CHAPTER 3

Hopf Algebras, Algebraic, Formal, and Quantum Groups

6. The bialgebra coend

Let $\omega : \mathcal{D} \to \mathcal{C}$ and $\omega' : \mathcal{D}' \to \mathcal{C}$ be diagrams in $\mathcal{C}$. We call the diagram $(\mathcal{D}, \omega) \otimes (\mathcal{D}', \omega') := (\mathcal{D} \times \mathcal{D}', \omega \otimes \omega')$ with $(\omega \otimes \omega')(X, Y) := \omega(X) \otimes \omega'(Y)$ the *tensor product* of these two diagrams. The new diagram consists of all possible tensor products of all objects and all morphisms of the original diagrams.

From now on we assume that the category $\mathcal{C}$ is the category of vector spaces and we use the symmetry $\tau : V \otimes W \to W \otimes V$ in **Vec**.

Proposition 3.6.1. *Let $(\mathcal{D}, \omega)$ and $(\mathcal{D}', \omega')$ be finite diagrams in* **Vec**. *Then*

$$\operatorname{coend}(\omega \otimes \omega') \cong \operatorname{coend}(\omega) \otimes \operatorname{coend}(\omega').$$

Proof. First observe the following. If two diagrams $\omega : \mathcal{D} \to \mathbf{Vec}$ and $\omega' : \mathcal{D}' \to \mathbf{Vec}$ are given then $\varinjlim_{\mathcal{D}} \varinjlim_{\mathcal{D}'} (\omega \otimes \omega') \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'} (\omega \otimes \omega') \cong \varinjlim_{\mathcal{D}} (\omega) \otimes \varinjlim_{\mathcal{D}'} (\omega')$ since the tensor product preserves colimits and colimits commute with colimits. For this consider the diagram

$$\begin{tikzcd}
\omega(X) \otimes \omega'(Y) \arrow[r] \arrow[d] & \omega(X) \otimes \varinjlim_{\mathcal{D}'}(\omega') \arrow[d] & \\
\varinjlim_{\mathcal{D}}(\omega) \otimes \omega'(Y) \arrow[r] & \varinjlim_{\mathcal{D}}(\omega) \otimes \varinjlim_{\mathcal{D}'}(\omega') & \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'}(\omega \otimes \omega').
\end{tikzcd}$$

The maps in the diagram are the injections for the corresponding colimits. In particular we have $\operatorname{coend}(\omega \otimes \omega') \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'} ((\omega \otimes \omega')^* \otimes (\omega \otimes \omega')) \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'} (\omega^* \otimes \omega \otimes \omega'^* \otimes \omega') \cong \varinjlim_{\mathcal{D}} (\omega^* \otimes \omega) \otimes \varinjlim_{\mathcal{D}'} (\omega'^* \otimes \omega') \cong \operatorname{coend}(\omega) \otimes \operatorname{coend}(\omega')$.

The (universal) morphism

$$(\iota(X) \otimes \iota'(Y))(1 \otimes \tau \otimes 1) : \omega(X)^* \otimes \omega'(Y)^* \otimes \omega(X) \otimes \omega'(Y) \to \varinjlim(\omega^* \otimes \omega) \otimes \varinjlim(\omega'^* \otimes \omega')$$

can be identified with the universal morphism

$$\iota(X, Y) : \omega(X)^* \otimes \omega'(Y)^* \otimes \omega(X) \otimes \omega'(Y) \to \varinjlim((\omega \otimes \omega')^* \otimes (\omega \otimes \omega')).$$

Hence the induced morphisms

$$(1 \otimes \tau \otimes 1)(\delta \otimes \delta') : \omega(X) \otimes \omega'(Y) \to \omega(X) \otimes \omega'(Y) \otimes \operatorname{coend}(\omega) \otimes \operatorname{coend}(\omega')$$

and

$$\delta : \omega(X) \otimes \omega'(Y) \to \omega(X) \otimes \omega'(Y) \otimes \operatorname{coend}(\omega \otimes \omega')$$

can be identified. □

Corollary 3.6.2. *For all finite diagrams $(\mathcal{D}, \omega)$ and $(\mathcal{D}', \omega')$ in $\mathcal{D}$ there is a universal natural transformation $\delta : \omega \otimes \omega' \to \omega \otimes \omega' \otimes \operatorname{coend}(\omega) \otimes \operatorname{coend}(\omega')$ so that for each object M and each natural transformation $\varphi : \omega \otimes \omega' \to \omega \otimes \omega' \otimes M$ there exists* *a unique morphism* $\widetilde{\varphi} : \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega') \to M$ *such that*

$$\begin{tikzcd}
\omega \otimes \omega' \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \omega \otimes \omega' \otimes \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega') \arrow[d, "1\otimes 1\otimes\widetilde{\varphi}"] \\
& \omega \otimes \omega' \otimes M
\end{tikzcd}$$

commutes.

Definition 3.6.3. Let $(\mathcal{D}, \omega)$ be a diagram in $\mathcal{C} = \mathbf{Vec}$. Then ω is called *reconstructive*

- if there is an object $\mathrm{coend}(\omega)$ in $\mathcal{C}$ and a universal natural transformation $\delta : \omega \to \omega \otimes \mathrm{coend}(\omega)$
- and if $(1 \otimes \tau \otimes 1)(\delta \otimes \delta) : \omega \otimes \omega \to \omega \otimes \omega \otimes \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega)$ is a univesarl natural transformation of bifunctors.

Definition 3.6.4. Let $(\mathcal{D}, \omega)$ be a diagram in $\mathbf{Vec}$. Let $\mathcal{D}$ be a monoidal category and ω be a monoidal functor. Then $(\mathcal{D}, \omega)$ is called a *monoidal diagram.*

Let $(\mathcal{D}, \omega)$ be a monoidal diagram $\mathbf{Vec}$. Let $A \in \mathbf{Vec}$ be an algebra. A natural transformation $\varphi : \omega \to \omega \otimes B$ is called monoidal *monoidal* if the diagrams

$$\begin{tikzcd}
\omega(X) \otimes \omega(Y) \arrow[r, "\varphi(X)\otimes\varphi(Y)"] \arrow[d, "\rho"'] & \omega(X) \otimes \omega(Y) \otimes B \otimes B \arrow[d, "\rho\otimes m"] \\
\omega(X \otimes Y) \arrow[r, "\varphi(X\otimes Y)"] & \omega(X \otimes Y) \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\cong"] \arrow[d] & \mathbb{K} \otimes \mathbb{K} \arrow[d] \\
\omega(I) \arrow[r, "\varphi(I)"] & \omega(I) \otimes B
\end{tikzcd}$$

commute.

We denote the set of monoidal natural transformations by $\mathrm{Nat}^{\otimes}(\omega, \omega \otimes B)$.

Problem 3.6.1. Show that $\mathrm{Nat}^{\otimes}(\omega, \omega \otimes B)$ is a functor in B.

Theorem 3.6.5. *Let* $(\mathcal{D}, \omega)$ *be a reconstructive, monoidal diagram in* $\mathbf{Vec}$. *Then* $\mathrm{coend}(\omega)$ *is a bialgebra and* $\delta : \omega \to \omega \otimes \mathrm{coend}(\omega)$ *is a monoidal natural transformation.*

If B *is a bialgebra and* $\partial : \omega \to \omega \otimes B$ *is a monoidal natural transformation, then there is a unique homomorphism of bialgebras* $f : coend(\omega) \to B$ *such that the* *diagram*

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[dr, "\partial"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes f"] \\
& \omega \otimes B
\end{tikzcd}$$

commutes.

Proof. The multiplication of $\mathrm{coend}(\omega)$ arises from the following diagram

$$\begin{tikzcd}
\omega(X)\otimes\omega(Y) \arrow[r, "\delta\otimes\delta"] \arrow[d] & \omega(X)\otimes\omega(Y)\otimes\mathrm{coend}(\omega)\otimes\mathrm{coend}(\omega) \arrow[d] \arrow[dr] & \\
\omega(X\otimes Y) \arrow[r, "\delta"] & \omega(X\otimes Y)\otimes\mathrm{coend}(\omega) \arrow[r, phantom, "\cong"] & \omega(X)\otimes\omega(Y)\otimes\mathrm{coend}(\omega)
\end{tikzcd}$$

For the construction of the unit we consider the diagram $\mathcal{D}_0 = (\{I\}, \{\mathrm{id}\})$ together with $\omega_0 : \mathcal{D}_0 \longrightarrow \mathbf{Vec}$, $\omega_0(I) = \mathbb{K}$, the monoidal unit object in the monoidal category of diagrams in **Vec**. Then $(\mathbb{K} \longrightarrow \mathbb{K}\otimes\mathbb{K}) = (\omega_0 \longrightarrow \omega_0 \otimes \mathrm{coend}(\omega_0))$ is the universal map. The following diagram then induced the unit for $\mathrm{coend}(\omega)$

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\cong"] \arrow[d] & \mathbb{K}\otimes\mathbb{K} \arrow[d] \arrow[dr] & \\
\omega(I) \arrow[r] & \omega(I)\otimes\mathrm{coend}(\omega) \arrow[r, phantom, "\cong"] & \mathbb{K}\otimes\mathrm{coend}(\omega)
\end{tikzcd}$$

By using the universal property one checks the laws for bialgebras.

The above diagrams show in particular that the natural transformation $\delta : \omega \longrightarrow \omega \otimes \mathrm{coend}(\omega)$ is monoidal. $\square$