G$ is

$$\Delta(x^*) = \sum_{y \in G} y^* \otimes (y^{-1}x)^* = \sum_{y,z \in G, yz = x} y^* \otimes z^*$$

since

$$\begin{aligned} \langle z \otimes u, \Delta(x^*) \rangle &= \langle zu, x^* \rangle = \delta_{x,zu} = \delta_{z^{-1}x,u} = \textstyle\sum_{y \in G} \delta_{y,z} \delta_{y^{-1}x,u} \\ &= \textstyle\sum_{y \in G} \langle z, y^* \rangle \langle u, (y^{-1}x)^* \rangle = \langle z \otimes u, \sum_{y \in G} y^* \otimes (y^{-1}x)^* \rangle. \end{aligned}$$

Let $a \in \quad G$. Then $a$ defines a map $\widetilde{a} : G \longrightarrow \quad \in \quad ^G$ by $a = \sum_{x \in G} \widetilde{a}(x) x$. For arbitrary $f \in \quad ^G$ and $a \in \quad G$ this gives

$$\langle a, f \rangle = f(\sum_{x \in G} \widetilde{a}(x) x) = \sum_{x \in G} \widetilde{a}(x) f(x).$$

The counit of $^G$ is given by $\varepsilon(x^*) = \delta_{x,e}$ where $e \in G$ is the unit element.

The antipode is, as above, $S(x^*) = (x^{-1})^*$.

We consider $H = \quad ^G$ as the function algebra on the finite group $G$ and $\quad G$ as the dual space of $H = \quad ^G$ hence as the set of distributions on $H$.

Then

$$\textstyle\int := \sum_{x \in G} x \in H^* = \quad G \tag{1}$$

is a (two sided) integral on $H$ since $\sum_{x \in G} yx = \sum_{x \in G} x = \varepsilon(y) \sum_{x \in G} x = \sum_{x \in G} yx$.

We write

$$\int f(x) dx := \langle \textstyle\int, f \rangle = \sum_{x \in G} f(x).$$

We have seen that there is a decomposition of the unit $1 \in \quad ^G$ into a set of primitive orthogonal idempotents $\{x^* | x \in G\}$ such that every element $f \in \quad ^G$ has a unique representation $f = \sum f(x) x^*$. Since $\int y^* = \sum_{x \in G} \langle x, y^* \rangle$ we get $\int f y^* = \sum_{x \in G} \langle x, f y^* \rangle = \sum f(x) y^*(x) = f(y)$ hence

$$f = \sum (\int f(x) y^*(x) dx) y^*.$$

**Problem 4.1.1.** Describe the group valued functor -**cAlg**( $^G, -)$ in terms of sets and their group structure.

**Definition and Remark 4.1.11.** Let be an algebraicly closed field and let $G$ be a finite abelian group (replacing above). Assume that the characteristic of does not divide the order of $G$. Let $H = \ ^G$. We identify $^G = \mathrm{Hom}(\ G, \ )$ along the linear expansion of maps as in Example 2.1.10.

Let us consider the set $\hat{G} := \{\chi : G \longrightarrow \ ^* | \chi$ group homomorphism$\}$. Since $^*$ is an abelian group, the set $\hat{G}$ is an abelian group by pointwise multiplication.

The group $\hat{G}$ is called the *character group* of $G$.

Obviously the character group is a multiplicative subset of $^G = \mathrm{Hom}(\ G, \ )$. Actually it is a subgroup of -**cAlg**( $G$, ) $\subseteq \mathrm{Hom}(\ G, \ )$ since the elements $\chi \in \hat{G}$ expand to algebra homomorphisms: $\chi(ab) = \chi(\sum \alpha_x x \sum \beta_y y) = \sum \alpha_x \beta_y \chi(xy) = \chi(a)\chi(b)$ and $\chi(1) = \chi(e) = 1$. Conversely an algebra homomorphism $f \in$ -**cAlg**( $G$, ) restricts to a character $f : G \longrightarrow \ ^*$. Thus $\hat{G} =$ -**cAlg**( $G$, ), the set of rational points of the affine algebraic group represented by $G$.

There is a more general observation behind this remark.

**Lemma 4.1.12.** *Let $H$ be a finite dimensional Hopf algebra. Then the set $Gr(H^*)$ of group like elements of $H^*$ is equal to* -**Alg**$(H, \ )$.

Proof. In fact $f : H \longrightarrow$ is an algebra homomorphism iff $\langle f \otimes f, a \otimes b\rangle = \langle f, a\rangle\langle f, b\rangle = \langle f, ab\rangle = \langle \Delta(f), a \otimes b\rangle$ and $1 = \langle f, 1\rangle = \varepsilon(f)$. $\square$

Hence there is a Hopf algebra homomorphism $\varphi : \ \hat{G} \longrightarrow \ ^G$ by 2.1.5.

**Proposition 4.1.13.** *The Hopf algebra homomorphism $\varphi : \ \hat{G} \longrightarrow \ ^G$ is bijective.*

Proof. We give the proof by several lemmas.

**Lemma 4.1.14.** *Any set of group like elements in a Hopf algebra $H$ is linearly independent.*

Proof. Assume there is a linearly dependent set $\{x_0, x_1, \ldots, x_n\}$ of group like elements in $H$. Choose such a set with $n$ minimal. Obviously $n \geq 1$ since all elements are non zero. Thus $x_0 = \sum_{i=1}^n \alpha_i x_i$ and $\{x_1, \ldots, x_n\}$ linearly independent. We get

$$\sum_{i,j} \alpha_i \alpha_j x_i \otimes x_j = x_0 \otimes x_0 = \Delta(x_0) = \sum_i \alpha_i x_i \otimes x_i.$$

Since all $\alpha_i \neq 0$ and the $x_i \otimes x_j$ are linearly independent we get $n = 1$ and $\alpha_1 = 1$ so that $x_0 = x_1$, a contradiction. $\square$

**Corollary 4.1.15.** *(Dedekind's Lemma) Any set of characters in $^G$ is linearly independent.*

Thus $\varphi : \quad \hat{G} \longrightarrow \quad {}^G$ is injective. Now we prove that the map $\varphi : \quad \hat{G} \longrightarrow \quad {}^G$ is surjective.

**Lemma 4.1.16.** *(Pontryagin duality) The evaluation $\hat{G} \times G \longrightarrow \quad {}^*$ is a non-degenerate bilinear map of abelian groups.*

Proof. First we observe that $\mathrm{Hom}(C_n, \quad {}^*) \cong C_n$ for a cyclic group of order $n$ since has a primitive $n$-th root of unity ($\mathrm{char}(\quad) \not| \; |G|$).

Since the direct product and the direct sum coincide in **Ab** we can use the fundamental theorem for finite abelian groups $G \cong C_{n_1} \times \ldots \times C_{n_t}$ to get $\mathrm{Hom}(G, \quad {}^*) \cong G$ for any abelian group $G$ with $\mathrm{char}(\quad) \not| \; |G|$. Thus $\hat{G} \cong G$ and $\hat{\hat{G}} = G$. In particular $\chi(x) = 1$ for all $x \in G$ iff $\chi = 1$. By the symmetry of the situation we get that the bilinear form $\langle ., . \rangle : \hat{G} \times G \longrightarrow \quad {}^*$ is non-degenerate. $\square$

Thus $|\hat{G}| = |G|$ hence $\dim(\quad \hat{G}) = \dim(\quad {}^G)$. This proves Proposition 2.1.13. $\square$

**Definition 4.1.17.** Let $H$ be a Hopf algebra. A -module $M$ that is a right $H$-module by $\rho : M \otimes H \longrightarrow M$ and a right $H$-comodule by $\delta : M \longrightarrow M \otimes H$ is called a *Hopf module* if the diagram

$$\begin{tikzcd}
M \otimes H \arrow[r, "\rho"] \arrow[d, "\delta\otimes\Delta"'] & H \arrow[r, "\delta"] & M \otimes H \\
M \otimes H \otimes H \otimes H \arrow[rr, "1\otimes\tau\otimes 1"] & & M \otimes H \otimes H \otimes H \arrow[u, "\rho\otimes\nabla"']
\end{tikzcd}$$

commutes, i.e. if $\delta(mh) = \sum m_{(M)} h_{(1)} \otimes m_{(1)} h_{(2)}$ holds for all $h \in H$ and all $m \in M$.

Observe that $H$ is an Hopf module over itself. Furthermore each module of the form $V \otimes H$ is a Hopf module by the induced structure. More generally there is a functor $\mathbf{Vec} \ni V \mapsto V \otimes H \in \mathbf{Hopf\text{-}Mod}\text{-}H$.

**Proposition 4.1.18.** *The two functors $\text{-}^{coH} : \mathbf{Hopf\text{-}Mod}\text{-}H \longrightarrow \mathbf{Vec}$ and $\text{-} \otimes H : \mathbf{Vec} \ni V \mapsto V \otimes H \in \mathbf{Hopf\text{-}Mod}\text{-}H$ are inverse equivalences of each other.*

Proof. Define natural isomorphisms

$$\alpha : M^{coH} \otimes H \ni m \otimes h \mapsto mh \in M$$

with inverse map

$$\alpha^{-1} : M \ni m \mapsto \sum m_{(M)} S(m_{(1)}) \otimes m_{(2)} \in M^{coH} \otimes H$$

and

$$\beta : V \ni v \mapsto v \otimes 1 \in (V \otimes H)^{coH}$$

with inverse map

$$(V \otimes H)^{coH} \ni v \otimes h \mapsto v\varepsilon(h) \in V.$$

Obviously these homomorphisms are natural transformations in $M$ and $V$. Furthermore $\alpha$ is a homomorphism of $H$-modules. $\alpha^{-1}$ is well-defined since

$$\begin{aligned}\delta(\textstyle\sum m_{(M)}S(m(1))) &= \textstyle\sum m_{(M)}S(m_{(3)}) \otimes m_{(1)}S(m_{(2)}) \\ &\quad \text{(since } M \text{ is a Hopf module)} \\ &= \textstyle\sum m_{(M)}S(m_{(2)}) \otimes \eta\varepsilon(m_{(1)}) \\ &= \textstyle\sum m_{(M)}S(m_{(1)}) \otimes 1\end{aligned}$$

hence $\sum m_{(M)}S(m_{(1)}) \in M^{coH}$. Furthermore $\alpha^{-1}$ is a homomorphism of comodules since

$$\begin{aligned}\delta\alpha^{-1}(m) &= \delta(\textstyle\sum m_{(M)}S(m_{(1)}) \otimes m_{(2)}) = \sum m_{(M)}S(m_{(1)}) \otimes m_{(2)} \otimes m_{(3)} \\ &= \textstyle\sum \alpha^{-1}(m_{(M)}) \otimes m_{(1)} = (\alpha^{-1} \otimes 1)\delta(m).\end{aligned}$$

Finally $\alpha$ and $\alpha^{-1}$ are inverse to each other by

$$\alpha\alpha^{-1}(m) = \alpha(\sum m_{(M)}S(m_{(1)}) \otimes m_{(2)}) = \sum m_{(M)}S(m_{(1)})m_{(2)} = m$$

and

$$\begin{aligned}\alpha^{-1}\alpha(m \otimes h) &= \alpha^{-1}(mh) = \textstyle\sum m_{(M)}h_{(1)}S(m_{(1)}h_{(2)}) \otimes m_{(2)}h_{(3)} \\ &= \textstyle\sum mh_{(1)}S(h_{(2)}) \otimes h_{(3)} \text{ ( by } \delta(m) = m \otimes 1 \text{ )} = m \otimes h.\end{aligned}$$

Thus $\alpha$ and $\alpha^{-1}$ are mutually inverse homomorphisms of Hopf modules.

The image of $\beta$ is in $(V \otimes H)^{coH}$ by $\delta(v \otimes 1) = v \otimes \Delta(1) = (v \otimes 1) \otimes 1$. Both $\beta$ and $\beta^{-1}$ are -linear maps. Furthermore we have

$$\beta^{-1}\beta(v) = \beta^{-1}(v \otimes 1) = v\varepsilon(1) = v$$

and

$$\begin{aligned}\beta\beta^{-1}(\textstyle\sum v_i \otimes h_i) &= \beta(\textstyle\sum v_i\varepsilon(h_i)) = \sum v_i\varepsilon(h_i) \otimes 1 = \sum v_i \otimes \varepsilon(h_i)1 \\ &= \textstyle\sum v_i \otimes \varepsilon(h_{i(1)})h_{i(2)} \text{ ( since } \sum v_i \otimes h_i \in (V \otimes H)^{coH} \text{ )} = \sum v_i \otimes h_i.\end{aligned}$$

Thus $\beta$ and $\beta^{-1}$ are mutually inverse homomorphisms. □

Since $H^* = \mathrm{Hom}(H, \ )$ and $S : H \longrightarrow H$ is an algebra antihomomorphism, the dual $H^*$ is an $H$-module in four different ways:

$$\begin{aligned}&\langle (f \rightharpoonup a), g\rangle := \langle a, gf\rangle, &&\langle (a \leftharpoonup f), g\rangle := \langle a, fg\rangle, \\ &\langle (f \rightharpoondown a), g\rangle := \langle a, S(f)g\rangle, &&\langle (a \leftharpoondown f), g\rangle := \langle a, gS(f)\rangle.\end{aligned} \tag{2}$$

If $H$ is finite dimensional then $H^*$ is a Hopf algebra. The equality $\langle (f \rightharpoonup a), g\rangle = \langle a, gf\rangle = \sum\langle a_{(1)}, g\rangle\langle a_{(2)}, f\rangle$ implies

$$(f \rightharpoonup a) = \sum a_{(1)}\langle a_{(2)}, f\rangle. \tag{3}$$

Analogously we have

$$(a \leftharpoonup f) = \sum\langle a_{(1)}, f\rangle a_{(2)}. \tag{4}$$

**Proposition 4.1.19.** *Let $H$ be a finite dimensional Hopf algebra. Then $H^*$ is a right Hopf module over $H$.*

PROOF. $H^*$ is a left $H^*$-module by left multiplication hence by 2.1.1 a right $H$-comodule by $\delta(a) = \sum_i b_i^* a \otimes b_i$. Let $f, g \in H$ and $a, b \in H^*$. The (left) multiplication of $H^*$ satisfies

$$ab = \sum b_{(H^*)} \langle a, b_{(1)} \rangle.$$

We use the right $H$-module structure

$$(a \leftharpoondown f) = \sum a_{(1)} \langle S(f), a_{(2)} \rangle.$$

on $H^* = \mathrm{Hom}(H, \ )$.

Now we check the Hopf module property. Let $a, b \in H^*$ and $f, g \in H$. We apply $H^* \otimes H$ to its dual $H \otimes H^*$ and get

$$\begin{aligned}
\delta(a \leftharpoondown f)(g \otimes b) &= \sum \langle (a \leftharpoondown f)_{(H^*)}, g \rangle \langle b, (a \leftharpoondown f)_{(1)} \rangle = \langle b(a \leftharpoondown f), g \rangle \\
&= \sum \langle b, g_{(1)} \rangle \langle (a \leftharpoondown f), g_{(2)} \rangle = \sum \langle b, g_{(1)} \rangle \langle a, g_{(2)} S(f) \rangle \\
&= \sum \langle b, g_{(1)} \varepsilon(f_{(2)}) \rangle \langle a, g_{(2)} S(f_{(1)}) \rangle = \sum \langle (f_{(3)} \rightharpoonup b), g_{(1)} S(f_{(2)}) \rangle \langle a, g_{(2)} S(f_{(1)}) \rangle \\
&= \sum \langle (f_{(2)} \rightharpoonup b) a, g S(f_{(1)}) \rangle = \sum \langle ((f_{(2)} \rightharpoonup b) a) \leftharpoondown f_{(1)}, g \rangle \\
&= \sum \langle (a_{(H^*)} \langle (f_{(2)} \rightharpoonup b), a_{(1)} \rangle) \leftharpoondown f_{(1)}, g \rangle \\
&= \sum \langle (a_{(H^*)} \leftharpoondown f_{(1)}) \langle (f_{(2)} \rightharpoonup b), a_{(1)} \rangle, g \rangle = \sum \langle (a_{(H^*)} \leftharpoondown f_{(1)}) \langle b, a_{(1)} f_{(2)} \rangle, g \rangle
\end{aligned}$$

hence $\delta(a \leftharpoondown f) = \sum (a_{(H^*)} \leftharpoondown f_{(1)}) \otimes a_{(1)} f_{(2)}$. $\square$

**Theorem 4.1.20.** *Let $H$ be a finite dimensional Hopf algebra. Then the antipode $S$ is bijective, the space of left integrals $\mathrm{Int}_l(H^*)$ has dimension 1, and the homomorphism*

$$H \ni f \mapsto (f \rightharpoonup \textstyle\int) = \sum \int_{(1)} \langle \int_{(2)}, f \rangle \ni H^*$$

*is bijective for any $0 \neq \int \in \mathrm{Int}_l(H^*)$.*

PROOF. By Proposition 2.1.19 $H^*$ is a right Hopf module over $H$. By Proposition 2.1.18 there is an isomorphism $\alpha : (H^*)^{coH} \otimes H \ni a \otimes f \mapsto (a \leftharpoondown f) = (S(f) \rightharpoonup a) \in H^*$. Since $(H^*)^{coH} \cong \mathrm{Int}_l(H^*)$ by 2.1.9 we get

$$\mathrm{Int}_l(H^*) \otimes H \cong H^*$$

as right $H$-Hopf modules by the given map. This shows $\dim(\mathrm{Int}_l(H^*)) = 1$. So we get an isomorphism $H \ni f \mapsto (\int \leftharpoondown f) \in H^*$ that is a composition of $S$ and $f \mapsto (f \rightharpoonup \int)$. Since $H$ is finite dimensional both of these maps are bijective. $\square$

If $G$ is a finite group then every generalized integral $a \in \ G$ can be written with a uniquely determined $g \in H = \ ^G$ as

$$\langle a, f \rangle = \int f(x) S(g)(x) dx = \sum_{x \in G} f(x) g(x^{-1}) \tag{5}$$

for all $f \in H$.

If $G$ is a finite Abelian group then each group element (rational integral) $y \in G \subseteq G$ can be written as

$$y = \sum_{x \in G} \sum_{\chi \in \hat{G}} \beta_\chi \langle x^{-1}, \chi \rangle x$$

since

$$\begin{aligned}\langle y, f \rangle &= \langle (\textstyle\int \leftharpoonup \sum_{\chi \in \hat{G}} \beta_\chi \chi), f \rangle = \langle \textstyle\int, f S(\sum_{\chi \in \hat{G}} \beta_\chi \chi) \rangle \\ &= \textstyle\sum_{x \in G} \langle x, f \rangle \sum_{\chi \in \hat{G}} \beta_\chi \langle x, S(\chi) \rangle = \langle \sum_{x \in G} \sum_{\chi \in \hat{G}} \beta_\chi \langle x^{-1}, \chi \rangle x, f \rangle.\end{aligned}$$

In particular the matrix $(\langle x^{-1}, \chi \rangle)$ is invertible.

Let $H$ be finite dimensional. Since $\langle \int, fg \rangle = \langle (\int \leftharpoonup f), g \rangle$ as a functional on $g$ is a generalized integral, there is a unique $\nu(f) \in H$ such that

$$\langle \textstyle\int, fg \rangle = \langle \textstyle\int, g\nu(f) \rangle \tag{6}$$

or

$$\int f(x) g(x) dx = \int g(x) \nu(f)(x) dx. \tag{7}$$

Although the functions $f, g \in H$ of the quantum group do not commute under multiplication, there is a simple commutation rule if the product is integrated.

**Proposition and Definition 4.1.21.** *The map $\nu : H \longrightarrow H$ is an algebra automorphism, called the* Nakayama automorphism.

Proof. It is clear that $\nu$ is a linear map. We have $\int f\nu(gh) = \int ghf = \int hf\nu(g) = \int f\nu(g)\nu(h)$ hence $\nu(gh) = \nu(g)\nu(h)$ and $\int f\nu(1) = \int f$ hence $\nu(1) = 1$. Furthermore if $\nu(g) = 0$ then $0 = \langle \int, f\nu(g) \rangle = \langle \int, gf \rangle = \langle (f \rightharpoonup \int), g \rangle$ for all $f \in H$ hence $\langle a, g \rangle = 0$ for all $a \in H^*$ hence $g = 0$. So $\nu$ is injective hence bijective. □

**Corollary 4.1.22.** *The map $H \ni f \mapsto (\int \leftharpoonup f) \in H^*$ is an isomorphism.*

Proof. We have

$$(\textstyle\int \leftharpoonup f) = (\nu(f) \rightharpoonup \textstyle\int)$$

since $\langle (\int \leftharpoonup f), g \rangle = \langle \int, fg \rangle = \langle \int, g\nu(f) \rangle = \langle (\nu(f) \rightharpoonup \int), g \rangle$. This implies the corollary. □

If $G$ is a finite group and $H = {}^G$ then $H$ is commutative hence $\nu = \mathrm{id}$.

**Definition 4.1.23.** An element $\delta \in H$ is called a *Dirac $\delta$-function* if $\delta$ is a left invariant integral in $H$ with $\langle \int, \delta \rangle = 1$, i.e. if $\delta$ satisfies

$$f\delta = \varepsilon(f)\delta \qquad \text{and} \qquad \int \delta(x) dx = 1$$

for all $f \in H$. If $H$ has a Dirac $\delta$-function then we write

$$\int^* a(x) dx = \textstyle\int^* a := \langle a, \delta \rangle. \tag{8}$$

**Proposition 4.1.24.**

1. *If $H$ is finite dimensional then there exists a unique Dirac $\delta$-function $\delta$.*
2. *If $H$ is infinite dimensional then there exists no Dirac $\delta$-function.*

PROOF. 1. Since $H \ni f \mapsto (f \rightharpoonup \int) \in H^*$ is an isomorphism there is a $\delta \in H$ such that $(\delta \rightharpoonup \int) = \varepsilon$. Then $(f\delta \rightharpoonup \int) = (f \rightharpoonup (\delta \rightharpoonup \int)) = (f \rightharpoonup \varepsilon) = \varepsilon(f)\varepsilon = \varepsilon(f)(\delta \rightharpoonup \int)$ which implies $f\delta = \varepsilon(f)\delta$. Furthermore we have $\langle \int, \delta\rangle = \langle \int, 1_H\delta\rangle = \langle(\delta \rightharpoonup \int), 1_H\rangle = \varepsilon(1_H) = 1$ .

2. is [Sweedler] exercise V.4. □

**Lemma 4.1.25.** *Let $H$ be a finite dimensional Hopf algebra. Then $\int \in H^*$ is a left integral iff*

$$a(\sum \int_{(1)} \otimes S(\int_{(2)})) = (\sum \int_{(1)} \otimes S(\int_{(2)}))a \tag{9}$$

*iff*

$$\sum S(a)\int_{(1)} \otimes \int_{(2)} = \sum \int_{(1)} \otimes a\int_{(2)} \tag{10}$$

*iff*

$$\sum f_{(1)}\langle \int, f_{(2)}\rangle = \langle \int, f\rangle 1_H. \tag{11}$$

PROOF. Let $\int$ be a left integral. Then

$$\sum a_{(1)}\int_{(1)} \otimes S(\int_{(2)})S(a_{(2)}) = \sum (a\int)_{(1)} \otimes S((a\int)_{(2)}) = \varepsilon(a)(\sum \int_{(1)} \otimes S(\int_{(2)}))$$

for all $a \in H$. Hence

$$\begin{aligned}(\sum \int_{(1)} \otimes S(\int_{(2)}))a &= \sum \varepsilon(a_{(1)})(\int_{(1)} \otimes S(\int_{(2)}))a_{(2)} \\ &= \sum a_{(1)}\int_{(1)} \otimes S(\int_{(2)})S(a_{(2)})a_{(3)} \\ &= \sum a_{(1)}\int_{(1)} \otimes S(\int_{(2)})\varepsilon(a_{(2)}) = a(\sum \int_{(1)} \otimes S(\int_{(2)})).\end{aligned}$$

Conversely $a(\sum \int_{(1)} \varepsilon(S(\int_{(2)}))) = (\sum \int_{(1)} \varepsilon(S(\int_{(2)})a)) = \varepsilon(a)(\sum \int_{(1)} \varepsilon(S(\int_{(2)})))$, hence $\int = \sum \int_{(1)} \varepsilon(S(\int_{(2)}))$ is a left integral.

Since $S$ is bijective the following holds

$$\begin{aligned}&\sum S(a)\int_{(1)} \otimes \int_{(2)} = \sum S(a)\int_{(1)} \otimes S^{-1}(S(\int_{(2)})) \\ &= \sum \int_{(1)} \otimes S^{-1}(S(\int_{(2)})S(a)) = \sum \int_{(1)} \otimes a\int_{(2)}.\end{aligned}$$

The converse follows easily.

If $\int \in \mathrm{Int}_l(H)$ is a left integral then $\sum \langle a, f_{(1)}\rangle\langle \int, f_{(2)}\rangle = \langle a\int, f\rangle = \langle a, 1_H\rangle\langle \int, f\rangle$.

Conversely if $\lambda \in H^*$ with (11) is given then $\langle a\lambda, f\rangle = \sum \langle a, f_{(1)}\rangle\langle \lambda, f_{(2)}\rangle = \langle a, 1_H\rangle\langle \lambda, f\rangle$ hence $a\lambda = \varepsilon(a)\lambda$. □

If $G$ is a finite group then

$$\delta(x) = \begin{cases} 0 \text{ if } x \neq e; \\ 1 \text{ if } x = e. \end{cases} \tag{12}$$

In fact since $\delta$ is left invariant we get $f(x)\delta(x) = f(e)\delta(x)$ for all $x \in G$ and $f \in {}^G$. Since $G \subset H^* = \quad G$ is a basis, we get $\delta(x) = 0$ if $x \neq e$. Furthermore $\int \delta(x)dx = \sum_{x\in G} \delta(x) = 1$ implies $\delta(e) = 1$. So we have $\delta = e^*$.

If $G$ is a finite Abelian group we get $\delta = \alpha \sum_{\chi\in\hat{G}} \chi$ for some $\alpha \in$ . The evaluation gives $1 = \langle \int, \delta\rangle = \alpha \sum_{x\in G, \chi\in\hat{G}} \langle \chi, x\rangle$. Now let $\lambda \in \widehat{G}$. Then $\sum_{\chi\in\hat{G}} \langle \chi, x\rangle = \sum_{\chi\in\hat{G}} \langle \lambda\chi, x\rangle = \langle \lambda, x\rangle \sum_{\chi\in\hat{G}} \langle \chi, x\rangle$. Since for each $x \in G \setminus \{e\}$ there is a $\lambda$ such that $\langle \lambda, x\rangle \neq 1$ and we get

$$\sum_{\chi\in\hat{G}} \langle \chi, x\rangle = |G|\delta_{e,x}.$$

Hence $\sum_{x\in G, \chi\in\hat{G}} \langle \chi, x\rangle = |G| = \alpha^{-1}$ and

$$\delta = |G|^{-1} \sum_{\chi\in\hat{G}} \chi. \tag{13}$$

Let $H$ be finite dimensional for the rest of this section. In Corollary 1.22 we have seen that the map $H \ni f \mapsto (\int \leftharpoonup f) \in H^*$ is an isomorphism. This map will be called the *Fourier transform.*

**Theorem 4.1.26.** *The Fourier transform $H \ni f \mapsto \widetilde{f} \in H^*$ is bijective with*

$$\widetilde{f} = (\textstyle\int \leftharpoonup f) = \sum \langle \int_{(1)}, f\rangle \int_{(2)} \tag{14}$$

*The inverse Fourier transform is defined by*

$$\widetilde{a} = \sum S^{-1}(\delta_{(1)})\langle a, \delta_{(2)}\rangle. \tag{15}$$

*Since these maps are inverses of each other the following formulas hold*

$$\begin{array}{ll} \langle \widetilde{f}, g\rangle = \displaystyle\int f(x)g(x)dx & \langle a, \widetilde{b}\rangle = \displaystyle\int^* S^{-1}(a)(x)b(x)dx \\ f = \sum S^{-1}(\delta_{(1)})\langle \widetilde{f}, \delta_{(2)}\rangle & a = \sum \langle \int_{(1)}, \widetilde{a}\rangle \int_{(2)}. \end{array} \tag{16}$$

Proof. We use the isomorphisms $H \longrightarrow H^*$ defined by $\widehat{f} := \widetilde{f} = (\int \leftharpoonup f) = \sum \langle \int_{(1)}, f\rangle \int_{(2)}$ and $H^* \longrightarrow H$ defined by $\widehat{a} := (a \rightharpoonup \delta) = \sum \delta_{(1)}\langle a, \delta_{(2)}\rangle$. Because of

$$\langle a, \widehat{b}\rangle = \langle a, (b \rightharpoonup \delta)\rangle = \langle ab, \delta\rangle \tag{17}$$

and

$$\langle \widetilde{f}, g\rangle = \langle (\textstyle\int \leftharpoonup f), g\rangle = \langle \int, fg\rangle \tag{18}$$

we get for all $a \in H^*$ and $f \in H$

$$\begin{aligned} \langle a, \widehat{\widehat{f}}\rangle &= \langle a\widehat{f}, \delta\rangle = \textstyle\sum \langle a, \delta_{(1)}\rangle\langle \widehat{f}, \delta_{(2)}\rangle = \sum \langle a, \delta_{(1)}\rangle\langle \int, f\delta_{(2)}\rangle \qquad \text{( by Lemma 1.25 )} \\ &= \textstyle\sum \langle a, S(f)\delta_{(1)}\rangle\langle \int, \delta_{(2)}\rangle = \langle a, S(f)\rangle\langle \int, \delta\rangle = \langle a, S(f)\rangle. \end{aligned}$$

This gives $\widehat{\widehat{f}} = S(f)$. So the inverse map of $H \longrightarrow H^*$ with $\widehat{f} = (\int \leftharpoonup f) = \widetilde{f}$ is $H^* \longrightarrow H$ with $S^{-1}(\widehat{a}) = \sum S^{-1}(\delta_{(1)})\langle a, \delta_{(2)}\rangle = \widetilde{a}$. Then the given inversion formulas are clear.

We note for later use $\langle a, \widetilde{b}\rangle = \langle a, S^{-1}(\widehat{b})\rangle = \langle S^{-1}(a), \widehat{b}\rangle = \langle S^{-1}(a)b, \delta\rangle$. $\square$

If $G$ is a finite group and $H = \quad {}^G$ then

$$\widetilde{f} = \sum_{x \in G} f(x)x.$$

Since $\Delta(\delta) = \sum_{x \in G} x^{-1^*} \otimes x^*$ where the $x^* \in \quad {}^G$ are the dual basis to the $x \in G$, we get

$$\widetilde{a} = \sum_{x \in G} \langle a, x^*\rangle x^*.$$

If $G$ is a finite Abelian group then the groups $G$ and $\widehat{G}$ are isomorphic so the Fourier transform induces a linear automorphism $\widetilde{-} : \quad {}^G \longrightarrow \quad {}^G$ and we have

$$\widetilde{a} = |G|^{-1} \sum_{\chi \in \hat{G}} \langle a, \chi\rangle \chi^{-1}$$

By substituting the formulas for the integral and the Dirac $\delta$-function (1) and (13) we get

$$\begin{array}{ll} \widetilde{f} = \sum_{x \in G} f(x)x, & \widetilde{a} = |G|^{-1} \sum_{\chi \in \hat{G}} a(\chi)\chi^{-1}, \\ f = |G|^{-1} \sum_{\chi \in \hat{G}} \widetilde{f}(\chi)\chi^{-1}, & a = \sum_{x \in G} \widetilde{a}(x)x. \end{array} \tag{19}$$

This implies

$$\widetilde{f}(\chi) = \sum_{x \in G} f(x)\chi(x) = \int f(x)\chi(x)dx \tag{20}$$

with inverse transform

$$\widetilde{a}(x) = |G|^{-1} \sum_{\chi \in \hat{G}} \chi(a)\chi^{-1}(x). \tag{21}$$

**Corollary 4.1.27.** *The Fourier transforms of the left invariant integrals in $H$ and $H^*$ are*

$$\widetilde{\delta} = \varepsilon\nu^{-1} \in H^* \qquad \text{and} \qquad \widetilde{\int} = 1 \in H. \tag{22}$$

Proof. We have $\langle \widetilde{\delta}, f\rangle = \langle \int, \delta f\rangle = \langle \int, \nu^{-1}(f)\delta\rangle = \varepsilon\nu^{-1}(f)\langle \int, \delta\rangle = \varepsilon\nu^{-1}(f)$ hence $\widetilde{\delta} = \varepsilon\nu^{-1}$. From $\widetilde{1} = (\int \leftharpoonup 1) = \int$ we get $\widetilde{\int} = 1$. $\square$

**Proposition 4.1.28.** *Define a* convolution multiplication *on $H^*$ by*

$$\langle a * b, f\rangle := \sum \langle a, S^{-1}(\delta_{(1)})f\rangle\langle b, \delta_{(2)}\rangle.$$

*Then the following transformation rule holds for $f, g \in H$:*

$$\widetilde{fg} = \widetilde{f} * \widetilde{g}. \tag{23}$$

*In particular $H^*$ with the convolution multiplication is an associative algebra with unit $\widetilde{1_H} = \int$, i.e.*

$$\int * a = a * \int = a. \tag{24}$$

Proof. Given $f, g, h \in H^*$. Then

$$\begin{aligned}\langle \widetilde{fg}, h\rangle &= \langle \textstyle\int, fgh\rangle = \langle \textstyle\int, fS^{-1}(1_H)gh\rangle\langle \textstyle\int, \delta\rangle \\ &= \textstyle\sum\langle \textstyle\int, fS^{-1}(\delta_{(1)})gh\rangle\langle \textstyle\int, \delta_{(2)}\rangle = \textstyle\sum\langle \textstyle\int, fS^{-1}(\delta_{(1)})h\rangle\langle \textstyle\int, g\delta_{(2)}\rangle \\ &= \textstyle\sum\langle \widetilde{f}, S^{-1}(\delta_{(1)})h\rangle\langle \widetilde{g}, \delta_{(2)}\rangle = \langle \widetilde{f} * \widetilde{g}, h\rangle.\end{aligned}$$

From (22) we get $\widetilde{1}_H = \int$. So we have $\widetilde{f} = \widetilde{1f} = \widetilde{1} * \widetilde{f} = \int * \widetilde{f}$. □

If $G$ is a finite Abelian group and $a, b \in H^* = \hat{G}$. Then

$$(a * b)(\mu) = |G|^{-1} \sum_{\chi, \lambda \in \hat{G}, \chi\lambda = \mu} a(\lambda)b(\chi).$$

In fact we have

$$\begin{aligned}(a * b)(\mu) &= \langle a * b, \mu\rangle = \textstyle\sum\langle a, S^{-1}(\delta_{(1)})\mu\rangle\langle b, \delta_{(2)}\rangle \\ &= |G|^{-1}\textstyle\sum_{\chi \in \hat{G}}\langle a, \chi^{-1}\mu\rangle\langle b, \chi\rangle = |G|^{-1}\textstyle\sum_{\chi, \lambda \in \hat{G}, \chi\lambda = \mu} a(\lambda)b(\chi).\end{aligned}$$

One of the most important formulas for Fourier transforms is the Plancherel formula on the invariance of the inner product under Fourier transforms. We have

**Theorem 4.1.29.** *(The Plancherel formula)*

$$\langle a, f\rangle = \langle \widetilde{f}, \nu(\widetilde{a})\rangle. \tag{25}$$

Proof. First we have from (16)

$$\begin{aligned}\langle a, f\rangle &= \textstyle\sum\langle \textstyle\int_{(1)}, \widetilde{a}\rangle\langle \textstyle\int_{(2)}, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle = \textstyle\sum\langle \textstyle\int, \widetilde{a}S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \textstyle\sum\langle \textstyle\int, S^{-1}(\delta_{(1)})\nu(\widetilde{a})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle = \textstyle\sum\langle \textstyle\int, S^{-1}(S(\nu(\widetilde{a}))\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \textstyle\sum\langle \textstyle\int, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \nu(\widetilde{a})\delta_{(2)}\rangle = \textstyle\sum\langle \textstyle\int, S^{-1}(\delta)_{(2)}\rangle\langle \widetilde{f}, \nu(\widetilde{a})S(S^{-1}(\delta)_{(1)})\rangle \\ &= \langle \textstyle\int, S^{-1}(\delta)\rangle\langle \widetilde{f}, \nu(\widetilde{a})\rangle.\end{aligned}$$

Apply this to $\langle \int, \delta\rangle$. Then we get

$$1 = \langle \textstyle\int, \delta\rangle = \langle \textstyle\int, S^{-1}(\delta)\rangle\langle \widetilde{\delta}, \nu(\widetilde{\textstyle\int})\rangle = \langle \textstyle\int, S^{-1}(\delta)\rangle\varepsilon\nu^{-1}\nu(1) = \langle \textstyle\int, S^{-1}(\delta)\rangle.$$

Hence we get $\langle a, f\rangle = \langle \widetilde{f}, \nu(\widetilde{a})\rangle$. □

**Corollary 4.1.30.** *If $H$ is unimodular then $\nu = S^2$.*

Proof. $H$ unimodular means that $\delta$ is left and right invariant. Thus we get

$$\begin{aligned}\langle a, f\rangle &= \sum\langle \textstyle\int_{(1)}, \widetilde{a}\rangle\langle \textstyle\int_{(2)}, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \sum\langle \textstyle\int, \widetilde{a}S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle = \sum\langle \textstyle\int, S^{-1}(\delta_{(1)}S(\widetilde{a}))\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \sum\langle \textstyle\int, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}S^2(\widetilde{a})\rangle \quad (\text{ since } \delta \text{ is right invariant}) \\ &= \langle \textstyle\int, S^{-1}(\delta)\rangle\langle \widetilde{f}, S^2(\widetilde{a})\rangle = \langle \widetilde{f}, S^2(\widetilde{a})\rangle.\end{aligned}$$

Hence $S^2 = \nu$. □

We also get a special representation of the inner product $H^* \otimes H \longrightarrow$ by both integrals:

**Corollary 4.1.31.**

$$\langle a, f\rangle = \int \widetilde{a}(x)f(x)dx = \int^* S^{-1}(a)(x)\widetilde{f}(x)dx. \tag{26}$$

Proof. We have the rules for the Fourier transform. From (18) we get $\langle a, f\rangle = \langle \int, \widetilde{a}f\rangle = \int \widetilde{a}(x)f(x)dx$ and from (17) $\langle a, f\rangle = \langle S^{-1}(a)\widetilde{f}, \delta\rangle = \int^* S^{-1}(a)(x)\widetilde{f}(x)dx$. □

The Fourier transform leads to an interesting integral transform on $H$ by double application.

**Proposition 4.1.32.** *The double transform $\breve{f} := (\delta \leftharpoonup (\int \leftharpoonup f))$ defines an automorphism $H \longrightarrow H$ with*

$$\breve{f}(y) = \int f(x)\delta(xy)dx.$$

Proof. We have

$$\begin{aligned}\langle y, \breve{f}\rangle &= \langle y, (\delta \leftharpoonup (\textstyle\int \leftharpoonup f))\rangle = \langle (\textstyle\int \leftharpoonup f)y, \delta\rangle \\ &= \sum\langle (\textstyle\int \leftharpoonup f), \delta_{(1)}\rangle\langle y, \delta_{(2)}\rangle = \sum\langle \textstyle\int, f\delta_{(1)}\rangle\langle y, \delta_{(2)}\rangle \\ &= \sum\langle \textstyle\int_{(1)}, f\rangle\langle \textstyle\int_{(2)}, \delta_{(1)}\rangle\langle y, \delta_{(2)}\rangle = \sum\langle \textstyle\int_{(1)}, f\rangle\langle \textstyle\int_{(2)} y, \delta\rangle \\ &= \sum\langle \textstyle\int_{(1)}, f\rangle\langle \textstyle\int_{(2)}, (y \rightharpoonup \delta)\rangle = \langle \textstyle\int, f(y \rightharpoonup \delta)\rangle \\ &= \textstyle\int f(x)\delta(xy)dx\end{aligned}$$

since $\langle x, (y \rightharpoonup \delta)\rangle = \langle xy, \delta\rangle$. □