CHAPTER 4

The Infinitesimal Theory

1. Integrals and Fourier Transforms

Assume for this chapter that   is a field.

Lemma 4.1.1. *Let C be a finite dimensional coalgebra. Every right C-comodule M is a left C^*-module by $c^*m = \sum m_{(M)}\langle c^*, m_{(1)}\rangle$ and conversely by $\delta(m) = \sum_i c_i^* m \otimes c_i$ where $\sum c_i^* \otimes c_i$ is the dual basis.*

Proof. We check that M becomes a left C^*-module

$$\begin{aligned}(c^*c'^*)m &= \sum m_{(M)}\langle c^*c'^*, m_{(1)}\rangle = \sum m_{(M)}\langle c^*, m_{(1)}\rangle\langle c'^*, m_{(2)}\rangle \\ &= c^* \sum m_{(M)}\langle c'^*, m_{(1)}\rangle = c^*(c'^* m).\end{aligned}$$

It is easy to check that the two constructions are inverses of each other. In particular assume that M is a right C-comodule. Choose m_i such that $\delta(m) = \sum m_i \otimes c_i$. Then $c_j^* m = \sum m_i \langle c_j^*, c_i\rangle = m_j$ and $\sum c_i^* m \otimes c_i = \sum m_i \otimes c_i = \delta(m)$. $\square$

Definition 4.1.2. 1. Let A be an algebra with augmentation $\varepsilon : A \longrightarrow$, an algebra homomorphism. Let M be a left A-module. Then ${}^A M = \{m \in M | am = \varepsilon(a)m\}$ is called the *space of left invariants* of M.

This defines a functor ^{A}- : $A-\mathbf{Mod} \longrightarrow \mathbf{Vec}$.

2. Let C be a coalgebra with a group-like element $1 \in C$. Let M be a right C-comodule. Then $M^{coC} := \{m \in M | \delta(m) = m \otimes 1\}$ is called the *space of right coinvariants* of M.

This defines a functor -^{coC} : $\mathbf{Comod}$-$C \longrightarrow \mathbf{Vec}$.

Lemma 4.1.3. *Let C be a finite dimensional coalgebra with a group like element $1 \in C$. Then $A := C^*$ is an augmented algebra with augmentation $\varepsilon : C^* \ni a \mapsto \langle a, 1\rangle \in$. Let M be a right C-comodule. Then M is a left C^*-module and we have*

$${}^{C^*}M = M^{coC}.$$

Proof. Since $1 \in C$ is group-like we have $\varepsilon_A(ab) = \langle ab, 1\rangle = \langle a, 1\rangle\langle b, 1\rangle = \varepsilon_A(a)\varepsilon_A(b)$ and $\varepsilon_A(1_A) = \langle 1_A, 1_C\rangle = \varepsilon_C(1_C) = 1$.

We have $m \in M^{coH}$ iff $\delta(m) = \sum m_{(M)} \otimes m_{(1)} = m \otimes 1$ iff $\sum m_{(M)}\langle a, m_{(1)}\rangle = m\langle a, 1\rangle$ for all $a \in A = C^*$ and by identifying $C^* \otimes C = \mathrm{Hom}(C^*, C^*)$ iff $am = \varepsilon_A(a)m$ iff $m \in {}^A M$. $\square$

Remark 4.1.4. The theory of Fourier transforms contains the following statements. Let H be the (Schwartz) space of infinitely differentiable functions $f :\ \ \longrightarrow\ \ ,$

such that f and all derivatives rapidly decrease at infinity. (f decreases rapidly at infinity if $|x|^m f(x)$ is bounded for all m.) This space is an algebra (without unit) under the multiplication of values. There is a second multiplication on H, the convolution

$$(f * g)(x) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} f(t)g(x-t)dt.$$

The Fourier transform is a homomorphism $\hat{} : H \longrightarrow H$ defined by

$$\hat{f}(x) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} f(t)e^{-itx}dt.$$

It satisfies the identity $(f * g)\hat{} = \hat{f}\hat{g}$ hence it is an algebra homomorphism. We want to find an analogue of this theory for finite quantum groups.

A similar example is the following. Let G be a locally compact topological group. Let μ be the (left) *Haar measure* on G and $\int f := \int_G f(x)d\mu(x)$ be the *Haar integral.*

The Haar measure is left invariant in the sense that $\mu(E) = \mu(gE)$ for all $g \in G$ and all compact subsets E of G. The Haar measure exists and is unique up to a positive factor. The Haar integral is translation invariant i.e. for all $y \in G$ we have $\int f(yx)d\mu(x) = \int f(x)d\mu(x)$.

If μ is a left-invariant Haar measure then there is a continuous homomorphism $\mathrm{mod} : G \longrightarrow (\quad^{+}, \cdot)$ such that $\int f(xy^{-1})d\mu(x) = \ \mathrm{mod}\,(y) \int (f(x)d\mu(x)$. The homomorphism μ does not depend on f and is called the *modulus* of G. The group G is called *unimodular* if the homomorphism mod is the identity.

If G is a compact, or discrete, or Abelian group, or a connected semisimple or nilpotent Lie group, then G is unimodular.

Let G be a quantum group (or a quantum monoid) with function algebra H an arbitrary Hopf algebra. We also use the algebra of linear functionals $H^* = \mathrm{Hom}(H, \quad)$ (called the bialgebra of G in the French literature). The operation $H^* \otimes H \ni a \otimes f \mapsto \langle a, f \rangle \in \quad$ is nondegenerate on both sides. We denote the elements of H by $f, g, h \in H$, the elements of H^* by $a, b, c \in H^*$, the (non existing) elements of the quantum group G by $x, y, z \in G$.

Remark 4.1.5. In 2.4.8 we have seen that the dual vector space H^* of a finite dimensional Hopf algebra H is again a Hopf algebra. The Hopf algebra structures are connected by the evaluation bilinear form

$$\mathrm{ev} : H^* \otimes H \ni a \otimes f \mapsto \langle a, f \rangle \in$$

as follows:

$$\begin{array}{cc} \langle a \otimes b, \sum f_{(1)} \otimes f_{(2)} \rangle = \langle ab, f \rangle, & \langle \sum a_{(1)} \otimes a_{(2)}, f \otimes g \rangle = \langle a, fg \rangle, \\ \langle a, 1 \rangle = \varepsilon(a), & \langle 1, f \rangle = \varepsilon(f), \\ \multicolumn{2}{c}{\langle a, S(f) \rangle = \langle S(a), f \rangle.} \end{array}$$

Definition 4.1.6. 1. The linear functionals $a \in H^*$ are called *generalized integrals on H* ([Riesz-Nagy] S.123).

2. An element $\int \in H^*$ is called a *left (invariant) integral on* H if

$$a\int = \langle a, 1_H\rangle \int$$

or $a\int = \varepsilon_{H^*}(a)\int$ for all $a \in H^*$.

3. An element $\delta \in H$ is called a *left integral in* H if

$$f\delta = \varepsilon(f)\delta$$

for all $f \in H$.

4. The set of left integrals in H is denoted by $\mathrm{Int}_l(H)$, the set of right integrals by $\mathrm{Int}_r(H)$. The set of left (right) integrals on H is $\mathrm{Int}_l(H^*)$ ($\mathrm{Int}_r(H^*)$).

5. A Hopf algebra H is called *unimodular* if $\mathrm{Int}_l(H) = \mathrm{Int}_r(H)$.

Lemma 4.1.7. *The left integrals* $\mathrm{Int}_l(H^*)$ *form a two sided ideal of* H^**. If the antipode* S *is bijective then* S *induces an isomorphism* $S : \mathrm{Int}_l(H^*) \longrightarrow \mathrm{Int}_r(H^*)$.

PROOF. For $\int$ in $\mathrm{Int}_l(H^*)$ we have $a\int = \varepsilon(a)\int \in \mathrm{Int}_l(H^*)$ and $a\int b = \varepsilon(a)\int b$ hence $\int b \in \mathrm{Int}_l(H^*)$. If S is bijective then the induced map $S : H^* \longrightarrow H^*$ is also bijective and satisfies $S(\int)b = S(\int)S(S^{-1}(b)) = S(S^{-1}(b)\int) = S(\int)\varepsilon(b)$ hence $S(\int) \in \mathrm{Int}_r(H^*)$. $\square$

Remark 4.1.8. Maschke's Theorem has an extension to finite dimensional Hopf algebras: $\varepsilon(\int) \neq 0$ iff H^* is semisimple.

Corollary 4.1.9. *Let* H *be a finite dimensional Hopf algebra. Then* H^* *is a left* H^**-module by the usual multiplication, hence a right* H*-comodule. We have*

$$(H^*)^{coH} = \mathrm{Int}_l(H^*).$$

PROOF. By definition we have $\mathrm{Int}_l(H^*) = {}^{H^*}H^*$. $\square$

Example 4.1.10. Let G be a finite group. Let $H := \mathrm{Map}(G, \;)$ be the Hopf algebra defined by the following isomorphism

$$\;^G = \mathrm{Map}(G, \;) \cong \mathrm{Hom}(\; G, \;) = (\; G)^*.$$

This isomorphism between the vector space $\;^G$ of all set maps from the group G to the base ring   and the dual vector space $(\; G)^*$ of the group algebra $\; G$ defines the structure of a Hopf algebra on $\;^G$.

We regard $H := \;^G$ as the function algebra on the set G. In the sense of algebraic geometry this is not quite true. The algebra $\;^G$ represents a functor from -**cAlg** to **Set** that has G as value for all connected algebras A in particular for all field extensions of .

As before we use the map $\mathrm{ev} : \; G \otimes \;^G \longrightarrow \;$. The multiplication of $\;^G$ is given by pointwise multiplication of maps since $\langle x, ff'\rangle = \langle \sum x_{(1)} \otimes x_{(2)}, f \otimes f'\rangle = \langle x \otimes x, f \otimes f'\rangle = \langle x, f\rangle\langle x, f'\rangle$ for all $f, f' \in \;^G$ and all $x \in G$. The unit element $1_{\;^G}$ of $\;^G$ is the map $\varepsilon : \; G \longrightarrow \;$ restricted to G, hence $\varepsilon(x) = 1 = \langle x, 1_{\;^G}\rangle$ for all $x \in G$. The antipode of $f \in \;^G$ is given by $S(f)(x) = \langle x, S(f)\rangle = f(x^{-1})$.

The elements of the dual basis $(x^*|x \in G)$ with $\langle x, y^* \rangle = \delta_{x,y}$ considered as maps from G to form a basis of ^{G}. They satisfy the conditions

$$x^*y^* = \delta_{x,y}x^* \text{ and } \sum_{x\in G} x^* = 1_{\ \,G}$$

since $\langle z, x^*y^* \rangle = \langle z, x^* \rangle \langle z, y^* \rangle = \delta_{z,x}\delta_{z,y} = \delta_{x,y}\langle z, x^* \rangle$ and $\langle z, \sum_{x\in G} x^* \rangle = 1 = \langle z, 1_{\ \,G} \rangle$.

Hence the dual basis $(x^*|x \in G)$ is a decomposition of the unit into a set of minimal orthogonal idempotents and the algebra of ^{G} has the structure

$$^G = \oplus_{x\in G} \quad x^* \cong \quad \times \ldots \times \quad .$$

In particular ^{G} is commutative and semisimple.

The diagonal of ^{G} is

$$\Delta(x^*) = \sum_{y\in G} y^* \otimes (y^{-1}x)^* = \sum_{y,z\in G, yz=x} y^* \otimes z^*$$

since

$$\begin{aligned}\langle z \otimes u, \Delta(x^*) \rangle &= \langle zu, x^* \rangle = \delta_{x,zu} = \delta_{z^{-1}x,u} = \textstyle\sum_{y\in G} \delta_{y,z}\delta_{y^{-1}x,u} \\ &= \textstyle\sum_{y\in G} \langle z, y^* \rangle \langle u, (y^{-1}x)^* \rangle = \langle z \otimes u, \sum_{y\in G} y^* \otimes (y^{-1}x)^* \rangle.\end{aligned}$$

Let $a \in \quad G$. Then a defines a map $\widetilde{a} : G \longrightarrow \quad \in \quad {}^G$ by $a = \sum_{x\in G} \widetilde{a}(x)x$. For arbitrary $f \in \quad {}^G$ and $a \in \quad G$ this gives

$$\langle a, f \rangle = f(\sum_{x\in G} \widetilde{a}(x)x) = \sum_{x\in G} \widetilde{a}(x)f(x).$$

The counit of ^{G} is given by $\varepsilon(x^*) = \delta_{x,e}$ where $e \in G$ is the unit element.

The antipode is, as above, $S(x^*) = (x^{-1})^*$.

We consider $H = \quad {}^G$ as the function algebra on the finite group G and $\quad G$ as the dual space of $H = \quad {}^G$ hence as the set of distributions on H.

Then

$$\int := \sum_{x\in G} x \in H^* = \quad G \tag{1}$$

is a (two sided) integral on H since $\sum_{x\in G} yx = \sum_{x\in G} x = \varepsilon(y)\sum_{x\in G} x = \sum_{x\in G} yx$.

We write

$$\int f(x)dx := \langle \textstyle\int, f \rangle = \displaystyle\sum_{x\in G} f(x).$$

We have seen that there is a decomposition of the unit $1 \in \quad {}^G$ into a set of primitive orthogonal idempotents $\{x^*|x \in G\}$ such that every element $f \in \quad {}^G$ has a unique representation $f = \sum f(x)x^*$. Since $\int y^* = \sum_{x\in G}\langle x, y^* \rangle$ we get $\int fy^* = \sum_{x\in G}\langle x, fy^* \rangle = \sum f(x)y^*(x) = f(y)$ hence

$$f = \sum (\int f(x)y^*(x)dx)y^*.$$

Problem 4.1.1. Describe the group valued functor $\ \text{-}\mathbf{cAlg}(\ ^G, -)$ in terms of sets and their group structure.

Definition and Remark 4.1.11. Let be an algebraicly closed field and let G be a finite abelian group (replacing above). Assume that the characteristic of does not divide the order of G. Let $H = \ ^G$. We identify $\ ^G = \mathrm{Hom}(\ G, \ )$ along the linear expansion of maps as in Example 2.1.10.

Let us consider the set $\hat{G} := \{\chi : G \longrightarrow \ ^* | \chi$ group homomorphism$\}$. Since $\ ^*$ is an abelian group, the set $\hat{G}$ is an abelian group by pointwise multiplication.

The group $\hat{G}$ is called the *character group* of G.

Obviously the character group is a multiplicative subset of $\ ^G = \mathrm{Hom}(\ G, \ )$. Actually it is a subgroup of $\ \text{-}\mathbf{cAlg}(\ G, \ ) \subseteq \mathrm{Hom}(\ G, \ )$ since the elements $\chi \in \hat{G}$ expand to algebra homomorphisms: $\chi(ab) = \chi(\sum \alpha_x x \sum \beta_y y) = \sum \alpha_x \beta_y \chi(xy) = \chi(a)\chi(b)$ and $\chi(1) = \chi(e) = 1$. Conversely an algebra homomorphism $f \in \ \text{-}\mathbf{cAlg}(\ G, \ )$ restricts to a character $f : G \longrightarrow \ ^*$. Thus $\hat{G} = \ \text{-}\mathbf{cAlg}(\ G, \ )$, the set of rational points of the affine algebraic group represented by $\ G$.

There is a more general observation behind this remark.

Lemma 4.1.12. *Let H be a finite dimensional Hopf algebra. Then the set $Gr(H^*)$ of group like elements of H^* is equal to $\ \text{-}\mathbf{Alg}(H, \ )$.*

Proof. In fact $f : H \longrightarrow \ $ is an algebra homomorphism iff $\langle f \otimes f, a \otimes b \rangle = \langle f, a \rangle \langle f, b \rangle = \langle f, ab \rangle = \langle \Delta(f), a \otimes b \rangle$ and $1 = \langle f, 1 \rangle = \varepsilon(f)$. $\square$

Hence there is a Hopf algebra homomorphism $\varphi : \ \hat{G} \longrightarrow \ ^G$ by 2.1.5.

Proposition 4.1.13. *The Hopf algebra homomorphism $\varphi : \ \hat{G} \longrightarrow \ ^G$ is bijective.*

Proof. We give the proof by several lemmas.

Lemma 4.1.14. *Any set of group like elements in a Hopf algebra H is linearly independent.*

Proof. Assume there is a linearly dependent set $\{x_0, x_1, \ldots, x_n\}$ of group like elements in H. Choose such a set with n minimal. Obviously $n \geq 1$ since all elements are non zero. Thus $x_0 = \sum_{i=1}^n \alpha_i x_i$ and $\{x_1, \ldots, x_n\}$ linearly independent. We get

$$\sum_{i,j} \alpha_i \alpha_j x_i \otimes x_j = x_0 \otimes x_0 = \Delta(x_0) = \sum_i \alpha_i x_i \otimes x_i.$$

Since all $\alpha_i \neq 0$ and the $x_i \otimes x_j$ are linearly independent we get $n = 1$ and $\alpha_1 = 1$ so that $x_0 = x_1$, a contradiction. $\square$

Corollary 4.1.15. *(Dedekind's Lemma) Any set of characters in $\ ^G$ is linearly independent.*

Thus $\varphi : \quad \hat{G} \longrightarrow \quad {}^G$ is injective. Now we prove that the map $\varphi : \quad \hat{G} \longrightarrow \quad {}^G$ is surjective.

Lemma 4.1.16. *(Pontryagin duality) The evaluation $\hat{G} \times G \longrightarrow \quad {}^*$ is a non-degenerate bilinear map of abelian groups.*

Proof. First we observe that $\mathrm{Hom}(C_n, \quad {}^*) \cong C_n$ for a cyclic group of order n since has a primitive n-th root of unity (char($) \not| \; |G|$).

Since the direct product and the direct sum coincide in **Ab** we can use the fundamental theorem for finite abelian groups $G \cong C_{n_1} \times \ldots \times C_{n_t}$ to get $\mathrm{Hom}(G, \quad {}^*) \cong G$ for any abelian group G with char($) \not| \; |G|$. Thus $\hat{G} \cong G$ and $\hat{\hat{G}} = G$. In particular $\chi(x) = 1$ for all $x \in G$ iff $\chi = 1$. By the symmetry of the situation we get that the bilinear form $\langle .,. \rangle : \hat{G} \times G \longrightarrow \quad {}^*$ is non-degenerate. $\square$

Thus $|\hat{G}| = |G|$ hence $\dim(\quad \hat{G}) = \dim(\quad {}^G)$. This proves Proposition 2.1.13. $\square$

Definition 4.1.17. Let H be a Hopf algebra. A -module M that is a right H-module by $\rho : M \otimes H \longrightarrow M$ and a right H-comodule by $\delta : M \longrightarrow M \otimes H$ is called a *Hopf module* if the diagram

$$\begin{tikzcd} M \otimes H \arrow[r, "\rho"] \arrow[d, "\delta\otimes\Delta"'] & H \arrow[r, "\delta"] & M \otimes H \\ M \otimes H \otimes H \otimes H \arrow[rr, "1\otimes\tau\otimes 1"] & & M \otimes H \otimes H \otimes H \arrow[u, "\rho\otimes\nabla"'] \end{tikzcd}$$

commutes, i.e. if $\delta(mh) = \sum m_{(M)}h_{(1)} \otimes m_{(1)}h_{(2)}$ holds for all $h \in H$ and all $m \in M$.

Observe that H is an Hopf module over itself. Furthermore each module of the form $V \otimes H$ is a Hopf module by the induced structure. More generally there is a functor $\mathbf{Vec} \ni V \mapsto V \otimes H \in \mathbf{Hopf\text{-}Mod}\text{-}H$.

Proposition 4.1.18. *The two functors $\text{-}^{coH} : \mathbf{Hopf\text{-}Mod}\text{-}H \longrightarrow \mathbf{Vec}$ and $\text{-} \otimes H : \mathbf{Vec} \ni V \mapsto V \otimes H \in \mathbf{Hopf\text{-}Mod}\text{-}H$ are inverse equivalences of each other.*

Proof. Define natural isomorphisms

$$\alpha : M^{coH} \otimes H \ni m \otimes h \mapsto mh \in M$$

with inverse map

$$\alpha^{-1} : M \ni m \mapsto \sum m_{(M)} S(m_{(1)}) \otimes m_{(2)} \in M^{coH} \otimes H$$

and

$$\beta : V \ni v \mapsto v \otimes 1 \in (V \otimes H)^{coH}$$

with inverse map

$$(V \otimes H)^{coH} \ni v \otimes h \mapsto v\varepsilon(h) \in V.$$

Obviously these homomorphisms are natural transformations in M and V. Furthermore α is a homomorphism of H-modules. α^{-1} is well-defined since

$$\begin{aligned}\delta(\sum m_{(M)}S(m(1))) &= \sum m_{(M)}S(m_{(3)}) \otimes m_{(1)}S(m_{(2)}) \\ &\quad \text{(since } M \text{ is a Hopf module)} \\ &= \sum m_{(M)}S(m_{(2)}) \otimes \eta\varepsilon(m_{(1)}) \\ &= \sum m_{(M)}S(m_{(1)}) \otimes 1\end{aligned}$$

hence $\sum m_{(M)}S(m_{(1)}) \in M^{coH}$. Furthermore α^{-1} is a homomorphism of comodules since

$$\begin{aligned}\delta\alpha^{-1}(m) &= \delta(\sum m_{(M)}S(m_{(1)}) \otimes m_{(2)}) = \sum m_{(M)}S(m_{(1)}) \otimes m_{(2)} \otimes m_{(3)} \\ &= \sum \alpha^{-1}(m_{(M)}) \otimes m_{(1)} = (\alpha^{-1} \otimes 1)\delta(m).\end{aligned}$$

Finally α and α^{-1} are inverse to each other by

$$\alpha\alpha^{-1}(m) = \alpha(\sum m_{(M)}S(m_{(1)}) \otimes m_{(2)}) = \sum m_{(M)}S(m_{(1)})m_{(2)} = m$$

and

$$\begin{aligned}\alpha^{-1}\alpha(m \otimes h) &= \alpha^{-1}(mh) = \sum m_{(M)}h_{(1)}S(m_{(1)}h_{(2)}) \otimes m_{(2)}h_{(3)} \\ &= \sum mh_{(1)}S(h_{(2)}) \otimes h_{(3)} \text{ (by } \delta(m) = m \otimes 1 \text{)} = m \otimes h.\end{aligned}$$

Thus α and α^{-1} are mutually inverse homomorphisms of Hopf modules.

The image of β is in $(V \otimes H)^{coH}$ by $\delta(v \otimes 1) = v \otimes \Delta(1) = (v \otimes 1) \otimes 1$. Both β and β^{-1} are -linear maps. Furthermore we have

$$\beta^{-1}\beta(v) = \beta^{-1}(v \otimes 1) = v\varepsilon(1) = v$$

and

$$\begin{aligned}\beta\beta^{-1}(\sum v_i \otimes h_i) &= \beta(\sum v_i\varepsilon(h_i)) = \sum v_i\varepsilon(h_i) \otimes 1 = \sum v_i \otimes \varepsilon(h_i)1 \\ &= \sum v_i \otimes \varepsilon(h_{i(1)})h_{i(2)} \text{ (since } \textstyle\sum v_i \otimes h_i \in (V \otimes H)^{coH} \text{)} = \sum v_i \otimes h_i.\end{aligned}$$

Thus β and β^{-1} are mutually inverse homomorphisms. $\square$

Since $H^* = \mathrm{Hom}(H,\ )$ and $S : H \longrightarrow H$ is an algebra antihomomorphism, the dual H^* is an H-module in four different ways:

$$\begin{aligned}&\langle(f \rightharpoonup a), g\rangle := \langle a, gf\rangle, &&\langle(a \leftharpoonup f), g\rangle := \langle a, fg\rangle, \\ &\langle(f \rightharpoondown a), g\rangle := \langle a, S(f)g\rangle, &&\langle(a \leftharpoondown f), g\rangle := \langle a, gS(f)\rangle.\end{aligned} \tag{2}$$

If H is finite dimensional then H^* is a Hopf algebra. The equality $\langle(f \rightharpoonup a), g\rangle = \langle a, gf\rangle = \sum\langle a_{(1)}, g\rangle\langle a_{(2)}, f\rangle$ implies

$$(f \rightharpoonup a) = \sum a_{(1)}\langle a_{(2)}, f\rangle. \tag{3}$$

Analogously we have

$$(a \leftharpoonup f) = \sum\langle a_{(1)}, f\rangle a_{(2)}. \tag{4}$$

Proposition 4.1.19. *Let H be a finite dimensional Hopf algebra. Then H^* is a right Hopf module over H.*

PROOF. H^* is a left H^*-module by left multiplication hence by 2.1.1 a right H-comodule by $\delta(a) = \sum_i b_i^* a \otimes b_i$. Let $f, g \in H$ and $a, b \in H^*$. The (left) multiplication of H^* satisfies

$$ab = \sum b_{(H^*)}\langle a, b_{(1)}\rangle.$$

We use the right H-module structure

$$(a \leftharpoonup f) = \sum a_{(1)}\langle S(f), a_{(2)}\rangle.$$

on $H^* = \mathrm{Hom}(H, \quad)$.

Now we check the Hopf module property. Let $a, b \in H^*$ and $f, g \in H$. We apply $H^* \otimes H$ to its dual $H \otimes H^*$ and get

$$\begin{aligned}
\delta(a \leftharpoonup f)(g \otimes b) &= \sum\langle(a \leftharpoonup f)_{(H^*)}, g\rangle\langle b, (a \leftharpoonup f)_{(1)}\rangle = \langle b(a \leftharpoonup f), g\rangle \\
&= \sum\langle b, g_{(1)}\rangle\langle(a \leftharpoonup f), g_{(2)}\rangle = \sum\langle b, g_{(1)}\rangle\langle a, g_{(2)}S(f)\rangle \\
&= \sum\langle b, g_{(1)}\varepsilon(f_{(2)})\rangle\langle a, g_{(2)}S(f_{(1)})\rangle = \sum\langle(f_{(3)} \rightharpoonup b), g_{(1)}S(f_{(2)})\rangle\langle a, g_{(2)}S(f_{(1)})\rangle \\
&= \sum\langle(f_{(2)} \rightharpoonup b)a, gS(f_{(1)})\rangle = \sum\langle((f_{(2)} \rightharpoonup b)a) \leftharpoonup f_{(1)}, g\rangle \\
&= \sum\langle(a_{(H^*)}\langle(f_{(2)} \rightharpoonup b), a_{(1)}\rangle) \leftharpoonup f_{(1)}, g\rangle \\
&= \sum\langle(a_{(H^*)} \leftharpoonup f_{(1)})\langle(f_{(2)} \rightharpoonup b), a_{(1)}\rangle, g\rangle = \sum\langle(a_{(H^*)} \leftharpoonup f_{(1)})\langle b, a_{(1)}f_{(2)}\rangle, g\rangle
\end{aligned}$$

hence $\delta(a \leftharpoonup f) = \sum(a_{(H^*)} \leftharpoonup f_{(1)}) \otimes a_{(1)}f_{(2)}$. $\square$

Theorem 4.1.20. *Let H be a finite dimensional Hopf algebra. Then the antipode S is bijective, the space of left integrals $\mathrm{Int}_l(H^*)$ has dimension 1, and the homomorphism*

$$H \ni f \mapsto (f \rightharpoonup \textstyle\int) = \sum \int_{(1)}\langle\int_{(2)}, f\rangle \ni H^*$$

is bijective for any $0 \neq \int \in \mathrm{Int}_l(H^*)$.

PROOF. By Proposition 2.1.19 H^* is a right Hopf module over H. By Proposition 2.1.18 there is an isomorphism $\alpha : (H^*)^{coH} \otimes H \ni a \otimes f \mapsto (a \leftharpoonup f) = (S(f) \rightharpoonup a) \in H^*$. Since $(H^*)^{coH} \cong \mathrm{Int}_l(H^*)$ by 2.1.9 we get

$$\mathrm{Int}_l(H^*) \otimes H \cong H^*$$

as right H-Hopf modules by the given map. This shows $\dim(\mathrm{Int}_l(H^*)) = 1$. So we get an isomorphism $H \ni f \mapsto (\int \leftharpoonup f) \in H^*$ that is a composition of S and $f \mapsto (f \rightharpoonup \int)$. Since H is finite dimensional both of these maps are bijective. $\square$

If G is a finite group then every generalized integral $a \in \quad G$ can be written with a uniquely determined $g \in H = \quad^G$ as

$$\langle a, f\rangle = \int f(x)S(g)(x)dx = \sum_{x \in G} f(x)g(x^{-1}) \tag{5}$$

for all $f \in H$.

If G is a finite Abelian group then each group element (rational integral) $y \in G \subseteq G$ can be written as

$$y = \sum_{x \in G} \sum_{\chi \in \hat{G}} \beta_\chi \langle x^{-1}, \chi \rangle x$$

since

$$\begin{aligned}\langle y, f \rangle &= \langle (\textstyle\int \leftharpoonup \sum_{\chi \in \hat{G}} \beta_\chi \chi), f \rangle = \langle \textstyle\int, f S(\sum_{\chi \in \hat{G}} \beta_\chi \chi) \rangle \\ &= \textstyle\sum_{x \in G} \langle x, f \rangle \sum_{\chi \in \hat{G}} \beta_\chi \langle x, S(\chi) \rangle = \langle \sum_{x \in G} \sum_{\chi \in \hat{G}} \beta_\chi \langle x^{-1}, \chi \rangle x, f \rangle.\end{aligned}$$

In particular the matrix $(\langle x^{-1}, \chi \rangle)$ is invertible.

Let H be finite dimensional. Since $\langle \int, fg \rangle = \langle (\int \leftharpoonup f), g \rangle$ as a functional on g is a generalized integral, there is a unique $\nu(f) \in H$ such that

$$\langle \textstyle\int, fg \rangle = \langle \textstyle\int, g\nu(f) \rangle \tag{6}$$

or

$$\int f(x) g(x) dx = \int g(x) \nu(f)(x) dx. \tag{7}$$

Although the functions $f, g \in H$ of the quantum group do not commute under multiplication, there is a simple commutation rule if the product is integrated.

Proposition and Definition 4.1.21. *The map $\nu : H \longrightarrow H$ is an algebra automorphism, called the* Nakayama automorphism.

Proof. It is clear that ν is a linear map. We have $\int f\nu(gh) = \int ghf = \int hf\nu(g) = \int f\nu(g)\nu(h)$ hence $\nu(gh) = \nu(g)\nu(h)$ and $\int f\nu(1) = \int f$ hence $\nu(1) = 1$. Furthermore if $\nu(g) = 0$ then $0 = \langle \int, f\nu(g) \rangle = \langle \int, gf \rangle = \langle (f \rightharpoonup \int), g \rangle$ for all $f \in H$ hence $\langle a, g \rangle = 0$ for all $a \in H^*$ hence $g = 0$. So ν is injective hence bijective. $\square$

Corollary 4.1.22. *The map $H \ni f \mapsto (\int \leftharpoonup f) \in H^*$ is an isomorphism.*

Proof. We have

$$(\textstyle\int \leftharpoonup f) = (\nu(f) \rightharpoonup \textstyle\int)$$

since $\langle (\int \leftharpoonup f), g \rangle = \langle \int, fg \rangle = \langle \int, g\nu(f) \rangle = \langle (\nu(f) \rightharpoonup \int), g \rangle$. This implies the corollary. $\square$

If G is a finite group and $H = {}^{G}$ then H is commutative hence $\nu = \mathrm{id}$.

Definition 4.1.23. An element $\delta \in H$ is called a *Dirac δ-function* if δ is a left invariant integral in H with $\langle \int, \delta \rangle = 1$, i.e. if δ satisfies

$$f\delta = \varepsilon(f)\delta \qquad \text{and} \qquad \int \delta(x) dx = 1$$

for all $f \in H$. If H has a Dirac δ-function then we write

$$\int^* a(x) dx = \textstyle\int^* a := \langle a, \delta \rangle. \tag{8}$$

Proposition 4.1.24.

1. *If H is finite dimensional then there exists a unique Dirac δ-function δ.*
2. *If H is infinite dimensional then there exists no Dirac δ-function.*

PROOF. 1. Since $H \ni f \mapsto (f \rightharpoonup \int) \in H^*$ is an isomorphism there is a $\delta \in H$ such that $(\delta \rightharpoonup \int) = \varepsilon$. Then $(f\delta \rightharpoonup \int) = (f \rightharpoonup (\delta \rightharpoonup \int)) = (f \rightharpoonup \varepsilon) = \varepsilon(f)\varepsilon = \varepsilon(f)(\delta \rightharpoonup \int)$ which implies $f\delta = \varepsilon(f)\delta$. Furthermore we have $\langle \int, \delta\rangle = \langle \int, 1_H\delta\rangle = \langle(\delta \rightharpoonup \int), 1_H\rangle = \varepsilon(1_H) = 1$.

2. is [Sweedler] exercise V.4. $\square$

Lemma 4.1.25. *Let H be a finite dimensional Hopf algebra. Then $\int \in H^*$ is a left integral iff*

$$a(\sum \int_{(1)} \otimes S(\int_{(2)})) = (\sum \int_{(1)} \otimes S(\int_{(2)}))a \tag{9}$$

iff

$$\sum S(a)\int_{(1)} \otimes \int_{(2)} = \sum \int_{(1)} \otimes a\int_{(2)} \tag{10}$$

iff

$$\sum f_{(1)}\langle \int, f_{(2)}\rangle = \langle \int, f\rangle 1_H. \tag{11}$$

PROOF. Let $\int$ be a left integral. Then

$$\sum a_{(1)}\int_{(1)} \otimes S(\int_{(2)})S(a_{(2)}) = \sum (a\int)_{(1)} \otimes S((a\int)_{(2)}) = \varepsilon(a)(\sum \int_{(1)} \otimes S(\int_{(2)}))$$

for all $a \in H$. Hence

$$\begin{aligned}(\sum \int_{(1)} \otimes S(\int_{(2)}))a &= \sum \varepsilon(a_{(1)})(\int_{(1)} \otimes S(\int_{(2)}))a_{(2)} \\ &= \sum a_{(1)}\int_{(1)} \otimes S(\int_{(2)})S(a_{(2)})a_{(3)} \\ &= \sum a_{(1)}\int_{(1)} \otimes S(\int_{(2)})\varepsilon(a_{(2)}) = a(\sum \int_{(1)} \otimes S(\int_{(2)})).\end{aligned}$$

Conversely $a(\sum \int_{(1)} \varepsilon(S(\int_{(2)}))) = (\sum \int_{(1)} \varepsilon(S(\int_{(2)})a)) = \varepsilon(a)(\sum \int_{(1)} \varepsilon(S(\int_{(2)})))$, hence $\int = \sum \int_{(1)} \varepsilon(S(\int_{(2)}))$ is a left integral.

Since S is bijective the following holds

$$\begin{aligned}&\sum S(a)\int_{(1)} \otimes \int_{(2)} = \sum S(a)\int_{(1)} \otimes S^{-1}(S(\int_{(2)})) \\ &= \sum \int_{(1)} \otimes S^{-1}(S(\int_{(2)})S(a)) = \sum \int_{(1)} \otimes a\int_{(2)}.\end{aligned}$$

The converse follows easily.

If $\int \in \mathrm{Int}_l(H)$ is a left integral then $\sum \langle a, f_{(1)}\rangle\langle \int, f_{(2)}\rangle = \langle a\int, f\rangle = \langle a, 1_H\rangle\langle \int, f\rangle$.

Conversely if $\lambda \in H^*$ with (11) is given then $\langle a\lambda, f\rangle = \sum\langle a, f_{(1)}\rangle\langle\lambda, f_{(2)}\rangle = \langle a, 1_H\rangle\langle\lambda, f\rangle$ hence $a\lambda = \varepsilon(a)\lambda$. $\square$

If G is a finite group then

$$\delta(x) = \begin{cases} 0 \text{ if } x \neq e; \\ 1 \text{ if } x = e.\end{cases} \tag{12}$$

In fact since δ is left invariant we get $f(x)\delta(x) = f(e)\delta(x)$ for all $x \in G$ and $f \in {}^{G}$. Since $G \subset H^* = \; G$ is a basis, we get $\delta(x) = 0$ if $x \neq e$. Furthermore $\int \delta(x)dx = \sum_{x\in G} \delta(x) = 1$ implies $\delta(e) = 1$. So we have $\delta = e^*$.

If G is a finite Abelian group we get $\delta = \alpha \sum_{\chi\in\hat{G}} \chi$ for some $\alpha \in \;$. The evaluation gives $1 = \langle \int, \delta\rangle = \alpha \sum_{x\in G, \chi\in\hat{G}} \langle \chi, x\rangle$. Now let $\lambda \in \widehat{G}$. Then $\sum_{\chi\in\hat{G}} \langle \chi, x\rangle = \sum_{\chi\in\hat{G}} \langle \lambda\chi, x\rangle = \langle \lambda, x\rangle \sum_{\chi\in\hat{G}} \langle \chi, x\rangle$. Since for each $x \in G \setminus \{e\}$ there is a λ such that $\langle \lambda, x\rangle \neq 1$ and we get

$$\sum_{\chi\in\hat{G}} \langle \chi, x\rangle = |G|\delta_{e,x}.$$

Hence $\sum_{x\in G, \chi\in\hat{G}} \langle \chi, x\rangle = |G| = \alpha^{-1}$ and

$$\delta = |G|^{-1} \sum_{\chi\in\hat{G}} \chi. \tag{13}$$

Let H be finite dimensional for the rest of this section. In Corollary 1.22 we have seen that the map $H \ni f \mapsto (\int \leftharpoonup f) \in H^*$ is an isomorphism. This map will be called the *Fourier transform*.

Theorem 4.1.26. *The Fourier transform $H \ni f \mapsto \widetilde{f} \in H^*$ is bijective with*

$$\widetilde{f} = (\textstyle\int \leftharpoonup f) = \sum \langle \int_{(1)}, f\rangle \int_{(2)} \tag{14}$$

The inverse Fourier transform is defined by

$$\widetilde{a} = \sum S^{-1}(\delta_{(1)})\langle a, \delta_{(2)}\rangle. \tag{15}$$

Since these maps are inverses of each other the following formulas hold

$$\begin{array}{ll} \langle \widetilde{f}, g\rangle = \int f(x)g(x)dx & \langle a, \widetilde{b}\rangle = \int^* S^{-1}(a)(x)b(x)dx \\ f = \sum S^{-1}(\delta_{(1)})\langle \widetilde{f}, \delta_{(2)}\rangle & a = \sum \langle \int_{(1)}, \widetilde{a}\rangle \int_{(2)}. \end{array} \tag{16}$$

Proof. We use the isomorphisms $H \longrightarrow H^*$ defined by $\widehat{f} := \widetilde{f} = (\int \leftharpoonup f) = \sum\langle \int_{(1)}, f\rangle \int_{(2)}$ and $H^* \longrightarrow H$ defined by $\widehat{a} := (a \rightharpoonup \delta) = \sum \delta_{(1)}\langle a, \delta_{(2)}\rangle$. Because of

$$\langle a, \widehat{b}\rangle = \langle a, (b \rightharpoonup \delta)\rangle = \langle ab, \delta\rangle \tag{17}$$

and

$$\langle \widetilde{f}, g\rangle = \langle (\textstyle\int \leftharpoonup f), g\rangle = \langle \int, fg\rangle \tag{18}$$

we get for all $a \in H^*$ and $f \in H$

$$\begin{aligned} \langle a, \widehat{\widehat{f}}\rangle &= \langle a\widehat{f}, \delta\rangle = \textstyle\sum \langle a, \delta_{(1)}\rangle\langle \widehat{f}, \delta_{(2)}\rangle = \sum\langle a, \delta_{(1)}\rangle\langle \int, f\delta_{(2)}\rangle \qquad (\text{ by Lemma 1.25 }) \\ &= \textstyle\sum\langle a, S(f)\delta_{(1)}\rangle\langle \int, \delta_{(2)}\rangle = \langle a, S(f)\rangle\langle \int, \delta\rangle = \langle a, S(f)\rangle. \end{aligned}$$

This gives $\widehat{\widehat{f}} = S(f)$. So the inverse map of $H \longrightarrow H^*$ with $\widehat{f} = (\int \leftharpoonup f) = \widetilde{f}$ is $H^* \longrightarrow H$ with $S^{-1}(\widehat{a}) = \sum S^{-1}(\delta_{(1)})\langle a, \delta_{(2)}\rangle = \widetilde{a}$. Then the given inversion formulas are clear.

We note for later use $\langle a, \widetilde{b}\rangle = \langle a, S^{-1}(\widehat{b})\rangle = \langle S^{-1}(a), \widehat{b}\rangle = \langle S^{-1}(a)b, \delta\rangle$. $\square$

If G is a finite group and $H = \ ^G$ then

$$\widetilde{f} = \sum_{x \in G} f(x)x.$$

Since $\Delta(\delta) = \sum_{x \in G} x^{-1*} \otimes x^*$ where the $x^* \in \ ^G$ are the dual basis to the $x \in G$, we get

$$\widetilde{a} = \sum_{x \in G} \langle a, x^*\rangle x^*.$$

If G is a finite Abelian group then the groups G and $\widehat{G}$ are isomorphic so the Fourier transform induces a linear automorphism $\widetilde{-}: \ ^G \longrightarrow \ ^G$ and we have

$$\widetilde{a} = |G|^{-1} \sum_{\chi \in \hat{G}} \langle a, \chi\rangle \chi^{-1}$$

By substituting the formulas for the integral and the Dirac δ-function (1) and (13) we get

$$\begin{array}{ll} \widetilde{f} = \sum_{x \in G} f(x)x, & \widetilde{a} = |G|^{-1} \sum_{\chi \in \hat{G}} a(\chi)\chi^{-1}, \\ f = |G|^{-1} \sum_{\chi \in \hat{G}} \widetilde{f}(\chi)\chi^{-1}, & a = \sum_{x \in G} \widetilde{a}(x)x. \end{array} \tag{19}$$

This implies

$$\widetilde{f}(\chi) = \sum_{x \in G} f(x)\chi(x) = \int f(x)\chi(x)dx \tag{20}$$

with inverse transform

$$\widetilde{a}(x) = |G|^{-1} \sum_{\chi \in \hat{G}} \chi(a)\chi^{-1}(x). \tag{21}$$

Corollary 4.1.27. *The Fourier transforms of the left invariant integrals in H and H^* are*

$$\widetilde{\delta} = \varepsilon\nu^{-1} \in H^* \qquad \text{and} \qquad \widetilde{\int} = 1 \in H. \tag{22}$$

Proof. We have $\langle \widetilde{\delta}, f\rangle = \langle \int, \delta f\rangle = \langle \int, \nu^{-1}(f)\delta\rangle = \varepsilon\nu^{-1}(f)\langle \int, \delta\rangle = \varepsilon\nu^{-1}(f)$ hence $\widetilde{\delta} = \varepsilon\nu^{-1}$. From $\widetilde{1} = (\int \leftharpoonup 1) = \int$ we get $\widetilde{\int} = 1$. $\square$

Proposition 4.1.28. *Define a* convolution multiplication *on H^* by*

$$\langle a * b, f\rangle := \sum \langle a, S^{-1}(\delta_{(1)})f\rangle\langle b, \delta_{(2)}\rangle.$$

Then the following transformation rule holds for $f, g \in H$:

$$\widetilde{fg} = \widetilde{f} * \widetilde{g}. \tag{23}$$

In particular H^* with the convolution multiplication is an associative algebra with unit $\widetilde{1_H} = \int$, i.e.

$$\int * a = a * \int = a. \tag{24}$$

Proof. Given $f, g, h \in H^*$. Then

$$\begin{aligned} \langle \widetilde{fg}, h\rangle &= \langle \textstyle\int, fgh\rangle = \langle \int, fS^{-1}(1_H)gh\rangle\langle \int, \delta\rangle \\ &= \textstyle\sum\langle \int, fS^{-1}(\delta_{(1)})gh\rangle\langle \int, \delta_{(2)}\rangle = \sum\langle \int, fS^{-1}(\delta_{(1)})h\rangle\langle \int, g\delta_{(2)}\rangle \\ &= \textstyle\sum\langle \widetilde{f}, S^{-1}(\delta_{(1)})h\rangle\langle \widetilde{g}, \delta_{(2)}\rangle = \langle \widetilde{f} * \widetilde{g}, h\rangle. \end{aligned}$$

From (22) we get $\widetilde{1}_H = \int$. So we have $\widetilde{f} = \widetilde{1f} = \widetilde{1} * \widetilde{f} = \int * \widetilde{f}$. □

If G is a finite Abelian group and $a, b \in H^* = \ \hat{G}$. Then

$$(a * b)(\mu) = |G|^{-1} \sum_{\chi, \lambda \in \hat{G}, \chi\lambda = \mu} a(\lambda) b(\chi).$$

In fact we have

$$\begin{aligned} (a * b)(\mu) &= \langle a * b, \mu\rangle = \textstyle\sum\langle a, S^{-1}(\delta_{(1)})\mu\rangle\langle b, \delta_{(2)}\rangle \\ &= |G|^{-1} \textstyle\sum_{\chi \in \hat{G}}\langle a, \chi^{-1}\mu\rangle\langle b, \chi\rangle = |G|^{-1} \sum_{\chi, \lambda \in \hat{G}, \chi\lambda = \mu} a(\lambda) b(\chi). \end{aligned}$$

One of the most important formulas for Fourier transforms is the Plancherel formula on the invariance of the inner product under Fourier transforms. We have

Theorem 4.1.29. *(The Plancherel formula)*

$$\langle a, f\rangle = \langle \widetilde{f}, \nu(\widetilde{a})\rangle. \tag{25}$$

Proof. First we have from (16)

$$\begin{aligned} \langle a, f\rangle &= \textstyle\sum\langle \int_{(1)}, \widetilde{a}\rangle\langle \int_{(2)}, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle = \sum\langle \int, \widetilde{a}S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \textstyle\sum\langle \int, S^{-1}(\delta_{(1)})\nu(\widetilde{a})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle = \sum\langle \int, S^{-1}(S(\nu(\widetilde{a}))\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \textstyle\sum\langle \int, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \nu(\widetilde{a})\delta_{(2)}\rangle = \sum\langle \int, S^{-1}(\delta)_{(2)}\rangle\langle \widetilde{f}, \nu(\widetilde{a})S(S^{-1}(\delta)_{(1)})\rangle \\ &= \langle \textstyle\int, S^{-1}(\delta)\rangle\langle \widetilde{f}, \nu(\widetilde{a})\rangle. \end{aligned}$$

Apply this to $\langle \int, \delta\rangle$. Then we get

$$1 = \langle \textstyle\int, \delta\rangle = \langle \int, S^{-1}(\delta)\rangle\langle \widetilde{\delta}, \nu(\widetilde{\int})\rangle = \langle \int, S^{-1}(\delta)\rangle\varepsilon\nu^{-1}\nu(1) = \langle \int, S^{-1}(\delta)\rangle.$$

Hence we get $\langle a, f\rangle = \langle \widetilde{f}, \nu(\widetilde{a})\rangle$. □

Corollary 4.1.30. *If H is unimodular then $\nu = S^2$.*

Proof. H unimodular means that δ is left and right invariant. Thus we get

$$\begin{aligned}
\langle a, f\rangle &= \sum \langle \textstyle\int_{(1)}, \widetilde{a}\rangle\langle \textstyle\int_{(2)}, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\
&= \sum \langle \textstyle\int, \widetilde{a}S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle = \sum \langle \textstyle\int, S^{-1}(\delta_{(1)}S(\widetilde{a}))\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\
&= \sum \langle \textstyle\int, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}S^2(\widetilde{a})\rangle \quad (\text{ since } \delta \text{ is right invariant}) \\
&= \langle \textstyle\int, S^{-1}(\delta)\rangle\langle \widetilde{f}, S^2(\widetilde{a})\rangle = \langle \widetilde{f}, S^2(\widetilde{a})\rangle.
\end{aligned}$$

Hence $S^2 = \nu$. □

We also get a special representation of the inner product $H^* \otimes H \longrightarrow$ by both integrals:

Corollary 4.1.31.

$$\langle a, f\rangle = \int \widetilde{a}(x)f(x)dx = \int^* S^{-1}(a)(x)\widetilde{f}(x)dx. \tag{26}$$

Proof. We have the rules for the Fourier transform. From (18) we get $\langle a, f\rangle = \langle \int, \widetilde{a}f\rangle = \int \widetilde{a}(x)f(x)dx$ and from (17) $\langle a, f\rangle = \langle S^{-1}(a)\widetilde{f}, \delta\rangle = \int^* S^{-1}(a)(x)\widetilde{f}(x)dx$. □

The Fourier transform leads to an interesting integral transform on H by double application.

Proposition 4.1.32. *The double transform* $\breve{f} := (\delta \leftharpoonup (\int \leftharpoonup f))$ *defines an automorphism* $H \longrightarrow H$ *with*

$$\breve{f}(y) = \int f(x)\delta(xy)dx.$$

Proof. We have

$$\begin{aligned}
\langle y, \breve{f}\rangle &= \langle y, (\delta \leftharpoonup (\textstyle\int \leftharpoonup f))\rangle = \langle (\textstyle\int \leftharpoonup f)y, \delta\rangle \\
&= \sum \langle (\textstyle\int \leftharpoonup f), \delta_{(1)}\rangle\langle y, \delta_{(2)}\rangle = \sum \langle \textstyle\int, f\delta_{(1)}\rangle\langle y, \delta_{(2)}\rangle \\
&= \sum \langle \textstyle\int_{(1)}, f\rangle\langle \textstyle\int_{(2)}, \delta_{(1)}\rangle\langle y, \delta_{(2)}\rangle = \sum \langle \textstyle\int_{(1)}, f\rangle\langle \textstyle\int_{(2)} y, \delta\rangle \\
&= \sum \langle \textstyle\int_{(1)}, f\rangle\langle \textstyle\int_{(2)}, (y \rightharpoonup \delta)\rangle = \langle \textstyle\int, f(y \rightharpoonup \delta)\rangle \\
&= \textstyle\int f(x)\delta(xy)dx
\end{aligned}$$

since $\langle x, (y \rightharpoonup \delta)\rangle = \langle xy, \delta\rangle$. □

2. Derivations

Definition 4.2.1. Let A be a -algebra and ${}_AM_A$ be an A-A-bimodule (with identical -action on both sides). A linear map $D : A \longrightarrow M$ is called a *derivation* if

$$D(ab) = aD(b) + D(a)b.$$

The set of derivations Der $(A, {}_AM_A)$ is a -module and a functor in ${}_AM_A$.

By induction one sees that D satisfies

$$D(a_1 \ldots a_n) = \sum_{i=1}^{n} a_1 \ldots a_{i-1} D(a_i) a_{i+1} \ldots a_n.$$

Let A be a commutative -algebra and ${}_AM$ be an A-module. Consider M as an A-A-bimodule by $ma := am$. We denote the set of derivations from A to M by Der $(A, M)_c$.

Proposition 4.2.2. 1. *Let A be a -algebra. Then the functor* Der $(A, \text{-})$: A-**Mod**-$A \longrightarrow$ **Vec** *is representable by the* module of differentials Ω_A.

2. *Let A be a commutative -algebra. Then the functor* Der $(A, \text{-})_c$: A-**Mod** $\longrightarrow$ **Vec** *is representable by the* module of commutative differentials Ω^c_A.

Proof. 1. Represent A as a quotient of a free -algebra $A := \langle X_i | i \in J\rangle / I$ where $B = \langle X_i | i \in J\rangle$ is the free algebra with generators X_i. We first prove the theorem for free algebras.

a) A representing module for Der $(B, \text{-})$ is $(\Omega_B, d : B \longrightarrow \Omega_B)$ with

$$\Omega_B := B \otimes F(dX_i | i \in J) \otimes B$$

where $F(dX_i | i \in J)$ is the free -module on the set of formal symbols $\{dX_i | i \in J\}$ as a basis.

We have to show that for every derivation $D : B \longrightarrow M$ there exists a unique homomorphisms $\varphi : \Omega_B \longrightarrow M$ of B-B-bimodules such that the diagram

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "D"'] & \Omega_B \arrow[d, "\varphi"] \\ & M \end{tikzcd}$$

commutes. The module Ω_B is a B-B-bimodule in the canonical way. The products $X_1 \ldots X_n$ of the generators X_i of B form a basis for B. For any product $X_1 \ldots X_n$ we define $d(X_1 \ldots X_n) := \sum_{i=1}^n X_1 \ldots X_{i-1} \otimes dX_i \otimes X_{i+1} \ldots X_n$ in particular $d(X_i) = 1 \otimes dX_i \otimes 1$. To see that d is a derivation it suffices to show this on the basis elements:

$$\begin{aligned} & d(X_1 \ldots X_k X_{k+1} \ldots X_n) \\ & \quad = \textstyle\sum_{j=1}^k X_1 \ldots X_{j-1} \otimes dX_j \otimes X_{j+1} \ldots X_k X_{k+1} \ldots X_n \\ & \quad + \textstyle\sum_{j=k+1}^n X_1 \ldots X_k X_{k+1} \ldots X_{j-1} \otimes dX_j \otimes X_{j+1} \ldots X_n \\ & \quad = d(X_1 \ldots X_k) X_{k+1} \ldots X_n + X_1 \ldots X_k d(X_{k+1} \ldots X_n) \end{aligned}$$

Now let $D : B \longrightarrow M$ be a derivation. Define φ by $\varphi(1 \otimes dX_i \otimes 1) := D(X_i)$. This map obviously extends to a homomorphism of B-B-bimodules. Furthermore we have

$$\begin{aligned}\varphi d(X_1 \dots X_n) &= \varphi(\textstyle\sum_j X_1 \dots X_{j-1} \otimes dX_j \otimes X_{j+1} \dots X_n)\\ &= \textstyle\sum_j X_1 \dots X_{j-1} \varphi(1 \otimes dX_j \otimes 1) X_{j+1} \dots X_n = D(X_1 \dots X_n)\end{aligned}$$

hence $\varphi d = D$.

To show the uniqueness of φ let $\psi : \Omega_B \longrightarrow M$ be a bimodule homomorphism such that $\psi d = D$. Then $\psi(1 \otimes dX_i \otimes 1) = \psi d(X_i) = D(X_i) = \varphi(1 \otimes dX_i \otimes 1)$. Since ψ and φ are B-B-bimodules homomorphisms this extends to $\psi = \varphi$.

b) Now let $A := \langle X_i | i \in J\rangle / I$ be an arbitrary algebra with $B = \langle X_i | i \in J\rangle$ free. Define

$$\Omega_A := \Omega_B / (I\Omega_B + \Omega_B I + Bd_B(I) + d_B(I)B).$$

We first show that $I\Omega_B + \Omega_B I + Bd_B(I) + d_B(I)B$ is a B-B-subbimodule. Since Ω_B and I are B-B-bimodules the terms $I\Omega_B$ and $\Omega_B I$ are bimodules. Furthermore we have $bd_B(i)b' = bd_B(ib') - bid_B(b') \in Bd_B(I) + I\Omega_B$ hence $I\Omega_B + \Omega_B I + Bd_B(I) + d_B(I)B$ is a bimodule.

Now $I\Omega_B$ and $\Omega_B I$ are subbimodules of $I\Omega_B + \Omega_B I + Bd_B(I) + d_B(I)B$. Hence $A = B/I$ acts on both sides on Ω_A so that Ω_A becomes an A-A-bimodule.

Let $\nu : \Omega_B \longrightarrow \Omega_A$ and also $\nu : B \longrightarrow A$ be the residue homomorphisms. Since $\nu d_B(i) \in \nu d_B(I) = 0 \subseteq \Omega_A$ we get a unique factorization map $d_A : A \longrightarrow \Omega_A$ such that

$$\begin{tikzcd} B \arrow[r, "d_B"] \arrow[d, "\nu"'] & \Omega_B \arrow[d, "\nu"] \\ A \arrow[r, "d_A"'] & \Omega_A \end{tikzcd}$$

commutes. Since $d_A(\overline{b}) = \overline{d_B(b)}$ it is clear that d_A is a derivation.

Let $D : A \longrightarrow M$ be a derivation. The A-A-bimodule M is also a B-B-bimodule by $bm = \overline{b}m$. Furthermore $D\nu : B \longrightarrow A \longrightarrow M$ is again a derivation. Let $\varphi_B : \Omega_B \longrightarrow M$ be the unique factorization map for the B-derivation $D\nu$. Consider the following diagram

$$\begin{tikzcd} B \arrow[r, "d_B"] \arrow[d] & \Omega_B \arrow[d] \arrow[dd, bend left, "\varphi"] \\ A \arrow[r, "d_A"] \arrow[dr, "D"'] & \Omega_A \arrow[d, "\psi"'] \\ & M \end{tikzcd}$$

We want to construct ψ such that the diagram commutes. Let $i\omega \in I\Omega_B$. Then $\varphi(i\omega) = \overline{i}\varphi(\omega) = 0$ and similarly $\varphi(\omega i) = 0$. Let $bd_B(i) \in Bd_B(I)$ then $\varphi(bd_B(i)) = \overline{b}\varphi d_B(i) = \overline{b}D(\overline{i}) = 0$ and similarly $\varphi(d_B(i)b) = 0$. Hence φ vanishes on $I\Omega_B + \Omega_B I +$

$Bd_B(I)+d_B(I)B$ and thus factorizes through a unique map $\psi : \Omega_A \longrightarrow M$. Obviously ψ is a homomorphism of A-A-bimodules. Furthermore we have $D\nu = \varphi d_B = \psi\nu d_B = \psi d_A\nu$ and, since ν is surjective, $D = \psi d_A$. It is clear that ψ is uniquely determined by this condition.

2. If A is commutative then we can write $A = \ [X_i|i \in J]/I$ and $\Omega^c_B = B \otimes F(dX_i)$. With $\Omega^c_A = \Omega^c_B/(I\Omega^c_B + Bd_B(I))$ the proof is analogous to the proof in the noncommutative situation. $\square$

Remark 4.2.3. 1. Ω_A is generated by $d(A)$ as a bimodule, hence all elements are of the form $\sum_i a_i d(a'_i)a''_i$. These elements are called *differentials*.

2. If $A = \ \langle X_i\rangle/I$, then Ω_A is generated as a bimodule by the elements $\{\overline{d(X_i)}\}$.

3. Let $f \in B = \ \langle X_i\rangle$. Let B^{op} be the algebra opposite to B (with opposite multiplication). Then $\Omega_B = B \otimes F(dX_i) \otimes B$ is the free $B \otimes B^{op}$ left module over the free generating set $\{d(X_i)\}$. Hence $d(f)$ has a unique representation

$$d(f) = \sum_i \frac{\partial f}{\partial X_i} d(X_i)$$

with uniquely defined coefficients

$$\frac{\partial f}{\partial X_i} \in B \otimes B^{op}.$$

In the commutative situation we have unique coefficients

$$\frac{\partial f}{\partial X_i} \in \ [X_i].$$

4. We give the following examples for part 3:

$$\frac{\partial X_i}{\partial X_j} = \delta_{ij},$$
$$\frac{\partial X_1X_2}{\partial X_1} = 1 \otimes X_2,$$
$$\frac{\partial X_1X_2}{\partial X_2} = X_1 \otimes 1,$$
$$\frac{\partial X_1X_2X_3}{\partial X_2} = X_1 \otimes X_3,$$
$$\frac{\partial X_1X_3X_2}{\partial X_2} = X_1X_3 \otimes 1.$$

This is obtained by direct calculation or by the *product rule*

$$\frac{\partial fg}{\partial X_i} = (1 \otimes g)\frac{\partial f}{\partial X_i} + (f \otimes 1)\frac{\partial g}{\partial X_i}.$$

The product rule follows from

$$d(fg) = d(f)g + fd(g) = \sum((1 \otimes g)\frac{\partial f}{\partial X_i} + (f \otimes 1)\frac{\partial g}{\partial X_i})d(X_i).$$

Let $A = \quad \langle X_i \rangle / I$. If $f \in I$ then $\overline{d(f)} = d_A(\overline{f}) = 0$ hence

$$\sum \frac{\partial f}{\partial X_i} d_A(\overline{X_i}) = 0.$$

These are the defining relations for the A-A-bimodule Ω_A with the generators $d_A(\overline{X_i})$.

For motivation of the quantum group case we consider an affine algebraic group G with representing commutative Hopf algebra A. Recall that $\mathrm{Hom}(A, R)$ is an algebra with the convolution multiplication for every $R \in \quad$-**cAlg** and that $G(R) = \quad$-**cAlg**$(A, R) \subseteq \mathrm{Hom}(A, R)$ is a subgroup of the group of units of the algebra $\mathrm{Hom}(A, R)$.

Definition and Remark 4.2.4. A linear map $T : A \longrightarrow A$ is called *left translation invariant*, if the following diagram functorial in $R \in \quad$-**cAlg** commutes:

$$\begin{tikzcd} G(R) \times \mathrm{Hom}(A, R) \arrow[r, "*"] \arrow[d, "1 \otimes \mathrm{Hom}(T,R)"'] & \mathrm{Hom}(A, R) \arrow[d, "\mathrm{Hom}(T,R)"] \\ G(R) \times \mathrm{Hom}(A, R) \arrow[r, "*"'] & \mathrm{Hom}(A, R) \end{tikzcd}$$

i. e. if we have

$$\forall g \in G(R), \forall x \in \mathrm{Hom}(A, R) : g * (x \circ T) = (g * x) \circ T.$$

This condition is equivalent to

$$\Delta_A \circ T = (1_A \otimes T) \circ \Delta_A. \tag{27}$$

In fact if (27) holds then $g * (x \circ T) = \nabla_R(g \otimes x)(1_A \otimes T)\Delta_A = \nabla_R(g \otimes x)\Delta_A T = (g * x) \circ T$.

Conversely if the diagram commutes, then take $R = A$, $g = 1_A$ and we get $\nabla_A(1_A \otimes x)(1_A \otimes T)\Delta_A = 1_A * (x \circ T) = (1_A * x) \circ T = \nabla_A(1_A \otimes x)\Delta_A T$ for all $x \in \mathrm{Hom}(A, A)$. To get (27) it suffices to show that the terms $\nabla_A(1_A \otimes x)$ can be cancelled in this equation. Let $\sum_{i=1}^n a_i \otimes b_i \in A \otimes A$ be given such that $\nabla_A(1_A \otimes x)(\sum a_i \otimes b_i) = 0$ for all $x \in \mathrm{Hom}(A, A)$ and choose such an element with a shortest representation (n minimal). Then $\sum a_i x(b_i) = 0$ for all x. Since the b_i are linearly independent in such a shortest representation, there are x_i with $x_j(b_i) = \delta_{ij}$. Hence $a_j = \sum a_i x_j(b_i) = 0$ and thus $\sum a_i \otimes b_i = 0$. From this follows (27).

Definition 4.2.5. Let H be an arbitrary Hopf algebra. An element $T \in \mathrm{Hom}(H, H)$ is called *left translation invariant* if it satisfies

$$\Delta_H T = (1_H \otimes T)\Delta_H.$$

Proposition 4.2.6. *Let H be an arbitrary Hopf algebra. Then $\Phi : H^* \longrightarrow \mathrm{End}(H)$ with $\Phi(f) := \mathrm{id} * u_H f$ is an algebra monomorphism satisfying*

$$\Phi(f * g) = \Phi(f) \circ \Phi(g).$$

The image of Φ is precisely the set of left translation invariant elements $T \in \mathrm{End}(H)$.

PROOF. For $f \in \mathrm{Hom}(H, \ )$ we have $u_H f \in \mathrm{End}(H)$ hence $\mathrm{id} * u_H f \in \mathrm{End}(H)$. Thus Φ is a well defined homomorphism. Observe that

$$\Phi(f)(a) = (\mathrm{id}_H * u_H f)(a) = \sum a_{(1)} f(a_{(2)}).$$

Φ is injective since it has a retraction $\mathrm{End}(H) \ni g \mapsto \varepsilon_H \circ g \in \mathrm{Hom}(H, \ )$. In fact we have $(\varepsilon\Phi(f))(a) = \varepsilon(\sum a_{(1)} f(a_{(2)})) = \sum \varepsilon(a_{(1)}) f(a_{(2)}) = f(\sum \varepsilon(a_{(1)}) a_{(2)}) = f(a)$ hence $\varepsilon\Phi(f) = f$.

The map Φ preserves the algebra unit since $\Phi(1_{H^*}) = \Phi(\varepsilon_H) = \mathrm{id}_H * u_H \varepsilon_H = \mathrm{id}_H$.

The map Φ is compatible with the multiplication: $\Phi(f * g)(a) = \sum a_{(1)}(f * g)(a_{(2)}) = \sum a_{(1)} f(a_{(2)}) g(a_{(3)}) = \sum (\mathrm{id} * u_H f)(a_{(1)}) g(a_{(2)}) = \Phi(f)(\sum a_{(1)} g(a_{(2)})) = \Phi(f)\Phi(g)(a)$ so that $\Phi(f * g) = \Phi(f) \circ \Phi(g)$.

For each $f \in H^*$ the element $\Phi(f)$ is left translation invariant since $\Delta\Phi(f)(a) = \Delta(\sum a_{(1)} f(a_{(2)})) = \sum a_{(1)} \otimes a_{(2)} f(a_{(3)}) = (1 \otimes \Phi(f))\Delta(a)$.

Let $T \in \mathrm{End}(H)$ be left translation invariant then $S * T = \nabla_H (S \otimes 1)(1 \otimes T)\Delta_H = \nabla_H (S \otimes 1)\Delta_H T = u_H \varepsilon_H T$. Thus $\Phi(\varepsilon T) = \mathrm{id} * u_H \varepsilon_H T = \mathrm{id} * S * T = T$, so that T is in the image of Φ. $\square$

Proposition 4.2.7. *Let $d \in \mathrm{Hom}(H, \ )$ and $\Phi(d) = D \in \mathrm{Hom}(H, H)$ be given. The following are equivalent:*

1. *$d : H \longrightarrow {}_\varepsilon \ {}_\varepsilon$ is a derivation.*
2. *$D : H \longrightarrow {}_H H_H$ is a (left translation invariant) derivation.*

In particular Φ induces an isomorphism between the set of derivations $d : H \longrightarrow {}_\varepsilon \ {}_\varepsilon$ and the set of left translation invariant derivations $D : H \longrightarrow {}_H H_H$.

PROOF. Assume that 1. holds so that d satisfies $d(ab) = \varepsilon(a)d(b) + d(a)\varepsilon(b)$. Then we get $D(ab) = \Phi(d)(ab) = \sum a_{(1)} b_{(1)} d(a_{(2)} b_{(2)}) = \sum a_{(1)} b_{(1)} \varepsilon(a_{(2)}) d(b_{(2)}) + \sum a_{(1)} b_{(1)} d(a_{(2)}) \varepsilon(b_{(2)}) = aD(b) + D(a)b$. Conversely assume that $D(ab) = aD(b) + D(a)b$. Then $d(ab) = \varepsilon D(ab) = \varepsilon(a)\varepsilon D(b) + \varepsilon D(a)\varepsilon(b) = \varepsilon(a)d(b) + d(a)\varepsilon(b)$. $\square$

3. The Lie Algebra of Primitive Elements

Lemma 4.3.1. *Let H be a Hopf algebra and H^o be its Sweedler dual. If $d \in$* $\mathrm{Der}\ (H, {}_\varepsilon\ \ {}_\varepsilon) \subseteq \mathrm{Hom}(H,\ \ )$ *is a derivation then d is a primitive element of H^o. Furthermore every primitive element $d \in H^o$ is a derivation in* $\mathrm{Der}\ (H, {}_\varepsilon\ \ {}_\varepsilon)$.

Proof. Let $d : H \longrightarrow\ \ $ be a derivation and let $a, b \in H$. Then $(b \rightharpoonup d)(a) = d(ab) = \varepsilon(a)d(b) + d(a)\varepsilon(b) = (d(b)\varepsilon + \varepsilon(b)d)(a)$ hence $(b \rightharpoonup d) = d(b)\varepsilon + \varepsilon(b)d$. Consequently we have $Hd = (H \rightharpoonup d) \subseteq\ \ \varepsilon +\ \ d$ so that $\dim Hd \leq 2 < \infty$. This shows $d \in H^o$. Furthermore we have $\langle \Delta d, a \otimes b\rangle = \langle d, ab\rangle = d(ab) = d(a)\varepsilon(b) + \varepsilon(a)d(b) = \langle d \otimes \varepsilon, a \otimes b\rangle + \langle \varepsilon \otimes d, a \otimes b\rangle = \langle 1_{H^o} \otimes d + d \otimes 1_{H^o}, a \otimes b\rangle$ hence $\Delta(d) = d \otimes 1_{H^o} + 1_{H^o} \otimes d$ so that d is a primitive element in H^o.

Conversely let $d \in H^o$ be primitive. then $d(ab) = \langle \Delta(d), a \otimes b\rangle = d(a)\varepsilon(b) + \varepsilon(a)d(b)$. □

Proposition and Definition 4.3.2. *Let H be a Hopf algebra. The set of primitive elements of H will be denoted by* $\mathbf{Lie}(H)$ *and is a Lie algebra. If* $\mathrm{char}(\ \ ) = p > 0$ *then* $\mathbf{Lie}(H)$ *is a restricted Lie algebra or a p-Lie algebra.*

Proof. Let $a, b \in H$ be primitive elements. Then $\Delta([a, b]) = \Delta(ab - ba) = (a \otimes 1 + 1 \otimes a)(b \otimes 1 + 1 \otimes b) - (b \otimes 1 + 1 \otimes b)(a \otimes 1 + 1 \otimes a) = (ab - ba) \otimes 1 + 1 \otimes (ab - ba)$ hence $\mathbf{Lie}(H) \subseteq H^L$ is a Lie algebra. If the characteristic of   is $p > 0$ then we have $(a \otimes 1 + 1 \otimes a)^p = a^p \otimes 1 + 1 \otimes a^p$. Thus $\mathbf{Lie}(H)$ is a restricted Lie subalgebra of H^L with the structure maps $[a, b] = ab - ba$ and $a^{[p]} = a^p$. □

Corollary 4.3.3. *Let H be a Hopf algebra. Then the set of left translation invariant derivations $D : H \longrightarrow H$ is a Lie algebra under $[D, D'] = DD' - D'D$. If* $\mathrm{char} = p$ *then these derivations are a restricted Lie algebra with $D^{[p]} = D^p$.*

Proof. The map $\Psi : H^o \longrightarrow H^* \xrightarrow{\Phi} \mathrm{End}(H)$ is a homomorphism of algebras by 4.2.6. Hence $\Psi(d * d' - d' * d) = \Phi(d * d' - d' * d) = \Phi(d)\Phi(d') - \Phi(d')\Phi(d)$. If d is a primitive element in H^o then by 4.2.7 and 4.3.1 the image $D := \Psi(d)$ in $\mathrm{End}(H)$ is a left translation invariant derivation and all left translation invariant derivations are of this form. Since $[d, d'] = d * d' - d' * d$ is again primitive we get that $[D, D'] = DD' - D'D$ is a left translation invariant derivation so that the set of left translation invariant derivations $\mathrm{Der}^H(H, H)$ is a Lie algebra resp. a restricted Lie algebra. □

Definition 4.3.4. Let H be a Hopf algebra. An element $c \in H$ is called *cocommutative* if $\tau\Delta(c) = \Delta(c)$, i. e. if $\sum c_{(1)} \otimes c_{(2)} = \sum c_{(2)} \otimes c_{(1)}$. Let $C(H) := \{c \in H | c \text{ is cocommutative }\}$.

Let $G(H)$ denote the set of group like elements of H.

Lemma 4.3.5. *Let H be a Hopf algebra. Then the set of cocommutative elements $C(H)$ is a subalgebra of H and the group like elements $G(H)$ form a linearly independent subset of $C(H)$. Furthermore $G(H)$ is a multiplicative subgroup of the group of units $U(C(H))$.*

PROOF. It is clear that $C(H)$ is a linear subspace of H. If $a, b \in C(H)$ then $\Delta(ab) = \Delta(a)\Delta(b) = (\tau\Delta)(a)(\tau\Delta)(b) = \tau(\Delta(a)\Delta(b)) = \tau\Delta(ab)$ and $\Delta(1) = 1 \otimes 1 = \tau\Delta(1)$. Thus $C(H)$ is a subalgebra of H.

The group like elements obviously are cocommutative and form a multiplicative group, hence a subgroup of $U(C(H))$. They are linearly independent by Lemma 2.1.14. $\square$

Proposition 4.3.6. *Let H be a Hopf algebra with $S^2 = \mathrm{id}_H$. Then there is a left module structure*

$$C(H) \otimes \mathbf{Lie}(H) \ni c \otimes a \mapsto c \cdot a \in \mathbf{Lie}(H)$$

with $c \cdot a := \nabla_H(\nabla_H \otimes 1)(1 \otimes \tau)(1 \otimes S \otimes 1)(\Delta \otimes 1)(c \otimes a) = \sum c_{(1)} a S(c_{(2)})$ *such that*

$$c \cdot [a, b] = \sum [c_{(1)} \cdot a, c_{(2)} \cdot b].$$

In particular $G(H)$ acts by Lie automorphisms on $\mathbf{Lie}(H)$.

PROOF. The given action is actually the action $H \otimes H \longrightarrow H$ with $h \cdot a = \sum h_{(1)} a S(h_{(2)})$, the so-called *adjoint action*.

We first show that the given map has image in $\mathbf{Lie}(H)$. For $c \in C(H)$ and $a \in \mathbf{Lie}(H)$ we have $\Delta(c \cdot a) = \Delta(\sum c_{(1)} a S(c_{(2)})) = \sum \Delta(c_{(1)})(a \otimes 1 + 1 \otimes a)\Delta(S(c_{(2)})) = \sum \Delta(c_{(1)})(a \otimes 1)\Delta(S(c_{(2)})) + \sum \Delta(c_{(2)})(1 \otimes a)\Delta(S(c_{(1)})) = \sum c_{(1)} a S(c_{(4)}) \otimes c_{(2)} S(c_{(3)}) + \sum c_{(3)} S(c_{(2)}) \otimes c_{(4)} a S(c_{(1)}) = c \cdot a \otimes 1 + 1 \otimes c \cdot a$ since c is cocommutative, $S^2 = \mathrm{id}_H$ and a is primitive.

We show now that $\mathbf{Lie}(H)$ is a $C(H)$-module. $(cd) \cdot a = \sum c_{(1)} d_{(1)} a S(c_{(2)} d_{(2)}) = \sum c_{(1)} d_{(1)} a S(d_{(2)}) S(c_{(2)}) = c \cdot (d \cdot a)$. Furthermore we have $1 \cdot a = 1 a S(1) = a$.

To show the given formula let $a, b \in \mathbf{Lie}(H)$ and $c \in C(H)$. Then $c \cdot [a, b] = \sum c_{(1)}(ab - ba)S(c_{(2)}) = \sum c_{(1)} a S(c_{(2)}) c_{(3)} b S(c_{(4)}) - \sum c_{(1)} b S(c_{(2)}) c_{(3)} a S(c_{(4)}) = \sum (c_{(1)} \cdot a)(c_{(2)} \cdot b) - \sum (c_{(1)} \cdot b)(c_{(2)} \cdot a) = \sum [c_{(1)} \cdot a, c_{(2)} \cdot b]$ again since $c \in C(H)$ is cocommutative.

Now let $g \in G(H)$. Then $g \cdot a = gaS(g) = gag^{-1}$ since $S(g) = g^{-1}$ for any group like element. Furthermore $g \cdot [a, b] = [g \cdot a, g \cdot b]$ hence g defines a Lie algebra automorphism of $\mathbf{Lie}(H)$. $\square$

Problem 4.3.2. Show that the *adjoint action* $H \otimes H \ni h \otimes a \mapsto \sum h_{(1)} a S(h_{(2)}) \in H$ makes H an H-module algebra.

Definition and Remark 4.3.7. The algebra $(\delta) = [\delta]/(\delta^2)$ is called the algebra of *dual numbers*. Observe that $(\delta) = \oplus \ \delta$ as a -module.

We consider δ as a "small quantity" whose square vanishes.

The maps $p : (\delta) \longrightarrow K$ with $p(\delta) = 0$ and $j : \ \longrightarrow (\delta)$ are algebra homomorphism satisfying $pj = \mathrm{id}$.

Let $(\delta, \delta') := [\delta, \delta']/(\delta^2, \delta'^2)$. Then $(\delta, \delta') = \oplus \ \delta \oplus \ \delta' \oplus \ \delta\delta'$. The map $(\delta) \ni \delta \mapsto \delta\delta' \in (\delta, \delta')$ is an injective algebra homomorphism. Furthermore for every $\alpha \in$ we have an algebra homomorphism $\varphi_\alpha : (\delta) \ni \delta \mapsto \alpha\delta \in (\delta)$.

These algebra homomorphisms induce algebra homomorphisms $H \otimes (\delta) \longrightarrow H \otimes (\delta)$ resp. $H \otimes (\delta) \longrightarrow H \otimes (\delta, \delta')$ for every Hopf algebra H.

Proposition 4.3.8. *The map*

$$e^{\delta -} : \mathbf{Lie}(H) \longrightarrow H \otimes \quad (\delta) \subseteq H \otimes \quad (\delta, \delta')$$

with $e^{\delta a} := 1 + a \otimes \delta = 1 + \delta a$ *is called the* exponential map *and satisfies*

$$e^{\delta(a+b)} = e^{\delta a} e^{\delta b},$$
$$e^{\delta \alpha a} = \varphi_\alpha(e^{\delta a}),$$
$$e^{\delta \delta'[a,b]} = e^{\delta a} e^{\delta' b} (e^{\delta a})^{-1} (e^{\delta' b})^{-1}.$$

Furthermore all elements $e^{\delta a} \in H \otimes \quad (\delta)$ *are group like in the* $\quad (\delta)$*-Hopf algebra* $H \otimes \quad (\delta)$*.*

Proof. 1. $e^{\delta(a+b)} = (1 + \delta(a+b)) = (1+\delta a)(1+\delta b) = e^{\delta a} e^{\delta b}$.

2. $e^{\delta\alpha a} = 1 + \delta\alpha a = \varphi_\alpha(1 + \delta a) = \varphi_\alpha(e^{\delta a})$.

3. Since $(1+\delta a)(1-\delta a) = 1$ we have $(e^{\delta a}) = 1 - \delta a$. So we get $e^{\delta\delta'[a,b]} = 1 + \delta[a,b] = 1 + \delta(a-a) + \delta'(b-b) + \delta\delta'(ab - ab - ba + ab) = (1+\delta a)(1+\delta' b)(1-\delta a)(1-\delta' b) = e^{\delta a} e^{\delta' b}(e^{\delta a})^{-1}(e^{\delta' b})^{-1}$.

4. $\Delta_{\ (\delta)}(e^{\delta a}) = \Delta(1 + a\otimes\delta) = 1 \otimes_{\ (\delta)} 1 + (a \otimes 1 + 1 \otimes a)\otimes \delta = 1 \otimes_{\ (\delta)} 1 + \delta a \otimes_{\ (\delta)} 1 + 1 \otimes_{\ (\delta)} \delta a + \delta a \otimes_{\ (\delta)} \delta a = (1+\delta a) \otimes_{\ (\delta)} (1 + \delta a) = e^{\delta a} \otimes_{\ (\delta)} e^{\delta a}$ and $\varepsilon_{\ (\delta)}(e^{\delta a}) = \varepsilon_{\ (\delta)}(1 + \delta a) = 1 + \delta\varepsilon(a) = 1$. $\square$

Corollary 4.3.9. $(\mathbf{Lie}(H), e^{\delta -})$ *is the kernel of the group homomorphism* $p : G_{\ (\delta)}(H \otimes \quad (\delta)) \longrightarrow G(H)$*.*

Proof. $p = 1 \otimes p : H \otimes \quad (\delta) \longrightarrow H \otimes \quad = H$ is a homomorphism of -algebras. We show that it preserves group like elements. Observe that group like elements in $H \otimes \quad (\delta)$ are defined by the Hopf algebra structure over $\quad (\delta)$. Let $g \in G_{\ (\delta)}(H \otimes \quad (\delta))$. Then $(\Delta_H \otimes 1)(g) = g \otimes_{\ (\delta)} g$ and $(\varepsilon_H \otimes 1)(g) = 1 \in \quad (\delta)$.

Since $p : \quad (\delta) \longrightarrow$ is an algebra homomorphism the following diagram commutes

$$\begin{array}{ccc}
(H \otimes \quad (\delta)) \otimes_{\ (\delta)} (H \otimes \quad (\delta)) & \xrightarrow{\cong} & H \otimes H \otimes \quad (\delta) \\
\Big\downarrow {\scriptstyle (1\otimes p)\otimes(1\otimes p)} & & \Big\downarrow {\scriptstyle 1\otimes p} \\
(H \otimes \quad) \otimes (H \otimes \quad) & \xrightarrow[\cong]{} & H \otimes H \otimes \quad .
\end{array}$$

We identify elements along the isomorphisms. Thus we get $(\Delta_H \otimes 1_{\ })(1_H \otimes p)(g) = (1_{H\otimes H} \otimes p)(\Delta_H \otimes 1_{\ (\delta)})(g) = ((1_H \otimes p) \otimes_{\ (\delta)} (1_H \otimes p))(g \otimes_{\ (\delta)} g) = (1_H \otimes p)(g) \otimes (1_H \otimes p)(g)$, so that $1_H \otimes p : G_{\ (\delta)}(H \otimes \quad (\delta)) \longrightarrow G(H)$. Now we have $(1_H \otimes p)(gg') = (1_H \otimes p)(g)(1_H \otimes p)(g')$ so that $1_H \otimes p$ is a group homomorphism.

Now let $g = g_0 \otimes 1 + g_1 \otimes \delta \in G_{\ (\delta)}(H \otimes \quad (\delta)) \subseteq H \otimes \quad \oplus H \otimes \quad \delta$. Then we have $(1_H \otimes p)(g) = 1$ iff $g_0 = 1$ iff $g = 1_H \otimes 1_{\ (\delta)} + g_1 \otimes \delta$. Furthermore we have

$$\begin{aligned}
&\Delta_{H\otimes\ (\delta)}(g) = g \otimes_{\ (\delta)} g \iff \\
&1_H \otimes 1_H \otimes 1_{\ (\delta)} + \Delta_H(g_1) \otimes \delta = (1_H \otimes 1_{\ (\delta)} + g_1 \otimes \delta) \otimes_{\ (\delta)} (1_H \otimes 1_{\ (\delta)} + g_1 \otimes \delta) \\
&\qquad\qquad = 1_H \otimes 1_H \otimes 1_{\ (\delta)} + (g_1 \otimes 1_H + 1_H \otimes g_1) \otimes \delta \iff \\
&\Delta_H(g_1) = g_1 \otimes 1_H + 1_H \otimes g_1.
\end{aligned}$$

Similarly we have $\varepsilon \ _{(\delta)})(g) = 1$ iff $1 \otimes 1 + \varepsilon(g_1) \otimes \delta = 1$ iff $\varepsilon(g_1) = 0$. $\square$

4. Derivations and Lie Algebras of Affine Algebraic Groups

Lemma and Definition 7.4.1. *Let $\mathcal{G}$:* **-cAlg $\longrightarrow$ Set** *be a group valued functor. The kernel $\mathcal{L}ie\,(\mathcal{G})(R)$ of the sequence*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}ie\,(\mathcal{G})(R) \arrow[r] & \mathcal{G}(R(\delta)) \arrow[r, shift left, "\mathcal{G}(p)"] & \mathcal{G}(R) \arrow[l, shift left, "\mathcal{G}(j)"] \arrow[r] & 0 \end{tikzcd}$$

is called the Lie algebra *of $\mathcal{G}$ and is a group valued functor in R.*

Proof. For every algebra homomorphism $f : R \longrightarrow S$ the following diagram of groups commutes

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}ie\,(\mathcal{G})(R) \arrow[r] \arrow[d] & \mathcal{G}(R(\delta)) \arrow[r, shift left, "\mathcal{G}(p)"] \arrow[d, "\mathcal{G}(f(\delta))"] & \mathcal{G}(R) \arrow[l, shift left, "\mathcal{G}(j)"] \arrow[r] \arrow[d, "\mathcal{G}(f)"] & 0 \\ 0 \arrow[r] & \mathcal{L}ie\,(\mathcal{G})(S) \arrow[r] & \mathcal{G}(S(\delta)) \arrow[r, shift left, "\mathcal{G}(p)"] & \mathcal{G}(S) \arrow[l, shift left, "\mathcal{G}(j)"] \arrow[r] & 0 \end{tikzcd}$$

□

Proposition 4.4.2. *Let $\mathcal{G}$:* **-cAlg $\longrightarrow$ Set** *be a group valued functor with multiplication $*$. Then there are functorial operations*

$$\mathcal{G}(R) \times \mathcal{L}ie\,(\mathcal{G})(R) \ni (g, x) \mapsto g \cdot x \in \mathcal{L}ie\,(\mathcal{G})(R)$$

$$R \times \mathcal{L}ie\,(\mathcal{G})(R) \ni (a, x) \mapsto ax \in \mathcal{L}ie\,(\mathcal{G})(R)$$

such that

$$\begin{aligned} g \cdot (x + y) &= g \cdot x + g \cdot y, \\ h \cdot (g \cdot x) &= (h * g) \cdot x, \\ a(x + y) &= ax + ay, \\ (ab)x &= a(bx), \\ g \cdot (ax) &= a(g \cdot x). \end{aligned}$$

Proof. First observe that the composition $+$ on $\mathcal{L}ie\,(\mathcal{G})(R)$ is induced by the multiplication $*$ of $\mathcal{G}(R(\delta))$ so it is not necessarily commutative.

We define $g \cdot x := \mathcal{G}(j)(g) * x * \mathcal{G}(j)(g)^{-1}$. Then $\mathcal{G}(p)(g \cdot x) = \mathcal{G}(p)\mathcal{G}(j)(g) * \mathcal{G}(p)(x) * \mathcal{G}(p)\mathcal{G}(j)(g)^{-1} = g * 1 * g^{-1} = 1$ hence $g \cdot x \in \mathcal{L}ie\,(\mathcal{G})(R)$.

Now let $a \in R$. To define $a : \mathcal{L}ie\,(\mathcal{G})(R) \longrightarrow \mathcal{L}ie\,(\mathcal{G})(R)$ we use $u_a : R(\delta) \longrightarrow R(\delta)$ defined by $u_a(\delta) := a\delta$ and thus $u_a(b + c\delta) := b + ac\delta$. Obviously u_a is a homomorphism of R-algebras. Furthermore we have $pu_a = p$ and $u_a j = j$. Thus we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}ie\,(\mathcal{G})(R) \arrow[r] \arrow[d, "a"] & \mathcal{G}(R(\delta)) \arrow[r, shift left, "\mathcal{G}(p)"] \arrow[d, "\mathcal{G}(u_a)"] & \mathcal{G}(R) \arrow[l, shift left, "\mathcal{G}(j)"] \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mathcal{L}ie\,(\mathcal{G})(R) \arrow[r] & \mathcal{G}(R(\delta)) \arrow[r, shift left, "\mathcal{G}(p)"] & \mathcal{G}(R) \arrow[l, shift left, "\mathcal{G}(j)"] \arrow[r] & 0 \end{tikzcd}$$

that defines a group homomorphism $a : \mathcal{L}ie(\mathcal{G})(R) \longrightarrow \mathcal{L}ie(\mathcal{G})(R)$ on the kernel of the exact sequences. In particular we have then $a(x+y) = ax + ay$.

Furthermore we have $u_{ab} = u_a u_b$ hence $(ab)x = a(bx)$.

The next formula follows from $g \cdot (x+y) = \mathcal{G}(j)(g) * x * y * \mathcal{G}(j)(g)^{-1} = \mathcal{G}(j)(g) * x * \mathcal{G}(j)(g)^{-1} * \mathcal{G}(j)(g) * y * \mathcal{G}(j)(g)^{-1} = g \cdot x + g \cdot y$.

We also see $(h*g)\cdot x = \mathcal{G}(j)(h*g)*x*\mathcal{G}(j)(h*g)^{-1} = \mathcal{G}(j)(h)*\mathcal{G}(j)(g)*x*\mathcal{G}(j)(g)^{-1}* \mathcal{G}(j)(h)^{-1} = h \cdot (g \cdot x)$. Finally we have $g \cdot (ax) = \mathcal{G}(j)(g) * \mathcal{G}(u_a)(x) * \mathcal{G}(j)(g^{-1}) = \mathcal{G}(u_a)(\mathcal{G}(j)(g) * x * \mathcal{G}(j)(g^{-1})) = a(g \cdot x)$. $\square$

Proposition 4.4.3. *Let $\mathcal{G} = \ $-$\mathbf{cAlg}(H,\text{-})$ be an affine algebraic group. Then $\mathcal{L}ie(\mathcal{G})(\ \ ) \cong \mathbf{Lie}(H^o)$ as additive groups. The isomorphism is compatible with the operations given in 4.4.2 and 4.3.6.*

Proof. We consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}ie(\mathcal{G})(\ \ ) \arrow[r] & \text{-}\mathbf{cAlg}(H,\ \ (\delta)) \arrow[r, shift left, "p"] \arrow[d, "\omega"] & \text{-}\mathbf{cAlg}(H,\ \ ) \arrow[l, shift left] \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \mathbf{Lie}(H^o) \arrow[r, "e^{\delta-}"] \arrow[u, "e"] & G_{\ (\delta)}(H^o \otimes\ \ (\delta)) \arrow[r, "p"] & G(H^o) \arrow[r] & 0 \end{tikzcd}$$

We know by definition that the top sequence is exact. The bottom sequence is exact by Corollary 4.3.9.

Let $f \in\ $-$\mathbf{cAlg}(H,\ \ )$. Since $\mathrm{Ker}(f)$ is an ideal of codimension 1 we get $f \in H^o$. The map f is an algebra homomorphism iff $\langle f, ab\rangle = \langle f \otimes f, a \otimes b\rangle$ and $\langle f, 1\rangle = 1$ iff $\Delta_{H^o}(f) = f \otimes f$ and $\varepsilon_{H^o}(f) = 1$ iff $f \in G(H^o)$. Hence we get the right hand vertical isomorphism $\ $-$\mathbf{cAlg}(H,\ \ ) \cong G(H^o)$.

Consider an element $f \in\ $-$\mathbf{cAlg}(H,\ \ (\delta)) \subseteq \mathrm{Hom}(H,\ \ (\delta))$. It can be written as $f = f_0 + f_1\delta$ with $f_0, f_1 \in \mathrm{Hom}(H,\ \ )$. The linear map f is an algebra homomorphism iff $f_0 : H \longrightarrow\ \ $ is an algebra homomorphism and f_1 satisfies $f_1(1) = 0$ and $f_1(ab) = f_0(a)f_1(b) + f_1(a)f_0(b)$. In fact we have $f(1) = f_0(1) + f_1(1)\delta = 1$ iff $f_0(1) = 1$ and $f_1(1) = 0$ (by comparing coefficients). Furthermore we have $f(ab) = f(a)f(b)$ iff $f_0(ab) + f_1(ab)\delta = (f_0(a) + f_1(a)\delta)(f_0(b) + f_1(b)\delta) = f_0(a)f_0(b) + f_0(a)f_1(b)\delta + f_1(a)f_0(b)\delta$ iff $f_0(ab) = f_0(a)f_0(b)$ and $f_1(ab) = f_0(a)f_1(b) + f_1(a)f_0(b)$.

Since f_0 is an algebra homomorphism we have as above $f_0 \in H^o$. For f_1 we have $(b \rightharpoonup f_1)(a) = f_1(ab) = f_0(a)f_1(b) + f_1(a)f_0(b) = (f_1(b)f_0 + f_0(b)f_1)(a)$ hence $(b \rightharpoonup f_1) = f_1(b)f_0 + f_0(b)f_1 \in\ \ f_0 +\ \ f_1$, a two dimensional subspace. Thus $f_1 \in H^o$.

In the following computations we will identify $(H^o \otimes\ \ (\delta)) \otimes_{\ (\delta)} (H^o \otimes\ \ (\delta))$ with $H^o \otimes H^o \otimes |K(\delta)$.

Let $f = f_0 + f_1\delta = f_0 \otimes 1 + f_1 \otimes \delta \in H^o \oplus H^o\delta = H^o \otimes\ \ (\delta)$. Then f is a homomorphism of algebras iff $f(ab) = f(a)f(b)$ and $f(1) = 1$ iff $f_0(ab) = f_0(a)f_0(b)$ and $f_1(ab) = f_0(a)f_1(b) + f_1(a)f_0(b)$ and $f_0(1) = 1$ and $f_1(1) = 0$ iff $\Delta_{H^o}(f_0) = f_0 \otimes f_0$ and $\Delta_{H^o}(f_1) = f_0 \otimes f_1 + f_1 \otimes f_0$ and $\varepsilon_{H^o}(f_0) = 1$ and $\varepsilon_{H^o}(f_1) = 0$ iff $(\Delta_{H^o} \otimes \mathrm{id}_{\ (\delta)})(f_0 \otimes 1 + f_1 \otimes \delta) = f_0 \otimes f_0 \otimes 1 + f_0 \otimes f_1 \otimes \delta + f_1 \otimes f_0 \otimes \delta = (f_0 \otimes 1 + f_1 \otimes \delta) \otimes_{\ (\delta)} (f_0 \otimes 1 + f_1 \otimes \delta)$

and $(\varepsilon_{H^o} \otimes \mathrm{id}_{\ (\delta)})(f_0 \otimes 1 + f_1 \otimes \delta) = 1 \otimes 1$ iff $(\Delta_{H^o} \otimes \mathrm{id}_{\ (\delta)})(f) = f \otimes_{\ (\delta)} f$ and $(\varepsilon_{H^o} \otimes \mathrm{id}_{\ (\delta)})(f) = 1$ iff $f \in G_{\ (\delta)}(H^o \otimes \ (\delta))$.

Hence we have a bijective map $\omega : \ \text{-}\mathbf{cAlg}(H, \ (\delta)) \ni f = f_0 + f_1\delta \mapsto f_0 \otimes 1 + f_1 \otimes \delta \in G_{\ (\delta)}(H^o \otimes \ (\delta))$. Since the group multiplication in $\ \text{-}\mathbf{cAlg}(H, \ (\delta)) \subseteq \mathrm{Hom}(H, \ (\delta))$ is the convolution $*$ and the group multiplication in $G_{\ (\delta)}(H^o \otimes \ (\delta)) \subseteq H^o \otimes \ (\delta)$ is the ordinary algebra multiplication, where the multiplication of H^o again is the convolution, it is clear that ω is a group homomorphism. Furthermore the right hand square of the above diagram commutes. Thus we get an isomorphism $e : \mathbf{Lie}(H^o) \longrightarrow \mathcal{L}ie(\mathcal{G})(\ )$ on the kernels. This map is defined by $e(d) = 1 + d\delta \in \ \text{-}\mathbf{cAlg}(H, \ (\delta))$.

To show that this isomorphism is compatible with the actions of resp. $G(H^o)$ let $\alpha \in \ $, $a \in H$, and $d \in \mathbf{Lie}(H^o)$. We have $e(\alpha d)(a) = \varepsilon(a) + \alpha d(a)\delta = u_\alpha(\varepsilon(a) + d(a)\delta) = (u_\alpha \circ (1 + d\delta))(a) = (u_\alpha \circ e(d))(a) = (\alpha e(d))(a)$ hence $e(\alpha d) = \alpha e(d)$.

Furthermore let $g \in G(H^o) = \ \text{-}\mathbf{cAlg}(H, \ )$, $a \in H$, and $d \in \mathbf{Lie}(H^o)$. Then we have $e(g \cdot d)(a) = e(gdg^{-1})(a) = (1 + gdg^{-1}\delta)(a) = \varepsilon(a) + gdg^{-1}(a)\delta = \sum g(a_{(1)})\varepsilon(a_{(2)})gS(a_{(3)}) + \sum g(a_{(1)})d(a_{(2)})gS(a_{(3)})\delta = \sum g(a_{(1)})e(d)(a_{(2)})gS(a_{(3)}) = (j \circ g * e(d) * j \circ g^{-1})(a) = (g \cdot e(d))(a)$ hence $e(g \cdot d) = g \cdot e(d)$. □

Proposition 4.4.4. *Let H be a Hopf algebra and let $I := \mathrm{Ker}(\varepsilon)$. Then $\mathrm{Der}_\varepsilon(H, \text{-}) : \mathbf{Vec} \longrightarrow \mathbf{Vec}$ is representable by I/I^2 and $d : H \xrightarrow{1-\varepsilon} I \xrightarrow{\nu} I/I^2$, in particular*

$$\mathrm{Der}_\varepsilon(H, \text{-}) \cong \mathrm{Hom}(I/I^2, \text{-}) \qquad \textit{and} \qquad \mathbf{Lie}(H^o) \cong \mathrm{Hom}(I/I^2, \ ).$$

Proof. Because of $\varepsilon(\mathrm{id} - u\varepsilon)(a) = \varepsilon(a) - \varepsilon u \varepsilon(a) = 0$ we have $\mathrm{Im}(\mathrm{id} - \varepsilon) \subseteq I$. Let $i \in I$. Then we have $i = i - \varepsilon(i) = (\mathrm{id} - \varepsilon)(i)$ hence $\mathrm{Im}(\mathrm{id} - \varepsilon) = \mathrm{Ker}(\varepsilon)$. We have $I^2 \ni (\mathrm{id} - \varepsilon)(a)(\mathrm{id} - \varepsilon)(b) = ab - \varepsilon(a)b - a\varepsilon(b) + \varepsilon(a)\varepsilon(b) = (\mathrm{id} - \varepsilon)(ab) - \varepsilon(a)(\mathrm{id} - \varepsilon)(b) - (\mathrm{id} - \varepsilon)(b)$. Hence we have in I/I^2 the equation $(\mathrm{id} - \varepsilon)(ab) = \varepsilon(a)(\mathrm{id} - \varepsilon)(b) + (\mathrm{id} - \varepsilon)(a)\varepsilon(b)$ so that $\nu(\mathrm{id} - \varepsilon) : H \longrightarrow I \longrightarrow I/I^2$ is an ε-derivation.

Now let $D : H \longrightarrow M$ be an ε-derivation. Then $D(1) = D(1\,1) = 1D(1) + D(1)1$ hence $D(1) = 0$. It follows $D(a) = D(\mathrm{id} - \varepsilon)(a)$. From $\varepsilon(I) = 0$ we get $D(I^2) \subseteq \varepsilon(I)D(I) + D(I)\varepsilon(I) = 0$ hence there is a unique factorization

$$\begin{tikzcd}
H \arrow[r, "\mathrm{id} - \varepsilon"] \arrow[drr, "D"'] & I \arrow[r, "\nu"] \arrow[dr, "D"'] & I/I^2 \arrow[d, "f"] \\
 & & M.
\end{tikzcd}$$

□

Corollary 4.4.5. *Let H be a Hopf algebra that is finitely generated a s an algebra. Then $\mathbf{Lie}(H^o)$ is finite dimensional.*

Proof. Let $H = \ \langle a_1, \ldots, a_n \rangle$. Since $H = \ \oplus I$ we can choose $a_1 = 1$ and $a_2, \ldots, a_n \in I$. Thus any element in $i \in I$ can be written as $\sum \alpha_J a_{j_1} \ldots a_{j_k}$ so that $I/I^2 = \ \overline{a_2} + \ldots + \overline{a_n}$. This gives the result. □

Proposition 4.4.6. *Let H be a commutative Hopf algebra and ${}_HM$ be an H-module. Then we have $\Omega_H \cong H \otimes I/I^2$ and $d : H \longrightarrow H \otimes I/I^2$ is given by $d(a) = \sum a_{(1)} \otimes \overline{(\mathrm{id} - \varepsilon)(a_{(2)})}$.*

Proof. Consider the algebra $B := H \oplus M$ with $(a, m)(a', m') = (aa', am' + a'm)$. Let $\mathcal{G} = \ $ -**cAlg**$(H, \text{-})$. Then we have $\mathcal{G}(B) \subseteq \mathrm{Hom}(H, B) \cong \mathrm{Hom}(H, H) \oplus \mathrm{Hom}(H, M)$. An element $(\varphi, D) \in \mathrm{Hom}(H, B)$ is in $\mathcal{G}(B)$ iff $(\varphi, D)(1) = (\varphi(1), D(1)) = (1, 0)$, hence $\varphi(1) = 1$ and $D(1) = 0$, and $(\varphi(ab), D(ab)) = (\varphi, D)(ab) = (\varphi, D)(a)(\varphi, D)(b) = (\varphi(a), D(a))(\varphi(b), D(b)) = (\varphi(a)\varphi(b), \varphi(a)D(b) + D(a)\varphi(b)$, hence $\varphi(ab) = \varphi(a)\varphi(b)$ and $D(ab) = \varphi(a)D(b) + D(a)\varphi(b)$. So (φ, D) is in $\mathcal{G}(B)$ iff $\varphi \in \mathcal{G}(H)$ and D is a φ-derivation. The $*$-multiplication in $\mathrm{Hom}(H, B)$ is given by $(\varphi, D) * (\varphi', D') = (\varphi * \varphi', \varphi * D' + D * \varphi')$ by applying this to an element $a \in H$. Since $(\varphi, 0) \in \mathcal{G}(B)$ and $(u\varepsilon, D) \in \mathcal{G}(B)$ for every ε-derivation D, there is a bijection $\mathrm{Der}_\varepsilon(H, M) \cong \{(u\varepsilon, D_\varepsilon) \in \mathcal{G}_\varepsilon(B)\} \cong \{(1_H, D_1) \in \mathcal{G}_1(B)\} \cong \mathrm{Der}\ (H, M)$ by $(u\varepsilon, D_\varepsilon) \mapsto (1, 0) * (u\varepsilon, D_\varepsilon) = (1, 1 * D_\varepsilon) \in \mathcal{G}_1(B)$ with inverse map $(1, D_1) \mapsto (S, 0) * (1, D_1) = (u\varepsilon, S * D_1) \in \mathcal{G}_\varepsilon(B)$. Hence we have isomorphisms $\mathrm{Der}\ (H, M) \cong \mathrm{Der}_\varepsilon(H, M) \cong \mathrm{Hom}(I/I^2, M) \cong \mathrm{Hom}_H(H \otimes I/I^2, M)$.

The universal ε-derivation for vector spaces is $\overline{\mathrm{id} - \varepsilon} : A \longrightarrow I/I^2$. The universal ε-derivation for H-modules is $D_\varepsilon(a) = 1 \otimes \overline{(\mathrm{id} - \varepsilon)(a)} \in A \otimes I/I^2$. The universal 1-derivation for H-modules is $1 * D_\varepsilon$ with $(1 * D_\varepsilon)(a) = \sum a_{(1)} \otimes \overline{(\mathrm{id} - \varepsilon)(a_{(2)})} \in A \otimes I/I^2$. $\square$