CHAPTER 3

Hopf Algebras, Algebraic, Formal, and Quantum Groups

5. The coalgebra coend

Proposition 3.5.1. *Let $\mathcal{C}$ be a monoidal category and $\omega : \mathcal{D} \longrightarrow \mathcal{C}$ be a diagram in $\mathcal{C}$. Assume that there is a universal object* $\mathrm{coend}(\omega)$ *and natural transformation $\delta : \omega \longrightarrow \omega \otimes \mathrm{coend}(\omega)$.*

Then there is exactly one coalgebra structure on $\mathrm{coend}(\omega)$ *such that the diagrams*

$$\begin{tikzcd} \omega \arrow[r, "\delta"] \arrow[d, "\delta"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\Delta"] \\ \omega \otimes \mathrm{coend}(\omega) \arrow[r, "\delta\otimes 1"'] & \omega \otimes \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega) \end{tikzcd}$$

and

$$\begin{tikzcd} \omega \arrow[r, "\delta"] \arrow[dr, "\mathrm{id}_\omega"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\epsilon"] \\ & \omega \otimes I \end{tikzcd}$$

commute.

Proof. Because of the universal property of $\mathrm{coend}(\omega)$ there are structure morphisms $\Delta : \mathrm{coend}(\omega) \longrightarrow \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega)$ and $\epsilon : \mathrm{coend}(\omega) \longrightarrow I$. This implies the coalgebra property similar to the proof of Corollary 3.3.8. □

Observe that by this construction all objects and all morphisms of the diagram $\omega : \mathcal{D} \longrightarrow \mathcal{C}_0 \subseteq \mathcal{C}$ are comodules or morphisms of comodules over the coalgebra $\mathrm{coend}(\omega)$. In fact $C := \mathrm{coend}(\omega)$ is the universal coalgebra over which the given diagram becomes a diagram of comodules.

Corollary 3.5.2. *Let $(\mathcal{D}, \omega)$ be a diagram $\mathcal{C}$ with objects in $\mathcal{C}_0$. Then all objects of the diagram are comodules over the coalgebra $C :=$* $\mathrm{coend}(\omega)$ *and all morphisms are morphisms of comodules. If D is another coalgebra and all objects of the diagram are D-comodules by $\varphi(X) : \omega(X) \longrightarrow \omega(X) \otimes D$ and all morphisms of the diagram are morphisms of D-comodules then there exists a unique morphism of coalgebras $\widetilde{\varphi} : \mathrm{coend}(\omega) \longrightarrow D$ such that the diagram*

$$\begin{tikzcd} \omega \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\widetilde{\varphi}"] \\ & \omega \otimes D \end{tikzcd}$$

commutes.

Proof. The morphisms $\varphi(X) : \omega(X) \longrightarrow \omega(X) \otimes D$ define a natural transformation since all morphisms of the diagram are morphisms of comodules. So the existence and the uniqueness of a morphism $\widetilde{\varphi} : \mathrm{coend}(\omega) \to D$ is clear. The only thing to show is that this is a morphism of coalgebras. This follows from the universal property of $C = \mathrm{coend}(\omega)$ and the diagram

$$\begin{tikzcd}
\omega \arrow[rr, "\delta"] \arrow[dr, "\delta"] \arrow[dd, "1"'] & & \omega \otimes C \arrow[dr, "1\otimes\Delta"] \arrow[dd, "1\otimes\widetilde{\varphi}" near start] & \\
& \omega \otimes C \arrow[rr, "\delta\otimes 1" near start] \arrow[dd, "1\otimes\widetilde{\varphi}" near start] & & \omega \otimes C \otimes C \arrow[dd, "1\otimes\widetilde{\varphi}\otimes\widetilde{\varphi}"] \\
\omega \arrow[rr, "\varphi"' near start] \arrow[dr, "\varphi"'] & & \omega \otimes D \arrow[dr, "1\otimes\Delta"'] & \\
& \omega \otimes D \arrow[rr, "\varphi\otimes 1"'] & & \omega \otimes D \otimes D
\end{tikzcd}$$

where the right side of the cube commutes by the universal property. Similarly we get that $\widetilde{\varphi}$ preserves the counit since the following diagram commutes

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[d, "1"'] \arrow[drr, "1" near start] & \omega \otimes C \arrow[d, "1\otimes\widetilde{\varphi}"'] \arrow[dr, "1\otimes\varepsilon"] & \\
\omega \arrow[r, "\varphi"] \arrow[drr, "1"'] & \omega \otimes D \arrow[dr, "1\otimes\varepsilon"] & \omega \otimes \mathbb{K} \arrow[d, "1"] \\
& & \omega \otimes \mathbb{K}
\end{tikzcd}$$

$\square$