CHAPTER 3

Hopf Algebras, Algebraic, Formal, and Quantum Groups

8. Reconstruction and $\mathcal{C}$-categories

Now we show that an arbitrary coalgebra C can be reconstructed by the methods introduced above from its (co-)representations or more precisely from the underlying functor $\omega : \textbf{Comod-}C \to \textbf{Vec}$. In this case one can not use the usual construction of $\mathrm{coend}(\omega)$ that is restricted to finite dimensional comodules.

The following Theorem is an example that shows that the restriction to finite dimensional comodules in general is too strong for Tannaka reconstruction. There may be universal coendomorphism bialgebras for more general diagrams. On the other hand the following Theorem also holds if one only considers finite dimensional corepresentations of C. However the proof then becomes somewhat more complicated.

Definition 3.8.1. Let $\mathcal{C}$ be a monoidal category. A category $\mathcal{D}$ together with a bifunctor $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ and natural isomorphisms $\beta : (A \otimes B) \otimes M \to A \otimes (B \otimes M)$, $\eta : I \otimes M \to M$ is called a *$\mathcal{C}$-category* if the following diagrams commute

$$\begin{tikzcd}
((A\otimes B)\otimes C)\otimes M \arrow[r, "\alpha(A,B,C)\otimes 1"] \arrow[d, "\beta(A\otimes B,C,M)"] & (A\otimes(B\otimes C))\otimes M \arrow[r, "\beta(A,B\otimes C,M)"] & A\otimes((B\otimes C)\otimes M) \arrow[d, "1\otimes\beta(B,C,M)"'] \\
(A\otimes B)\otimes(C\otimes M) \arrow[rr, "\beta(A,B,C\otimes M)"] & & A\otimes(B\otimes(C\otimes M))
\end{tikzcd}$$

$$\begin{tikzcd}
(A\otimes I)\otimes M \arrow[rr, "\beta(A,I,M)"] \arrow[dr, "\rho(A)\otimes 1"'] & & A\otimes(I\otimes M) \arrow[dl, "1\otimes\eta(M)"] \\
& A\otimes M &
\end{tikzcd}$$

A $\mathcal{C}$-category is called *strict* if the morphisms β, η are the identities.

Let $(\mathcal{D}, \otimes)$ and $(\mathcal{D}', \otimes)$ be $\mathcal{C}$-categories. A functor $\mathcal{F} : \mathcal{D} \to \mathcal{D}'$ together with a natural transformation $\zeta(A, M) : A \otimes \mathcal{F}(M) \to \mathcal{F}(A \otimes M)$ is called a *weak $\mathcal{C}$-functor* if the following diagrams commute

$$\begin{tikzcd}
(A\otimes B)\otimes\mathcal{F}(M) \arrow[rr, "\zeta"] \arrow[d, "\beta"'] & & \mathcal{F}((A\otimes B)\otimes M) \arrow[d, "\mathcal{F}(\beta)"] \\
A\otimes(B\otimes\mathcal{F}(M)) \arrow[r, "1\otimes\zeta"] & A\otimes\mathcal{F}(B\otimes M) \arrow[r, "\zeta"] & \mathcal{F}(A\otimes(B\otimes M))
\end{tikzcd}$$

$$\begin{tikzcd}
I\otimes\mathcal{F}(M) \arrow[rr, "\zeta"] \arrow[dr, "\eta"'] & & \mathcal{F}(I\otimes M) \arrow[dl, "\mathcal{F}(\eta)"] \\
& \mathcal{F}(M) &
\end{tikzcd}$$

If, in addition, ζ is an isomorphism then we call $\mathcal{F}$ a $\mathcal{C}$-functor. The functor is called a *strict $\mathcal{C}$-functor* if ζ is the identity morphism.

A natural transformation $\varphi : \mathcal{F} \to \mathcal{F}'$ between (weak) $\mathcal{C}$-functors is called a *$\mathcal{C}$-transformation* if

$$\begin{CD} A \otimes \mathcal{F}(M) @>{\zeta}>> \mathcal{F}(A \otimes M) \\ @V{1_A \otimes \varphi(M)}VV @VV{\varphi(A\otimes M)}V \\ A \otimes \mathcal{F}'(M) @>>{\zeta'}> \mathcal{F}'(A \otimes M) \end{CD}$$

commutes.

Example 3.8.2. Let C be a coalgebra and $\mathcal{C} := \mathbf{Vec}$. Then the category $\mathbf{Comod}$-C of right C-comodules is a $\mathcal{C}$-category since $N \in \mathbf{Comod}$-C and $V \in \mathcal{C} = \mathbf{Vec}$ implies that $V \otimes N$ is a comodules with the comodule structure of N.

The underlying functor $\omega : \mathbf{Comod}\text{-}C \to \mathbf{Vec}$ is a strict $\mathcal{C}$-functor since we have $V \otimes \omega(N) = \omega(V \otimes N)$. Similarly $\omega \otimes M : \mathbf{Comod}\text{-}C \to \mathbf{Vec}$ is a $\mathcal{C}$-functor since $V \otimes (\omega(N) \otimes M) \cong \omega(V \otimes N) \otimes M$.

Lemma 3.8.3. *Let C be a coalgebra. Let $\omega : \mathbf{Comod}\text{-}C \to \mathbf{Vec}$ be the underlying functor. Let $\varphi : \omega \to \omega \otimes M$ be a natural transformation. Then φ is a $\mathcal{C}$-transformation with $\mathcal{C} = \mathbf{Vec}$.*

Proof. It suffices to show $1_V \otimes \varphi(N) = \varphi(V \otimes N)$ for an arbitrary comodule N. We show that the diagram

$$\begin{CD} V \otimes N @>{\varphi(V\otimes N)}>> V \otimes N \otimes M \\ @V{1}VV @VV{1}V \\ V \otimes N @>>{1_V \otimes \varphi(N)}> V \otimes N \otimes M \end{CD}$$

commutes. Let (v_i) be a basis of V. For an arbitrary vector space W let $p_i : V \otimes W \to W$ be the projections defined by $p_i(t) = p_i(\sum_j v_j \otimes w_j) = w_i$ where $\sum_j v_j \otimes w_j$ is the unique representation of an arbitrary tensor in $V \otimes W$. So we get

$$t = \sum_i v_i \otimes p_i(t)$$

for all $t \in V \otimes W$. Now we consider $V \otimes N$ as a comodule by the comodule structure of N. Then the $p_i : V \otimes N \to N$ are homomorphisms of comodules. Hence all diagrams of the form

$$\begin{CD} V \otimes N @>{\varphi(V\otimes N)}>> V \otimes N \otimes M \\ @V{p_i}VV @VV{p_i \otimes M}V \\ N @>>{\varphi(N)}> N \otimes M. \end{CD}$$

commute. Expressed in formulas this means $\varphi(N)p_i(t) = p_i\varphi(V \otimes N)(t)$ for all $t \in V \otimes N$. Hence we have

$$\begin{aligned}(1_V \otimes \varphi(N))(t) &= (1_V \otimes \varphi(N))(\sum v_i \otimes p_i(t)) = \sum v_i \otimes \varphi(N)p_i(t) \\ &= \sum_i v_i \otimes p_i\varphi(V \otimes N)(t) = \varphi(V \otimes N)(t)\end{aligned}$$

So we have $1_V \otimes \varphi(N) = \varphi(V \otimes N)$ as claimed. $\square$

We prove the following Theorem only for the category $\mathcal{C} = \mathbf{Vec}$ of vector spaces. The Theorem holds in general and says that in an arbitrary symmetric monoidal category $\mathcal{C}$ the coalgebra C represents the functor $\mathcal{C}\text{-}\operatorname{Nat}(\omega, \omega \otimes M) \cong \operatorname{Mor}_{\mathcal{C}}(C, M)$ of natural $\mathcal{C}$-transformations.

Theorem 3.8.4. (Reconstruction of coalgebras) *Let C be a coalgebra. Let ω : $\mathbf{Comod}\text{-}C \to \mathbf{Vec}$ be the underlying functor. Then $C \cong \operatorname{coend}(\omega)$.*

Proof. Let M in $\mathbf{Vec}$ and let $\varphi : \omega \to \omega \otimes M$ be a natural transformation. We define the homomorphism $\widetilde{\varphi} : C \to M$ by $\widetilde{\varphi} = (\epsilon \otimes 1)\varphi(C)$ using the fact that C is a comodule.

Let N be a C-comodule. Then N is a subcomodule of $N \otimes C$ by $\delta : N \to N \otimes C$ since the diagram

$$\begin{tikzcd} N \arrow[r, "\delta"] \arrow[d, "\delta"'] & N \otimes C \arrow[d, "1\otimes\Delta"] \\ N \otimes C \arrow[r, "\delta\otimes 1"'] & N \otimes C \otimes C \end{tikzcd}$$

commutes. Thus the following diagram commutes

$$\begin{tikzcd} N \arrow[r, "\delta"] \arrow[d, "\varphi(N)"'] & N \otimes C \arrow[d, "\varphi(N\otimes C)=1_N\otimes\varphi(C)"'] \arrow[ddr, "1\otimes\widetilde{\varphi}"] & \\ N \otimes M \arrow[r, "\delta\otimes 1"] \arrow[drr, "1"] & N \otimes C \otimes M \arrow[dr, "1\otimes\epsilon\otimes 1"] & \\ & & N \otimes M \end{tikzcd}$$

In particular we have shown that the diagram

$$\begin{tikzcd} \omega \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \omega \otimes C \arrow[d, "1\otimes\widetilde{\varphi}"] \\ & \omega \otimes M \end{tikzcd}$$

commutes.

To show the uniqueness of $\widetilde{\varphi}$ let $g : C \longrightarrow M$ be another homomorphism with $(1 \otimes g)\delta = \varphi$. For $c \in C$ we have $g(c) = g(\epsilon \otimes 1)\Delta(c) = (\epsilon \otimes 1)(1 \otimes g)\Delta(c) = (\epsilon \otimes 1)\varphi(C)(c) = \widetilde{\varphi}(c)$.

The coalgebra structure from Corollary 3.5.1 is the original coalgebra structure of C. This can be seen as follows. The comultiplication $\delta : \omega \longrightarrow \omega \otimes C$ is a natural transformation hence $(\delta \otimes 1_C)\delta : \omega \longrightarrow \omega \otimes C \otimes C$ is also a natural transformation. As in Corollary 3.5.1 this induced a unique homomorphism $\Delta : C \longrightarrow C \otimes C$ so that the diagram

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[d, "\delta"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\Delta"] \\
\omega \otimes \mathrm{coend}(\omega) \arrow[r, "\delta\otimes 1"'] & \omega \otimes \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega)
\end{tikzcd}$$

commutes. In a similar way the natural isomorphism $\omega \cong \omega \otimes \mathbb{K}$ induces a unique homomorphism $\epsilon : C \longrightarrow \mathbb{K}$ so that the diagram

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[dr, "\mathrm{id}_\omega"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\epsilon"] \\
 & \omega \otimes I
\end{tikzcd}$$

commutes. Because of the uniqueness these must be the structure homomorphisms of C. $\square$

We need a more general version of this Theorem in the next chapter. So let C be a coalgebra. Let $\omega : \mathbf{Comod}\text{-}C \longrightarrow \mathbf{Vec}$ be the underlying functor and $\delta : \omega \longrightarrow \omega \otimes C$ the universal natural transformation for $C \cong \mathrm{coend}(\omega)$.

We use the permutation map τ on the tensor product that gives the natural isomorphism

$$\tau : N_1 \otimes T_1 \otimes N_2 \otimes T_2 \otimes \ldots \otimes N_n \otimes T_n \cong N_1 \otimes N_2 \otimes \ldots \otimes N_n \otimes T_1 \otimes T_2 \ldots \otimes \otimes T_n$$

which is uniquely determined by the coherence theorems and is constructed by suitable applications of the flip $\tau : N \otimes T \cong T \otimes N$.

Let $\omega^n : \mathbf{Comod}\text{-}C \times \mathbf{Comod}\text{-}C \times \ldots \times \mathbf{Comod}\text{-}C \longrightarrow \mathbf{Vec}$ be the functor $\omega^n(N_1, N_2, \ldots, N_n) = \omega(N_1) \otimes \omega(N_2) \otimes \ldots \otimes \omega(N_n)$. For notational convenience we abbreviate $\{N\}^n := N_1 \otimes N_2 \otimes \ldots \otimes N_n$, similarly $\{C\}^n = C \otimes C \otimes \ldots \otimes C$ and $\{f\}^n := f_1 \otimes f_2 \otimes \ldots \otimes f_n$. So we get $\tau : \{N \otimes T\}^n \cong \{N\}^n \otimes \{T\}^n$.

Lemma 3.8.5. *Let $\varphi : \omega^n \to \omega^n \otimes M$ be a natural transformation. Then φ is a $\mathcal{C}$-transformation in the sense that the diagrams*

$$\begin{tikzcd}
\{V \otimes N\}^n \arrow[r, "\varphi(V_1 \otimes N_1, \ldots, V_n \otimes N_n)"] \arrow[d, "\tau"'] & \{V \otimes N\}^n \otimes M \arrow[d, "\tau \otimes M"] \\
\{V\}^n \otimes \{N\}^n \arrow[r, "\{V\}^n \otimes \varphi(N_1, \ldots, N_n)"'] & \{V\}^n \otimes \{N\}^n \otimes M
\end{tikzcd}$$

commute for all vector spaces V_i and C-comodules N_i.

PROOF. Choose bases $\{v_{ij}\}$ of the vector spaces V_i with corresponding projections $p_{ij} : V_i \otimes N_i \to N_i$. Then we have $\tau(t_1 \otimes \ldots \otimes t_n) = \sum v_{1i_1} \otimes \ldots \otimes v_{ni_n} \otimes p_{1i_1}(t_1) \otimes \ldots \otimes p_{ni_n}(t_n)$ so $\tau = \sum v_{1i_1} \otimes \ldots \otimes v_{ni_n} \otimes \{p\}^n$.

The $p_{ij_i} : V_i \otimes N_i \to N_i$ are homomorphisms of C-comodules. Hence the diagrams

$$\begin{tikzcd}
\{V \otimes N\}^n \arrow[r, "\varphi(V_1 \otimes N_1, \ldots, V_n \otimes N_n)"] \arrow[d, "\{p\}^n"'] & \{V \otimes N\}^n \otimes M \arrow[d, "\{p\}^n \otimes M"] \\
\{N\}^n \arrow[r, "\varphi(N_1, \ldots, N_n)"'] & \{N\}^n \otimes M
\end{tikzcd}$$

commute for all choices of $\{p\}^n = p_{1i_1} \otimes \ldots \otimes p_{ni_n}$.

So we get for all $t_i \in V_i \otimes N_i$

$$\begin{aligned}
&(\{V\}^n \otimes \varphi(N_1, \ldots, N_n))\tau(t_1 \otimes \ldots \otimes t_n) = \\
&= (\{V\}^n \otimes \varphi(N_1, \ldots, N_n))(\textstyle\sum v_{1i_1} \otimes \ldots \otimes v_{ni_n} \otimes p_{1i_1}(t_1) \otimes \ldots \otimes p_{ni_n}(t_n)) \\
&= \textstyle\sum v_{1i_1} \otimes \ldots \otimes v_{ni_n} \otimes \varphi(N_1, \ldots, N_n)\{p\}^n(t_1 \otimes \ldots \otimes t_n) \\
&= \textstyle\sum v_{1i_1} \otimes \ldots \otimes v_{ni_n} \otimes (\{p\}^n \otimes M)\varphi(V_1 \otimes N_1, \ldots, V_n \otimes N_n)(t_1 \otimes \ldots \otimes t_n) \\
&= (\tau \otimes M)\varphi(V_1 \otimes N_1, \ldots, V_n \otimes N_n)(t_1 \otimes \ldots \otimes t_n).
\end{aligned}$$

□

Theorem 3.8.6. *With the notation given above we have*

$$\mathrm{coend}(\omega^n) \cong C \otimes C \otimes \ldots \otimes C$$

with the universal natural transformation

$$\begin{gathered}
\delta^{(n)}(N_1, N_2, \ldots, N_n) := \tau(\delta(N_1) \otimes \delta(N_2) \otimes \ldots \otimes \delta(N_n)) : \\
\omega(N_1) \otimes \omega(N_2) \otimes \ldots \otimes \omega(N_n) \to \omega(N_1) \otimes C \otimes \omega(N_2) \otimes C \otimes \ldots \otimes \omega(N_n) \otimes C \\
\cong \omega(N_1) \otimes \omega(N_2) \otimes \ldots \otimes \omega(N_n) \otimes C \otimes C \otimes \ldots \otimes C.
\end{gathered}$$

PROOF. We proceed as in the proof of the previous Theorem.

Let M in **Vec** and let $\varphi : \omega^n \to \omega^n \otimes M$ be a natural transformation. We define the homomorphism $\widetilde{\varphi} : C^n = \omega(C) \otimes \omega(C) \otimes \ldots \otimes \omega(C) = C \otimes C \otimes \ldots \otimes C \to M$ by $\widetilde{\varphi} = (\varepsilon^n \otimes 1_M)\varphi(C, \ldots, C)$ using the fact that C is a comodule.

As in the preceding proof we get that $\delta : N_i \longrightarrow N_i \otimes C$ are homomorphisms of C-comodules. Thus the following diagram commutes

$$\begin{tikzcd}
N_1 \otimes \ldots \otimes N_n \arrow[r, "\delta\otimes\ldots\otimes\delta"] \arrow[d, "\varphi(N_1\otimes\ldots\otimes N_n)"'] & N_1 \otimes C \otimes \ldots \otimes N_n \otimes C \arrow[r, "\tau"] \arrow[d, "\varphi(N_1\otimes C,\ldots,N_n\otimes C)"'] & N_1 \otimes \ldots \otimes N_n \otimes C \otimes \ldots \otimes C \arrow[d, "N_1\otimes\ldots\otimes N_n\otimes\varphi(C,\ldots,C)"'] \\
N_1 \otimes \ldots \otimes N_n \otimes M \arrow[r, "\delta\otimes\ldots\otimes\delta\otimes M"] \arrow[drr, "1"] & N_1 \otimes C \otimes \ldots \otimes N_n \otimes C \otimes M \arrow[r, "\tau\otimes M"] \arrow[dr, "\{1\otimes\varepsilon\}^n\otimes 1"] & N_1 \otimes \ldots \otimes N_n \otimes C \otimes \ldots \otimes C \otimes M \arrow[d, "1\otimes\{\epsilon\}^n\otimes 1"'] \\
 & & N_1 \otimes \ldots \otimes N_n \otimes M
\end{tikzcd}$$

Hence we get the commutative diagram

$$\begin{tikzcd}
\omega^n \arrow[r, "\delta^{(n)}"] \arrow[dr, "\varphi"'] & \omega^n \otimes \{C\}^n \arrow[d, "1\otimes\widetilde{\varphi}"] \\
 & \omega^n \otimes M
\end{tikzcd}$$

To show the uniqueness of $\widetilde{\varphi}$ let $g : C^n \longrightarrow M$ be another homomorphism with $(1_{\omega^n} \otimes g)\delta^{(n)} = \varphi$. We have $g = g(\varepsilon^n \otimes 1_{C^n})\tau\Delta^n = g(\varepsilon^n \otimes 1_{C^n})\delta^{(n)}(C, \ldots, C) = (\varepsilon^n \otimes 1_M)(1_{C^n} \otimes g)\delta^{(n)}(C, \ldots, C) = (\varepsilon^n \otimes 1_M)\varphi(C, \ldots, C) = \widetilde{\varphi}$. $\square$

Now we prove the finite dimensional case of reconstruction of coalgebras.

Proposition 3.8.7. (Reconstruction) *Let C be a coalgebra. Let* $\mathbf{Comod}_0$*-C be the category of finite dimensional C-comodules and ω :* $\mathbf{Comod}_0$*-$C \longrightarrow$* $\mathbf{Vec}$ *be the underlying functor. Then we have $C \cong \mathrm{coend}(\omega)$.*

Proof. Let M be in **Vec** and let $\varphi : \omega \longrightarrow \omega \otimes M$ be a natural transformation. We define the homomorphism $\widetilde{\varphi} : C \longrightarrow M$ as follows. Let $c \in C$. Let N be a finite dimensional C-subcomodule of C containing c. Then we define $g(c) := (\epsilon|_N \otimes 1)\varphi(N)(c)$. If N' is another finite dimensional subcomodule of C with $c \in N'$ and with $N \subseteq N'$ then the following commutes

$$\begin{tikzcd}
N \arrow[r, "\varphi(N)"] \arrow[d] & N \otimes M \arrow[d] \arrow[dr] & & \\
 & & C \otimes M \arrow[r, "\epsilon\otimes 1"] & M \\
N' \arrow[r, "\varphi(N')"] & N' \otimes M \arrow[ur] & &
\end{tikzcd}$$

Thus the definition of $\widetilde{\varphi}(c)$ is independent of the choice of N. Furthermore $\widetilde{\varphi} : N \longrightarrow M$ is obviously a linear map. For any two elements $c, c' \in C$ there is a finite dimensional subcomodule $N \subseteq C$ with $c, c' \in N$ e. g. the sum of the finite dimensional subcomodules containing c and c' separately. Thus $\widetilde{\varphi}$ can be extended to all of C.

The rest of the proof is essentially the same as the proof of the first reconstruction theorem. ☐

The representations allow to reconstruct further structure of the coalgebra. We prove a reconstruction theorem about bialgebras. Recall that the category of B-comodules over a bialgebra B is a monoidal category, furthermore that the underlying functor $\omega : \mathbf{Comod}\text{-}B \to \mathbf{Vec}$ is a monoidal functor. From this information we can reconstruct the full bialgebra structure of B. We have

Theorem 3.8.8. *Let B be a coalgebra. Let $\mathbf{Comod}$-B be a monoidal category such that the underlying functor $\omega : \mathbf{Comod}\text{-}B \to \mathbf{Vec}$ is a monoidal functor. Then there is a unique bialgebra structure on B that induces the given monoidal structure on the corepresentations.*

Proof. First we prove the uniqueness of the multiplication $\nabla : B \otimes B \to B$ and of the unit $\eta : \mathbb{K} \to B$. The natural transformation $\delta : \omega \to \omega \otimes B$ becomes a monoidal natural transformation with $\nabla : B \otimes B \to B$ and $\eta : \mathbb{K} \to B$ We show that ∇ and η are uniquely determined by ω and δ.

Let $\nabla' : B \otimes B \to B$ and $\eta' : B \to \mathbb{K}$ be morphisms that make δ a monoidal natural transformation. The diagrams

$$\begin{tikzcd} \omega(X) \otimes \omega(Y) \arrow[r, "\delta(X)\otimes\delta(Y)"] \arrow[d, "\rho"'] & \omega(X) \otimes \omega(Y) \otimes B \otimes B \arrow[d, "\rho\otimes\nabla'"] \\ \omega(X \otimes Y) \arrow[r, "\delta(X\otimes Y)"] & \omega(X \otimes Y) \otimes B \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{K} \arrow[r, "\cong"] \arrow[d] & \mathbb{K} \otimes \mathbb{K} \arrow[d, "1 \otimes \eta'"] \\ \omega(\mathbb{K}) \arrow[r, "\delta(\mathbb{K})"] & \omega(\mathbb{K}) \otimes B \end{tikzcd}$$

commute. In particular the following diagrams commute

$$\begin{tikzcd} \omega(B) \otimes \omega(B) \arrow[r, "\delta(B)\otimes\delta(B)"] \arrow[d, "\rho"'] & \omega(B) \otimes \omega(B) \otimes B \otimes B \arrow[d, "\rho\otimes\nabla'"] \\ \omega(B \otimes B) \arrow[r, "\delta(B\otimes B)"] & \omega(B \otimes B) \otimes B \end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\cong"] \arrow[d] & \mathbb{K} \otimes \mathbb{K} \arrow[d, "1 \otimes \eta'"] \\
\omega(\mathbb{K}) \arrow[r, "\delta(\mathbb{K})"] & \omega(\mathbb{K}) \otimes B
\end{tikzcd}$$

Hence we get $\sum b_{(1)} \otimes c_{(1)} \otimes b_{(2)} c_{(2)} = \sum b_{(1)} \otimes c_{(1)} \otimes \nabla'(b_{(2)} \otimes c_{(2)})$ and $1 \otimes 1 = 1 \otimes \eta'(1)$. This implies $bc = \sum \epsilon(b_{(1)})\epsilon(c_{(1)}) b_{(2)} c_{(2)} = \sum \epsilon(b_{(1)})\epsilon(c_{(1)}) \nabla'(b_{(2)} \otimes c_{(2)}) = \nabla'(b \otimes c)$ and $1 = \eta'(1)$.

Now we show the existence of a bialgebra structure. Let B be a coalgebra only and let $\omega : \mathbf{Comod}\text{-}B \to \mathbf{Vec}$ be a monoidal functor with $\xi : \omega(M) \otimes \omega(N) \to \omega(M \otimes N)$ and $\xi_0 : \mathbb{K} \to \omega(\mathbb{K})$. First we observe that the new tensor product between the comodules M and N coincides with the tensor product of the underlying vector spaces (up to an isomorphism ξ). Because of the coherence theorems for monoidal categories (that also hold in our situation) we may identify along the maps ξ and ξ_0.

We define $\eta := (\mathbb{K} \xrightarrow{\delta(\mathbb{K})} \mathbb{K} \otimes B \cong B)$ and $\nabla := (B \otimes B \xrightarrow{\delta(B \otimes B)} B \otimes B \otimes B \xrightarrow{\epsilon \otimes \epsilon \otimes 1_B} \mathbb{K} \otimes \mathbb{K} \otimes B \cong B)$.

Since the structural morphism for the comodule $\delta : M \to M \otimes B$ is a homomorphism of of B comodules where the comodule structure on $M \otimes B$ is only given by the diagonal of B that is the $\mathcal{C}$-structure on $\omega : \mathbf{Comod}\text{-}B \to \mathbf{Vec}$ we get that also $\delta(M) \otimes \delta(N) : M \otimes N \to M \otimes N \otimes B$ is a comodule homomorphism. Hence the first square in the following diagram commutes

$$\begin{tikzcd}
M \otimes N \arrow[r, "\delta(M) \otimes \delta(N)"] \arrow[d, "\delta(M \otimes N)"'] & M \otimes B \otimes N \otimes B \arrow[r, "1 \otimes \tau \otimes 1"] \arrow[d, "\delta(M \otimes B \otimes N \otimes B)"'] & M \otimes N \otimes B \otimes B \arrow[d, "1 \otimes 1 \otimes \delta(B \otimes B)"'] \\
M \otimes N \otimes B \arrow[r, "\delta(M) \otimes \delta(N) \otimes 1_B"] & M \otimes B \otimes N \otimes B \otimes B \arrow[r, "1 \otimes \tau \otimes 1 \otimes 1"] & M \otimes N \otimes B \otimes B \otimes B
\end{tikzcd}$$

The second square commutes by a similar reasoning since the comodule structure on $M \otimes B$ resp. $N \otimes B$ is given by the diagonal on B hence $M \otimes N$ can be factored out of the natural ($\mathcal{C}$-)transformation. Now we attach

$$1_M \otimes 1_N \otimes \epsilon \otimes \epsilon \otimes 1_B : M \otimes N \otimes B \otimes B \otimes B \to M \otimes N \otimes B$$

to the commutative rectangle and obtain $\delta(M \otimes N) = (1_M \otimes 1_N \otimes \nabla)(1 \otimes \tau \otimes 1)(\delta(M) \otimes \delta(N)$. Thus the comodule structure on $M \otimes N$ is induced by the multiplication $\nabla : B \otimes B \to B$ defined above.

So the following diagrams commute

$$\begin{tikzcd}
B \otimes B \arrow[rr, "\Delta\otimes\Delta"] \arrow[d, "1_B\otimes 1_B"] & & B \otimes B \otimes B \otimes B \arrow[rr, "1\otimes\tau\otimes 1"] & & B \otimes B \otimes B \otimes B \arrow[d, "1\otimes\nabla"] \\
B \otimes B \arrow[rrrr, "\delta(B\otimes B)"] \arrow[dr, "\delta(B\otimes B)"] \arrow[dd, "\nabla"'] & & & & B \otimes B \otimes B \arrow[dl, "\delta(B\otimes B)\otimes 1"'] \arrow[dd, "\nabla\otimes 1"] \\
& B \otimes B \otimes B \arrow[rr, "1\otimes 1\otimes\Delta"] \arrow[dl, "\epsilon\otimes\epsilon\otimes 1"] & & B \otimes B \otimes B \otimes B \arrow[dr, "\epsilon\otimes\epsilon\otimes 1\otimes 1"'] & \\
B \arrow[rrrr, "\Delta"] & & & & B \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
B \otimes B \arrow[rr, "\nabla"] \arrow[dr, "\delta(B\otimes B)"] \arrow[dd, "1"'] & & B \arrow[ddl, "\epsilon"] \\
& B \otimes B \otimes B \arrow[ur, "\epsilon\otimes\epsilon\otimes 1"] \arrow[dl, "1\otimes 1\otimes\epsilon"] & \\
B \otimes B \arrow[r, "\epsilon\otimes\epsilon"] & \mathbb{K} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{K} \arrow[rrr, "\eta"] \arrow[drr, "\delta(\mathbb{K})"] \arrow[ddr, "\delta(\mathbb{K})"] \arrow[ddd, "\eta"'] & & & B \arrow[dd, "\Delta"] \\
& & \mathbb{K} \otimes B \arrow[ur, "\cong"'] \arrow[d, "1\otimes\Delta"] & \\
& \mathbb{K} \otimes B \arrow[r, "\delta(\mathbb{K})\otimes 1"] \arrow[dl, "\cong"] \arrow[dr, "\eta\otimes 1"] & \mathbb{K} \otimes B \otimes B \arrow[r, "\cong"] \arrow[d, "\cong"] & B \otimes B \arrow[dl, "1_B"] \\
B \arrow[rr, "\eta\otimes 1"] & & B \otimes B &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \arrow[rr, "1"] \arrow[dr, "\delta(\mathbb{K})"] \arrow[ddr, "\eta"'] & & \mathbb{K} \\
& \mathbb{K} \otimes B \arrow[ur, "1\otimes\epsilon"] \arrow[d, "\cong"] & \\
& B \arrow[uur, "\epsilon"'] &
\end{tikzcd}$$

Hence η and ∇ are coalgebra homomorphisms.

To show the associativity of ∇ we identify along the maps $\alpha : (M \otimes N) \otimes P \cong M \otimes (N \otimes P)$ and furthermore simplify the relevant diagram by fixing that σ represents a suitable permutation of the tensor factors. Then the following commute

$$\begin{tikzcd}
B \otimes B \otimes B \arrow[r, "\sigma(\delta(B)\otimes\delta(B)\otimes\delta(B))"] \arrow[d, "1"] & B \otimes B \otimes B \otimes B \otimes B \otimes B \arrow[r, "\epsilon\otimes\epsilon\otimes\epsilon\otimes 1"] \arrow[d, shift right, "1\otimes(\nabla\otimes 1)"'] \arrow[d, shift left, "1\otimes(1\otimes\nabla)"] & B \otimes B \otimes B \arrow[d, shift right, "\nabla\otimes 1"'] \arrow[d, shift left, "1\otimes\nabla"] \\
B \otimes B \otimes B \arrow[r, shift left, "\sigma(\delta(B\otimes B)\otimes\delta(B))"] \arrow[r, shift right, "\sigma(\delta(B)\otimes\delta(B\otimes B))"'] \arrow[d, "1"] & B \otimes B \otimes B \otimes B \otimes B \otimes B \arrow[r, "\epsilon\otimes\epsilon\otimes\epsilon\otimes 1"] \arrow[d, "1\otimes\nabla"] & B \otimes B \arrow[d, "\nabla"] \\
B \otimes B \otimes B \arrow[r, "\delta(B\otimes B\otimes B)"] & B \otimes B \otimes B \otimes B \arrow[r, "\epsilon\otimes\epsilon\otimes\epsilon\otimes 1"] & B
\end{tikzcd}$$

The upper row is the identity hence we get the associative law.

For the proof that η has the properties of a unit we must explicitly consider the coherence morphisms λ and ρ By reasons of symmetry we will only show one half of of the unit axiom. This axiom follows from the commutativity of the following diagram

$$\begin{tikzcd}
B \arrow[r, "\delta(B)"] \arrow[d, "\rho^{-1}"'] & B \otimes B \arrow[r, "\rho^{-1}"] & B \otimes B \otimes \mathbb{K} \arrow[rr, "\rho"] \arrow[d, "1\otimes 1\otimes\delta(\mathbb{K})"] & & B \otimes B \arrow[r, "\epsilon\otimes 1"] \arrow[d, "1\otimes 1\otimes\eta"'] & B \arrow[d, "1\otimes\eta"'] \\
B \otimes \mathbb{K} \arrow[rr, "\delta(B)\otimes\delta(\mathbb{K})"] \arrow[urr, "\delta(B)\otimes 1"] \arrow[d, "="'] & & B \otimes B \otimes \mathbb{K} \otimes B \arrow[r, "1\otimes\tau\otimes 1"] & B \otimes \mathbb{K} \otimes B \otimes B \arrow[r, "\rho\otimes 1\otimes 1"] \arrow[d, "1\otimes 1\otimes\nabla"] & B \otimes B \otimes B \arrow[r, "\epsilon\otimes 1\otimes 1"] \arrow[dd, "1\otimes\nabla"'] & B \otimes B \arrow[dd, "\nabla"'] \\
B \otimes \mathbb{K} \arrow[rrr, "\delta(B\otimes\mathbb{K})"] \arrow[d, "\rho"'] & & & B \otimes \mathbb{K} \otimes B \arrow[dr, "\rho\otimes 1"] & & \\
B \arrow[rrrr, "\delta(B)"] & & & & B \otimes B \arrow[r, "\epsilon\otimes 1"] & B
\end{tikzcd}$$

$\square$