CHAPTER 3

Hopf Algebras, Algebraic, Formal, and Quantum Groups

3. Dual Objects

At the end of the first section in Corollary 3.1.15 we saw that the dual of an H-module can be constructed. We did not show the corresponding result for comodules. In fact such a construction for comodules needs some finiteness conditions. With this restriction the notion of a dual object can be introduced in an arbitrary monoidal category.

Definition 3.3.1. Let $(\mathcal{C}, \otimes)$ be a monoidal category $M \in \mathcal{C}$ be an object. An object $M^* \in \mathcal{C}$ together with a morphism $\mathrm{ev} : M^* \otimes M \longrightarrow I$ is called a *left dual* for M if there exists a morphism $\mathrm{db} : I \longrightarrow M \otimes M^*$ in $\mathcal{C}$ such that

$$(M \xrightarrow{\mathrm{db}\otimes 1} M \otimes M^* \otimes M \xrightarrow{1\otimes \mathrm{ev}} M) = 1_M$$
$$(M^* \xrightarrow{1\otimes \mathrm{db}} M^* \otimes M \otimes M^* \xrightarrow{\mathrm{ev}\otimes 1} M^*) = 1_{M^*}.$$

A monoidal category is called *left rigid* if each object $M \in \mathcal{C}$ has a left dual.

Symmetrically we define: an object ${}^*M \in \mathcal{C}$ together with a morphism $\mathrm{ev} : M \otimes {}^*M \longrightarrow I$ is called a *right dual* for M if there exists a morphism $\mathrm{db} : I \longrightarrow {}^*M \otimes M$ in $\mathcal{C}$ such that

$$(M \xrightarrow{1\otimes \mathrm{db}} M \otimes {}^*M \otimes M \xrightarrow{\mathrm{ev}\otimes 1} M) = 1_M$$
$$({}^*M \xrightarrow{\mathrm{db}\otimes 1} {}^*M \otimes M \otimes {}^*M \xrightarrow{1\otimes \mathrm{ev}} {}^*M) = 1_{{}^*M}.$$

A monoidal category is called *right rigid* if each object $M \in \mathcal{C}$ has a left dual.

The morphisms ev and db are called the *evaluation* respectively the *dual basis*.

Remark 3.3.2. If (M^*, ev) is a left dual for M then obviously (M, ev) is a right dual for M^* and conversely. One uses the same morphism $\mathrm{db} : I \longrightarrow M \otimes M^*$.

Lemma 3.3.3. *Let (M^*, ev) be a left dual for M. Then there is a natural isomorphism*

$$\mathrm{Mor}_{\mathcal{C}}(\text{-} \otimes M, \text{-}) \cong \mathrm{Mor}_{\mathcal{C}}(\text{-}, \text{-} \otimes M^*),$$

i. e. the functor $\text{-} \otimes M : \mathcal{C} \longrightarrow \mathcal{C}$ is left adjoint to the functor $\text{-} \otimes M^* : \mathcal{C} \longrightarrow \mathcal{C}$.

PROOF. We give the unit and the counit of the pair of adjoint functors. We define $\Phi(A) := 1_A \otimes \mathrm{db} : A \longrightarrow A \otimes M \otimes M^*$ and $\Psi(B) := 1_B \otimes \mathrm{ev} : B \otimes M^* \otimes M \longrightarrow B$. These are obviously natural transformations. We have commutative diagrams

$$(A \otimes M \xrightarrow[1_A \otimes \mathrm{db} \otimes 1_M]{\mathcal{F}\Phi(A)=} A \otimes M \otimes M^* \otimes M \xrightarrow[1_A \otimes 1_M \otimes \mathrm{ev}]{\Psi\mathcal{F}(A)=} A \otimes M) = 1_{A\otimes M}$$

and

$$(B \otimes M^* \xrightarrow[1_B \otimes 1_{M^*} \otimes \mathrm{db}]{\Phi\mathcal{G}(B)=} B \otimes M^* \otimes M \otimes M^* \xrightarrow[1_B \otimes \mathrm{ev} \otimes 1_{M^*}]{\mathcal{G}\Psi(B)=} B \otimes M^*) = 1_{B\otimes M^*}$$

thus the Lemma has been proved by Corollary A.9.11. □

The converse holds as well. If $\text{-} \otimes M$ is left adjoint to $\text{-} \otimes M^*$ then the unit Φ gives a morphism $\mathrm{db} := \Phi(I) : I \longrightarrow M \otimes M^*$ and the counit Ψ gives a morphism $\mathrm{ev} := \Psi(I) : M^* \otimes M \longrightarrow I$ satisfying the required properties. Thus we have

Corollary 3.3.4. *If* $\text{-}\otimes M : \mathcal{C} \to \mathcal{C}$ *is left adjoint to* $\text{-}\otimes M^* : \mathcal{C} \to \mathcal{C}$ *then* M^* *is a left dual for* M.

Corollary 3.3.5. (M^*, ev) *is a left dual for* M *if and only if there is a natural isomorphism*

$$\mathrm{Mor}_{\mathcal{C}}(M^* \otimes \text{-}, \text{-}) \cong \mathrm{Mor}_{\mathcal{C}}(\text{-}, M \otimes \text{-}),$$

i. e. the functor $M^* \otimes \text{-} : \mathcal{C} \to \mathcal{C}$ *is left adjoint to the functor* $M \otimes \text{-} : \mathcal{C} \to \mathcal{C}$. *The natural isomorphism if given by*

$$(f : M^* \otimes N \to P) \mapsto ((1_M \otimes f)(\mathrm{db} \otimes 1_N) : N \to M \otimes M^* \otimes N \to M \otimes P)$$

and

$$(g : N \to M \otimes P) \mapsto ((\mathrm{ev} \otimes 1_P)(1_{M^*} \otimes g) : M^* \otimes N \to M^* \otimes M \otimes P \to P).$$

Proof. We have a natural isomorphism

$$\mathrm{Mor}_{\mathcal{C}}(M^* \otimes \text{-}, \text{-}) \cong \mathrm{Mor}_{\mathcal{C}}(\text{-}, M \otimes \text{-}),$$

iff (M, ev) is a right dual for M^* (as a symmetric statement to Lemma 3.3.3) iff (M^*, ev) is a left dual for M. □

Corollary 3.3.6. *If* M *has a left dual then this is unique up to isomorphism.*

Proof. Let (M^*, ev) and $(M^!, \mathrm{ev}^!)$ be left duals for M. Then the functors $\text{-}\otimes M^*$ and $\text{-}\otimes M^!$ are isomorphic by Lemma A.9.7. In particular we have $M^* \cong I \otimes M^* \cong I \otimes M^! \cong M^!$. If we consider the construction of the isomorphism then we get in particular that $(\mathrm{ev}^! \otimes 1)(1 \otimes \mathrm{db}) : M^! \to M^! \otimes M \otimes M^* \to M^*$ is the given isomorphism. □

Problem 3.3.1. Let (M^*, ev) be a left dual for M. Then there is a *unique* morphism $\mathrm{db} : I \to M \otimes M^*$ satisfying the conditions of Definition 3.3.1.

Definition 3.3.7. Let (M^*, ev_M) and (N^*, ev_N) be left duals for M resp. N. For each morphism $f : M \to N$ we define the *transposed morphism*

$$(f^* : N^* \to M^*) := (N^* \xrightarrow{1 \otimes \mathrm{db}_M} N^* \otimes M \otimes M^* \xrightarrow{1 \otimes f \otimes 1} N^* \otimes N \otimes M^* \xrightarrow{\mathrm{ev}_N \otimes 1} M^*).$$

With this definition we get that the left dual is a contravariant functor, since we have

Lemma 3.3.8. *Let* (M^*, ev_M), (N^*, ev_N), *and* (P^*, ev_P) *be left duals for* M, N *and* P *respectively.*

1. We have $(1_M)^* = 1_{M^*}$.

2. If $f : M \to N$ *and* $g : N \to P$ *are given then* $(gf)^* = f^* g^*$ *holds.*

Proof. 1. $(1_M)^* = (\mathrm{ev} \otimes 1)(1 \otimes 1 \otimes 1)(1 \otimes \mathrm{db}) = 1_{M^*}$.

2. The following diagram commutes

$$\begin{tikzcd}
M \arrow[r, "\mathrm{db}_N \otimes 1"] \arrow[d, "f"] & N \otimes N^* \otimes M \arrow[d, "1 \otimes 1 \otimes f"] & \\
N \arrow[r, "\mathrm{db}_N \otimes 1"] & N \otimes N^* \otimes N \arrow[r, "1 \otimes \mathrm{ev}_N"] \arrow[d, "g \otimes 1 \otimes 1"] & N \arrow[d, "g"] \\
& P \otimes N^* \otimes N \arrow[r, "1 \otimes \mathrm{ev}_N"] & P
\end{tikzcd}$$

Hence we have $gf = (1 \otimes \mathrm{ev}_N)(g \otimes 1 \otimes f)(\mathrm{db}_N \otimes 1)$. Thus the following diagram commutes

$$\begin{tikzcd}
P^* \arrow[r, "1 \otimes \mathrm{db}"] \arrow[d, "1 \otimes \mathrm{db}"] & P^* \otimes N \otimes N^* \arrow[r, "1 \otimes g \otimes 1"] \arrow[d, "1 \otimes \mathrm{db}"] & P^* \otimes P \otimes N^* \arrow[dd, "1 \otimes \mathrm{db}"] \arrow[ddr, "\mathrm{ev} \otimes 1"] & \\
P^* \otimes M \otimes M^* \arrow[r, "1 \otimes \mathrm{db} \otimes 1"] \arrow[dd, "1 \otimes gf \otimes 1"] & P^* \otimes N \otimes N^* \otimes M \otimes M^* \arrow[dr, "1 \otimes g \otimes 1"] \arrow[dd, "1 \otimes g \otimes 1 \otimes f \otimes 1"'] & & \\
& & P^* \otimes P \otimes N^* \otimes M \otimes M^* \arrow[dl, "1 \otimes f \otimes 1"] \arrow[dd, "\mathrm{ev} \otimes 1"] & N^* \arrow[ddl, "1 \otimes \mathrm{db}"] \\
P^* \otimes P \otimes M^* \arrow[d, "\mathrm{ev} \otimes 1"] & P^* \otimes P \otimes N^* \otimes N \otimes M^* \arrow[l, "1 \otimes \mathrm{ev} \otimes 1"'] \arrow[d, "\mathrm{ev} \otimes 1"] & & \\
M^* & N^* \otimes N \otimes M^* \arrow[l, "1 \otimes \mathrm{ev}"'] & N^* \otimes M \otimes M^* \arrow[l, "1 \otimes f \otimes 1"'] &
\end{tikzcd}$$

$\square$

Problem 3.3.2. 1. In the category of $\mathbb{N}$-graded vector spaces determine all objects M that have a left dual.

2. In the category of chain complexes $\mathbb{K}$-**Comp** determine all objects M that have a left dual.

3. In the category of cochain complexes $\mathbb{K}$-**Cocomp** determine all objects M that have a left dual.

4. Let (M^*, ev) be a left dual for M. Show that $\mathrm{db} : I \longrightarrow M \otimes M^*$ is uniquely determined by M, M^*, and ev. (Uniqueness of the dual basis.)

5. Let (M^*, ev) be a left dual for M. Show that $\mathrm{ev} : M^* \otimes M \longrightarrow I$ is uniquely determined by M, M^*, and db.

Corollary 3.3.9. *Let M, N have the left duals (M^*, ev_M) and (N^*, ev_N) and let $f : M \longrightarrow N$ be a morphism in $\mathcal{C}$. Then the following diagram commutes*

$$\begin{tikzcd} I \arrow[r, "\mathrm{db}_M"] \arrow[d, "\mathrm{db}_N"'] & M \otimes M^* \arrow[d, "f\otimes 1"] \\ N \otimes N^* \arrow[r, "1\otimes f^*"'] & N \otimes M^*. \end{tikzcd}$$

PROOF. The following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\mathrm{db}\otimes 1"] \arrow[d, "f"'] & N \otimes N^* \otimes M \arrow[d, "1\otimes 1\otimes f"] \\ N \arrow[r, "\mathrm{db}\otimes 1"] \arrow[dr, "1"'] & N \otimes N^* \otimes N \arrow[d, "1\otimes \mathrm{ev}"] \\ & N \end{tikzcd}$$

This implies $(f \otimes 1_{M^*}) \mathrm{db}_M = ((1_N \otimes \mathrm{ev}_N)(1_N \otimes 1_{N^*} \otimes f)(\mathrm{db}_N \otimes 1_M) \otimes 1_{M^*}) \mathrm{db}_M = (1_N \otimes \mathrm{ev}_N \otimes 1_{M^*})(1_N \otimes 1_{N^*} \otimes f \otimes 1_{M^*})(\mathrm{db}_N \otimes 1_M \otimes 1_{M^*}) \mathrm{db}_M = (1_N \otimes \mathrm{ev}_N \otimes 1_{M^*})(1_N \otimes 1_{N^*} \otimes f \otimes 1_{M^*})(1_N \otimes 1_{N^*} \otimes \mathrm{db}_M) \mathrm{db}_N = (1_N \otimes (\mathrm{ev}_N \otimes 1_{M^*})(1_{N^*} \otimes f \otimes 1_{M^*})(1_{N^*} \otimes \mathrm{db}_M)) \mathrm{db}_N = (1_N \otimes f^*) \mathrm{db}_N$. $\square$

Corollary 3.3.10. *Let M, N have the left duals (M^*, ev_M) and (N^*, ev_N) and let $f : M \longrightarrow N$ be a morphism in $\mathcal{C}$. Then the following diagram commutes*

$$\begin{tikzcd} N^* \otimes M \arrow[r, "f^*\otimes 1"] \arrow[d, "1\otimes f"'] & M^* \otimes M \arrow[d, "\mathrm{ev}_M"] \\ N^* \otimes N \arrow[r, "\mathrm{ev}_N"'] & I. \end{tikzcd}$$

PROOF. This statement follows immediately from the symmetry of the definition of a left dual. $\square$

Example 3.3.11. Let $M \in {}_R\mathcal{M}_R$ be an R-R-bimodule. Then $\mathrm{Hom}_R(M., R.)$ is an R-R-bimodule by $(rfs)(x) = rf(sx)$. Furthermore we have the morphism $\mathrm{ev} : \mathrm{Hom}_R(M., R.) \otimes_R M \longrightarrow R$ defined by $\mathrm{ev}(f \otimes_R m) = f(m)$.

(Dual Basis Lemma:) A module $M \in \mathcal{M}_R$ is called *finitely generated and projective* if there are elements $m_1, \ldots, m_n \in M$ und $m^1, \ldots, m^n \in \mathrm{Hom}_R(M., R.)$ such that

$$\forall m \in M : \sum_{i=1}^{n} m_i m^i(m) = m.$$

Actually this is a consequence of the dual basis lemma. But this definition is equivalent to the usual definition.

Let $M \in {}_R\mathcal{M}_R$. M as a right R-module is finitely generated and projective iff M has a left dual. The left dual is isomorphic to $\mathrm{Hom}_R(M., R.)$.

If M_R is finitely generated projective then we use $\mathrm{db} : R \longrightarrow M \otimes_R \mathrm{Hom}_R(M., R.)$ with $\mathrm{db}(1) = \sum_{i=1}^n m_i \otimes_R m^i$. In fact we have $(1 \otimes_R \mathrm{ev})(\mathrm{db} \otimes_R 1)(m) = (1 \otimes_R \mathrm{ev})(\sum m_i \otimes_R m^i \otimes_R m) = \sum m_i m^i(m) = m$. We have furthermore $(\mathrm{ev} \otimes_R 1)(1 \otimes_R \mathrm{db})(f)(m) = (\mathrm{ev} \otimes_R 1)(\sum_{i=1}^n f \otimes_R m_i \otimes_R m^i)(m) = \sum f(m_i)m^i(m) = f(\sum m_i m^i(m))$ $= f(m)$ for all $m \in M$ hence $(\mathrm{ev} \otimes_R 1)(1 \otimes_R \mathrm{db})(f) = f$.

Conversely if M has a left dual M^* then $\mathrm{ev} : M^* \otimes_R M \longrightarrow R$ defines a homomorphism $\iota : M^* \longrightarrow \mathrm{Hom}_R(M., R.)$ in ${}_R\mathcal{M}_R$ by $\iota(m^*)(m) = \mathrm{ev}(m^* \otimes_R m)$. We define $\sum_{i=1}^n m_i \otimes m^i := \mathrm{db}(1) \in M \otimes M^*$, then $m = (1 \otimes \mathrm{ev})(\mathrm{db} \otimes 1)(m) = (1 \otimes \mathrm{ev})(\sum m_i \otimes m^i \otimes m) = \sum m_i \iota(m^i)(m)$ so that $m_1, \ldots, m_n \in M$ and $\iota(m^1), \ldots, \iota(m^n) \in \mathrm{Hom}_R(M., R.)$ form a dual basis for M, i. e. M is finitely generated and projective as an R-module. Thus M^* and $\mathrm{Hom}_R(M., R.)$ are isomorphic by the map ι.

Analogously $\mathrm{Hom}_R(.M, .R)$ is a right dual for M iff M is finitely generated and projective as a left R-module.

Problem 3.3.3. Find an example of an object M in a monoidal category $\mathcal{C}$ that has a left dual but no right dual.

Definition 3.3.12. Given objects M, N in $\mathcal{C}$. An object $[M, N]$ is called a *left inner Hom* of M and N if there is a natural isomorphism $\mathrm{Mor}_{\mathcal{C}}(\text{-} \otimes M, N) \cong \mathrm{Mor}_{\mathcal{C}}(\text{-}, [M, N])$, i. e. if it represents the functor $\mathrm{Mor}_{\mathcal{C}}(\text{-} \otimes M, N)$.

If there is an isomorphism $\mathrm{Mor}_{\mathcal{C}}(P \otimes M, N) \cong \mathrm{Mor}_{\mathcal{C}}(P, [M, N])$ natural in the three variable M, N, P then the category $\mathcal{C}$ is called *monoidal and left closed.*

If there is an isomorphism $\mathrm{Mor}_{\mathcal{C}}(M \otimes P, N) \cong \mathrm{Mor}_{\mathcal{C}}(P, [M, N])$ natural in the three variable M, N, P then the category $\mathcal{C}$ is called *monoidal and right closed.*

If M has a left dual M^* in $\mathcal{C}$ then there are inner Homs $[M, \text{-}]$ defined by $[M, N] := N \otimes M^*$. In particular left rigid monoidal categories are left closed.

Example 3.3.13.
1. The category of finite dimensional vector spaces is a monoidal category where each object has a (left and right) dual. Hence it is (left and right) closed and rigid.
2. Let **Ban** be the category of (complex) Banach spaces where the morphisms satisfy $\| f(m) \| \leq \| m \|$ i. e. the maps are bounded by 1 or contracting. **Ban** is a monoidal category by the complete tensor product $M \widehat{\otimes} N$. In **Ban** exists an inner Hom functor $[M, N]$ that consists of the set of bounded linear maps from M to N made into a Banach space by an appropriate topology. Thus **Ban** is a monoidal closed category.
3. Let H be a Hopf algebra. The category H-**Mod** of left H-modules is a monoidal category (see Example 3.2.4 2.). Then $\mathrm{Hom}_{\mathbb{K}}(M, N)$ is an object in H-**Mod** by the multiplication
$$(hf)(m) := \sum h_{(1)} f(mS(h_{(2)})$$

as in Proposition 3.1.14.

$\mathrm{Hom}_{\mathbb{K}}(M,N)$ is an inner Hom functor in the monoidal category H-**Mod**. The isomorphism $\phi : \mathrm{Hom}_{\mathbb{K}}(P, \mathrm{Hom}_{\mathbb{K}}(M,N)) \cong \mathrm{Hom}_{\mathbb{K}}(P \otimes M, N)$ can be restricted to an isomorphism

$$\mathrm{Hom}_H(P, \mathrm{Hom}_{\mathbb{K}}(M,N)) \cong \mathrm{Hom}_H(P \otimes M, N),$$

because $\phi(f)(h(p \otimes m)) = \phi(f)(\sum h_{(1)}p \otimes h_{(2)}m) = \sum f(h_{(1)}p)(h_{(2)}m) = \sum(h_{(1)}(f(p)))(h_{(2)}m) = \sum h_{(1)}(f(p)(S(h_{(2)})h_{(3)}m)) = h(f(p)(m)) = h(\phi(f)(p \otimes m))$ and conversely $(h(f(p)))(m) = \sum h_{(1)}(f(p)(S(h_{(2)})m)) = \sum h_{(1)}(\phi(f)(p \otimes S(h_{(2)})m)) = \sum \phi(f)(h_{(1)}p \otimes h_{(2)}S(h_{(3)})m) = \phi(f)(hp \otimes m) = f(hp)(m)$. Thus H-**Mod** is left closed.

If $M \in H$-**Mod** is a finite dimensional vector space then the dual vector space $M^* := \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$ again is an H-module by $(hf)(m) := f(S(h)m)$. Furthermore M^* is a left dual for M with the morphisms

$$\mathrm{db} : \mathbb{K} \ni 1 \mapsto \sum_i m_i \otimes m^i \in M \otimes M^*$$

and

$$\mathrm{ev} : M^* \otimes M \ni f \otimes m \mapsto f(m) \in \mathbb{K}$$

where m_i and m^i are a dual basis of the vector space M. Clearly we have $(1 \otimes \mathrm{ev})(\mathrm{db} \otimes 1) = 1_M$ and $(\mathrm{ev} \otimes 1)(1 \otimes \mathrm{db}) = 1_{M^*}$ since M is a finite dimensional vector space. We have to show that db and ev are H-module homomorphisms. We have

$$\begin{aligned}&(h\,\mathrm{db}(1))(m) = (h(\sum m_i \otimes m^i))(m) = (\sum h_{(1)}m_i \otimes h_{(2)}m^i)(m) = \\ &\sum(h_{(1)}m_i)((h_{(2)}m^i)(m)) = \sum(h_{(1)}m_i)(m^i(S(h_{(2)})m)) = \\ &\sum h_{(1)}S(h_{(2)})m = \varepsilon(h)m = \varepsilon(h)(\sum m_i \otimes m^i)(m) = \varepsilon(h)\,\mathrm{db}(1)(m) = \\ &\mathrm{db}(\varepsilon(h)1)(m) = \mathrm{db}(h1)(m),\end{aligned}$$

hence $h\,\mathrm{db}(1) = \mathrm{db}(h1)$. Furthermore we have

$$\begin{aligned}&h\,\mathrm{ev}(f \otimes m) = hf(m) = \sum h_{(1)}f(S(h_{(2)})h_{(3)}m) = \sum(h_{(1)}f)(h_{(2)}m) = \\ &\sum \mathrm{ev}(h_{(1)}f \otimes h_{(2)}m) = \mathrm{ev}(h(f \otimes m)).\end{aligned}$$

4. Let H be a Hopf algebra. Then the category of left H-comodules (see Example 3.2.4 3.) is a monoidal category. Let $M \in H$-**Comod** be a finite dimensional vector space. Let m_i be a basis for M and let the comultiplication of the comodule be $\delta(m_i) = \sum h_{ij} \otimes m_j$. Then we have $\Delta(h_{ik}) = \sum h_{ij} \otimes h_{jk}$. $M^* := \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$ becomes a left H-comodule $\delta(m^j) := \sum S(h_{ij}) \otimes m^i$. One verifies that M^* is a left dual for M.

Lemma 3.3.14. *Let $M \in \mathcal{C}$ be an object with left dual (M^*, ev). Then*

1. *$M \otimes M^*$ is an algebra with multiplication*

$$\nabla := 1_M \otimes \mathrm{ev} \otimes 1_{M^*} : M \otimes M^* \otimes M \otimes M^* \to M \otimes M^*$$

and unit

$$u := \mathrm{db} : I \longrightarrow M \otimes M^*;$$

2. $M^* \otimes M$ *is a coalgebra with comultiplication*

$$\Delta := 1_{M^*} \otimes \mathrm{db} \otimes 1_M : M^* \otimes M \longrightarrow M^* \otimes M \otimes M^* \otimes M$$

and counit

$$\varepsilon := \mathrm{ev} : M^* \otimes M \longrightarrow I.$$

Proof. 1. The associativity is given by $(\nabla \otimes 1)\nabla = (1_M \otimes \mathrm{ev} \otimes 1_{M^*} \otimes 1_M \otimes 1_{M^*})(1_M \otimes \mathrm{ev} \otimes 1_{M^*}) = 1_M \otimes \mathrm{ev} \otimes \mathrm{ev} \otimes 1_{M^*} = (1_M \otimes 1_{M^*} \otimes 1_M \otimes \mathrm{ev} \otimes 1_{M^*})(1_M \otimes \mathrm{ev} \otimes 1_{M^*}) = (1 \otimes \nabla)\nabla$. The axiom for the left unit is $\nabla(u \otimes 1) = (1_M \otimes \mathrm{ev} \otimes 1_{M^*})(\mathrm{db} \otimes 1_M \otimes 1_{M^*}) = 1_M \otimes 1_{M^*}$.

2. is dual to the statement for algebras. $\square$

Lemma 3.3.15. 1. *Let A be an algebra in $\mathcal{C}$ and left $M \in \mathcal{C}$ be a left rigid object with left dual (M^*, ev). There is a bijection between the set of morphisms $f : A \otimes M \longrightarrow M$ making M a left A-module and the set of algebra morphisms $\widetilde{f} : A \longrightarrow M \otimes M^*$.*
2. *Let C be a coalgebra in $\mathcal{C}$ and left $M \in \mathcal{C}$ be a left rigid object with left dual (M^*, ev). There is a bijection between the set of morphisms $f : M \longrightarrow M \otimes C$ making M a right C-comodule and the set of coalgebra morphisms $\widetilde{f} : M^* \otimes M \longrightarrow C$.*

Proof. 1. By Lemma 3.3.14 the object $M \otimes M^*$ is an algebra. Given $f : A \otimes M \longrightarrow M$ such that M becomes an A-module. By Lemma 3.3.3 we associate $\widetilde{f} := (f \otimes 1)(1 \otimes \mathrm{db}) : A \longrightarrow A \otimes M \otimes M^* \longrightarrow M \otimes M^*$. The compatibility of $\widetilde{f}$ with the multiplication is given by the commutative diagram

$$\begin{tikzcd}
A \otimes A \arrow[rrrr, "\nabla"] \arrow[dr, "1\otimes 1\otimes \mathrm{db}"] \arrow[ddr, "1\otimes\widetilde{f}"'] \arrow[dddd, "\widetilde{f}\otimes\widetilde{f}"'] & & & & A \arrow[dl, "1\otimes \mathrm{db}"'] \arrow[dddd, "\widetilde{f}"] \\
& A \otimes A \otimes M \otimes M^* \arrow[rr, "\nabla\otimes 1\otimes 1"] \arrow[d, "1\otimes f\otimes 1"] & & A \otimes M \otimes M^* \arrow[d, "f\otimes 1"'] & \\
& A \otimes M \otimes M^* \arrow[rr, "f\otimes 1"] \arrow[d, "1\otimes \mathrm{db}\otimes 1\otimes 1"] \arrow[drr, "1"] \arrow[ddl, "\widetilde{f}\otimes 1\otimes 1"'] & & M \otimes M^* \arrow[ddr, "1"] & \\
& A \otimes M \otimes M^* \otimes M \otimes M^* \arrow[rr, "1\otimes 1\otimes \mathrm{ev}\otimes 1"'] \arrow[dl, "f\otimes 1\otimes 1"] & & A \otimes M \otimes M^* \arrow[dr, "f\otimes 1"'] & \\
M \otimes M^* \otimes M \otimes M^* \arrow[rrrr, "1\otimes \mathrm{ev}\otimes 1"'] & & & & M \otimes M^*
\end{tikzcd}$$

The unit axiom is given by

$$\begin{tikzcd}
I \arrow[r, "\mathrm{db}"] \arrow[d, "u"'] & M \otimes M^* \arrow[d, "u\otimes 1"'] \arrow[dr, "1\otimes 1"] & \\
A \arrow[r, "1\otimes \mathrm{db}"] & A \otimes M \otimes M^* \arrow[r, "f\otimes 1"] & M \otimes M^*
\end{tikzcd}$$

Conversely let $g : A \longrightarrow M \otimes M^*$ be an algebra homomorphism and consider $\widetilde{g} := (1 \otimes \mathrm{ev})(g \otimes 1) : A \otimes M \longrightarrow M \otimes M^* \otimes M \longrightarrow M$. Then M becomes a left A-module since

$$\begin{tikzcd}
A \otimes A \otimes M \arrow[rrrr, "\nabla\otimes 1"] \arrow[dr, "1\otimes g\otimes 1"] \arrow[drr, "g\otimes g\otimes 1"] \arrow[ddd, "1\otimes\widetilde{g}"'] & & & & A \otimes M \arrow[dl, "g\otimes 1"'] \arrow[ddd, "\widetilde{g}"] \\
& A \otimes M \otimes M^* \otimes M \arrow[r, "g\otimes 1\otimes 1\otimes 1"'] \arrow[ddl, "1\otimes 1\otimes \mathrm{ev}"] & M \otimes M^* \otimes M \otimes M^* \otimes M \arrow[r, "1\otimes \mathrm{ev}\otimes 1\otimes 1"] \arrow[d, "1\otimes 1\otimes 1\otimes \mathrm{ev}"] & M \otimes M^* \otimes M \arrow[ddr, "1\otimes \mathrm{ev}"'] & \\
& & M \otimes M^* \otimes M \arrow[drr, "1\otimes \mathrm{ev}"] & & \\
A \otimes M \arrow[urr, "g\otimes 1"] \arrow[rrrr, "\widetilde{g}"] & & & & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \arrow[d, "u\otimes 1"'] \arrow[dr, "\mathrm{db}\otimes 1"] \arrow[drr, "1"] & & \\
A \otimes M \arrow[r, "g\otimes 1"'] & M \otimes M^* \otimes M \arrow[r, "1\otimes \mathrm{ev}"'] & M
\end{tikzcd}$$

commute.

2. (M^*, ev) is a left dual for M in the category $\mathcal{C}$ if and only if (M^*, db) is the right dual for M in the dual category $\mathcal{C}^{op}$. So if we dualize the result of part 1. we have to change sides, hence 2. $\square$