CHAPTER 3

Hopf Algebras, Algebraic, Formal, and Quantum Groups

7. The quantum monoid of a quantum space

Problem 3.7.1. If A is a finite dimensional algebra and $\delta : A \longrightarrow M(A) \otimes A$ the universal cooperation of the Tambara bialgebra on A from the left then $\tau\delta : A \longrightarrow A \otimes M(A)$ (with the same multiplication on $M(A)$) is a universal cooperation of $M(A)$ on A from the right. The comultiplication defined by this cooperation is $\tau\Delta : M(A) \longrightarrow M(A) \otimes M(A)$. Thus we have to distinguish between the left and the right Tambara bialgebra on A and we have $M_r(A) = M_l(A)^{cop}$.

Now consider the special monoidal diagram scheme $\mathcal{D} := \mathcal{D}[X; m, u]$. To make things simpler we assume that **Vec** is strict monoidal. The category $\mathcal{D}[X; m, u]$ has the objects $X \otimes \ldots \otimes X = X^{\otimes n}$ for all $n \in \mathbb{N}$ (and $I := X^{\otimes 0}$) and the morphisms $m : X \otimes X \longrightarrow X$, $u : I \longrightarrow X$ and all morphisms formally constructed from m, u, id by taking tensor products and composition of morphisms.

Let A be an algebra with multiplication $m_A : A \otimes A \longrightarrow A$ and unit $u_A : \mathbb{K} \longrightarrow A$. Then $\omega_A : \mathcal{D} \longrightarrow \mathcal{C}$ defined by $\omega(X) = A$, $\omega(X^{\otimes n}) = A^{\otimes n}$, $\omega(m) = m_A$ and $\omega(u) = u_A$ is a strict monoidal functor. If A is finite dimensional then the diagram is finite. We get

Theorem 3.7.1. *Let A be a finite dimensional algebra. Then the algebra $M(A)$ coacting universally from the right on A (the right Tambara bialgebra) $M(A)$ and* $\mathrm{coend}(\omega_A)$ *are isomorphic as bialgebras.*

Proof. We have studied the Tambara bialgebra for left coaction $f : A \longrightarrow M(A) \otimes A$ but here we need the analogue for universal right coaction $f : A \longrightarrow A \otimes M(A)$ (see Problem 3.9).

Let B be an algebra and $f : A \longrightarrow A \otimes B$ be a homomorphism of algebras. For $\omega = \omega_A$ we define

$$\varphi(X^{\otimes n}) : \omega(X^{\otimes n}) = A^{\otimes n} \xrightarrow{f^{\otimes n}} A^{\otimes n} \otimes B^{\otimes n} \xrightarrow{1 \otimes m_B^n} A^{\otimes n} \otimes B = \omega(X^{\otimes n}) \otimes B,$$

where $m_B^n : B^{\otimes n} \longrightarrow B$ is the n-fold multiplication on B. The map φ is a natural transformation since the diagrams

$$\begin{tikzcd} \mathbb{K} \arrow[r, "\varphi(I)"] \arrow[d, "u"'] & \mathbb{K} \otimes B \arrow[d, "1 \otimes u"] \\ A \arrow[r, "\varphi(X)"'] & A \otimes B \end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\varphi(X \otimes X)"] \arrow[dr, "f \otimes f"] \arrow[ddd, "m"] & & A \otimes A \otimes B \arrow[ddd, "m \otimes 1"] \\
& A \otimes A \otimes B \otimes B \arrow[ur, "1 \otimes m"] \arrow[d, "m \otimes m"] & \\
& A \otimes B \arrow[dr, "1 \otimes 1"] & \\
A \arrow[ur, "f"] \arrow[rr, "\varphi(X)"'] & & A \otimes B
\end{tikzcd}$$

commute. Furthermore the following commute

$$\begin{tikzcd}
A^{\otimes r} \otimes A^{\otimes s} \arrow[rr, "\varphi(X^{\otimes r}) \otimes \varphi(X^{\otimes s})"] \arrow[dr] \arrow[ddd] & & A^{\otimes r} \otimes A^{\otimes s} \otimes B \otimes B \arrow[ddd] \\
& A^{\otimes r} \otimes A^{\otimes s} \otimes B^{\otimes r} \otimes B^{\otimes s} \arrow[ur] \arrow[d] & \\
& A^{\otimes (r+s)} \otimes B^{\otimes (r+s)} \arrow[dr] & \\
A^{\otimes (r+s)} \arrow[ur] \arrow[rr, "\varphi(X^{\otimes (r+s)})"'] & & A^{\otimes (r+s)} \otimes B
\end{tikzcd}$$

so that $\varphi : \omega_A \longrightarrow \omega_A \otimes B$ is a monoidal natural transformation.

Conversely let $\varphi : \omega_A \longrightarrow \omega_A \otimes B$ be a natural transformation. Let $f := \varphi(X) : A \longrightarrow A \otimes B$. Then the following commute

$$\begin{tikzcd}
A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "="'] & A \otimes A \otimes B \otimes B \arrow[d, "1 \otimes m"] \\
A \otimes A \arrow[r, "\varphi(X \otimes X)"] \arrow[d, "m"'] & A \otimes A \otimes B \arrow[d, "m \otimes 1"] \\
A \arrow[r, "f"] & A \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\cong"] \arrow[d, "="'] & \mathbb{K} \otimes \mathbb{K} \arrow[d] \\
\mathbb{K} \arrow[r] \arrow[d, "u"'] & \mathbb{K} \otimes B \arrow[d, "u \otimes 1"] \\
A \arrow[r, "f"] & A \otimes B.
\end{tikzcd}$$

Hence $f : A \longrightarrow A \otimes B$ is a homomorphism of algebras.

Thus we have defined an isomorphism

$$\mathbb{K}\mathbf{Alg}(A, A \otimes B) \cong \mathrm{Nat}^{\otimes}(\omega_A, \omega_A \otimes B)$$

that is natural in B. If A is finite dimensional then the left hand side is represented by the Tambara bialgebra $M_r(A)$ and the right hand side by the bialgebra $\mathrm{coend}(\omega_A)$. Thus both bialgebras must be isomorphic. $\square$

Corollary 3.7.2. *There is a unique isomorphism of bialgebras* $M_r(A) \cong \mathrm{coend}(\omega_A)$ *such that the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A \otimes M_r(A) \arrow[d] \\ & A \otimes \mathrm{coend}(\omega_A) \end{tikzcd}$$

commutes

PROOF. This is a direct consequence of the universal property. $\square$

Thus the Tambara bialgebra that represents the universal quantum monoid acting on a finite quantum space may be reconstructed by the Tannaka-Krein reconstruction from representation theory. Similar reconstructions can be given for more complicated quantum spaces such as so called quadratic quantum spaces.