CHAPTER 3

Hopf Algebras, Algebraic, Formal, and Quantum Groups

2. Monoidal Categories

For our further investigations we need a generalized version of the tensor product that we are going to introduce in this section. This will give us the possibility to study more general versions of the notion of algebras and representations.

Definition 3.2.1. A *monoidal category* (or *tensor category*) consists of
a category $\mathcal{C}$,
a covariant functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the *tensor product*,
an object $I \in \mathcal{C}$, called the *unit*,
natural isomorphisms

$$\begin{aligned} &\alpha(A, B, C) : (A \otimes B) \otimes C \to A \otimes (B \otimes C), \\ &\lambda(A) : I \otimes A \to A, \\ &\rho(A) : A \otimes I \to A, \end{aligned}$$

called *associativity, left unit* and *right unit*, such that the following diagrams commute:

$$\begin{tikzcd} ((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha(A,B,C)\otimes 1"] \arrow[d, "\alpha(A\otimes B,C,D)"] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha(A,B\otimes C,D)"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "1\otimes\alpha(B,C,D)"'] \\ (A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha(A,B,C\otimes D)"] & & A \otimes (B \otimes (C \otimes D)) \end{tikzcd}$$

$$\begin{tikzcd} (A \otimes I) \otimes B \arrow[rr, "\alpha(A,I,B)"] \arrow[dr, "\rho(A)\otimes 1"'] & & A \otimes (I \otimes B) \arrow[dl, "1\otimes\lambda(B)"] \\ & A \otimes B & \end{tikzcd}$$

These diagrams are called *coherence diagrams* or *constraints.*

A monoidal category is called a *strict monoidal category*, if the morphisms α, λ, ρ are the identity morphisms.

Remark 3.2.2. We define $A_1 \otimes \ldots \otimes A_n := (\ldots(A_1 \otimes A_2) \otimes \ldots) \otimes A_n$.

There is an important theorem of S. MacLane that says that all diagrams whose morphisms are constructed by using copies of α, λ, ρ, identities, inverses, tensor products and compositions of such commute. We will not prove this theorem. It implies that each monoidal category can be replaced by (is monoidally equivalent to) a strict monoidal category. That means that we may omit in diagrams the morphisms α, λ, ρ or replace them by identities. In particular there is only one automorphism of $A_1 \otimes \ldots \otimes A_n$ formed by coherence morphisms namely the identity.

Remark 3.2.3. For each monoidal category $\mathcal{C}$ we can construct the monoidal category $\mathcal{C}^{symm}$ symmetric to $\mathcal{C}$ that coincides with $\mathcal{C}$ as a category and has tensor product $A \boxtimes B := B \otimes A$ and the coherence morphisms

$$\begin{aligned} &\alpha(C, B, A)^{-1} : (A \boxtimes B) \boxtimes C \to A \boxtimes (B \boxtimes C), \\ &\rho(A) : I \boxtimes A \to A, \\ &\lambda(A) : A \boxtimes I \to A. \end{aligned}$$

Then the coherence diagrams are commutative again, so that $\mathcal{C}^{symm}$ is a monoidal category.

Example 3.2.4. 1. Let R be an arbitrary ring. The category ${}_R\mathcal{M}_R$ of R-R-bimodules with the tensor product $M \otimes_R N$ is a monoidal category. In particular the $\mathbb{K}$-modules form a monoidal category. This is our most important example of a monoidal category.

2. Let B be a bialgebra and B-**Mod** be the category of left B-modules. We define the structure of a B-module on the tensor product $M \otimes N = M \otimes_{\mathbb{K}} N$ by

$$B \otimes M \otimes N \xrightarrow{\Delta \otimes 1_M \otimes 1_N} B \otimes B \otimes M \otimes N \xrightarrow{1_B \otimes \tau \otimes 1_N} B \otimes M \otimes B \otimes N \xrightarrow{\rho_M \otimes \rho_N} M \otimes N$$

as in the previous section. So B-**Mod** is a monoidal category by 3.1.7

3. Let B be a bialgebra and B-**Comod** be the category of B-comodules. The tensor product $M \otimes N = M \otimes_{\mathbb{K}} N$ carries the structure of a B-comodule by

$$M \otimes N \xrightarrow{\delta_M \otimes \delta_N} B \otimes M \otimes B \otimes N \xrightarrow{1_B \otimes \tau \otimes 1_N} B \otimes B \otimes M \otimes N \xrightarrow{\nabla \otimes 1_M \otimes 1_N} B \otimes M \otimes N.$$

as in the previous section. So B-**Comod** is a monoidal category by 3.1.8

4. Let G be a monoid. A $\mathbb{K}$-module together with a family of submodules $(V_g | g \in G)$ is called *G-graded* if $V = \oplus_{g \in G} V_g$.

Let V and W be G-graded $\mathbb{K}$-modules. A homomorphism of $\mathbb{K}$-modules $f : V \to W$ is called G-graded if $f(V_g) \subseteq W_g$ for all $g \in G$.

The G-graded $\mathbb{K}$-modules and their homomorphisms form the category $(\mathbb{K}\text{-}\mathbf{Mod})^G$ of *G-graded $\mathbb{K}$-modules.*

There is a monoidal structure on $(\mathbb{K}\text{-}\mathbf{Mod})^G$ given by the ordinary tensor product $V \otimes W$. The submodules on the tensor product $V \otimes W$ are given by $(V \otimes W)_g := \sum_{h \in G} V_h \otimes W_{h^{-1}g} = \sum_{h,k \in G, hk=g} V_h \otimes W_k$.

5. A *chain complex* of $\mathbb{K}$-modules

$$M = (\ldots \xrightarrow{\partial_3} M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0)$$

consists of a family of a family of $\mathbb{K}$-modules M_i and a family of homomorphisms $\partial_n : M_n \to M_{n-1}$ with $\partial_{n-1}\partial_n = 0$. This chain complex is indexed by the monoid $\mathbb{N}_0$. One may also consider more general chain complexes indexed by an arbitrary cyclic monoid. Chain complexes indexed by $\mathbb{N}_0 \times \mathbb{N}_0$ are called double complexes. So much more general chain complexes may be considered. We restrict ourselves to chain complexes over $\mathbb{N}_0$.

Let M and N be chain complexes. A *homomorphism of chain complexes* $f : M \to N$ consists of a family of homomorphisms of $\mathbb{K}$-modules $f_n : M_n \to N_n$ such that $f_n \partial_{n+1} = \partial_{n+1} f_{n+1}$ for all $n \in \mathbb{N}_0$.

The chain complexes with these homomorphisms form the category of chain complexes $\mathbb{K}$-**Comp**.

If M and N are chain complexes then we form a new chain complex $M \otimes N$ with $(M \otimes N)_n := \bigoplus_{i=0}^{n} M_i \otimes N_{n-i}$ and $\partial : (M \otimes N)_n \to (M \otimes N)_{n-1}$ given by

$\partial(m_i \otimes n_{n-i}) := (-1)^i \partial_M(m_i) \otimes n_{n-i} + m_i \otimes \partial(n_{n-i})$. This is often called the total complex associated with the double complex of the tensor product of M and N. Then it is easily checked that $\mathbb{K}$-**Comp** is a monoidal category with this tensor product.

Problem 3.2.1. 1. Prove that the category $(\mathbb{K}\text{-}\mathbf{Mod})^G$ of G-graded $\mathbb{K}$-modules is equivalent to the category $\mathbb{K}G$-**Comod** of $\mathbb{K}G$-comodules by the following construction. If V is a G-graded $\mathbb{K}$-module the V becomes a $\mathbb{K}G$-comodule by the map $\delta : V \longrightarrow \mathbb{K}G \otimes V$, $\delta(v) := g \otimes v$ for all $v \in V_g$ and all $g \in G$. Conversely if $V, \delta : V \longrightarrow \mathbb{K}G \otimes V$ is a $\mathbb{K}G$-comodule then V together with the submodules $V_g := \{v \in V | \delta(v) = g \otimes v\}$ is a G-graded $\mathbb{K}$-module.

Since $\mathbb{K}G$ is a bialgebra the category of $\mathbb{K}G$-comodules is a monoidal category. Show that the equivalence defined above between $(\mathbb{K}\text{-}\mathbf{Mod})^G$ and $\mathbb{K}G$-**Comod** preserves the tensor products, hence that it is a monoidal equivalence.

2. Let $B = \mathbb{K}\langle x, y\rangle / I$ where I is generated by $x^2, xy + yx$. Then B is a bialgebra with the diagonal $\Delta(y) = y \otimes y$, $\Delta(x) = x \otimes 1 + y \otimes x$. The counit is $\varepsilon(y) = 1, \varepsilon(x) = 0$. We introduced (the coopposite bialgebra of) this bialgebra in A.7 2.

Show that the category $\mathbb{K}$-**Comp** of chain complexes is equivalent to the category B-**Comod** of B-comodules by the following construction. If M is a chain complex then define a B-comodule on $M = \oplus_{i \in \mathbb{N}} M_i$ with the structure map $\delta : M \longrightarrow B \otimes M$, $\delta(m) := y^i \otimes m + xy^{i-1} \otimes \partial_i(m)$ for all $m \in M_i$ and for all $i \in \mathbb{N}$ resp. $\delta(m) := 1 \otimes m$ for $m \in M_0$. Conversely if $M, \delta : M \longrightarrow B \otimes M$ is a B-comodule then we define $\mathbb{K}$-modules $M_i := \{m \in M | \exists m' \in M[\delta(m) = y^i \otimes m + xy^{i-1} \otimes m']\}$ and $\mathbb{K}$-linear maps $\partial_i : M_i \longrightarrow M_{i-1}$ by $\partial_i(m) := m'$ for $\delta(m) = y^i \otimes m + xy^{i-1} \otimes m'$. Check that this defines an equivalence of categories.

(Hint: Let $m \in M \in B$-**Comod**. Since y^i, xy^i form a basis of B we have $\delta(m) = \sum_i y^i \otimes m_i + \sum_i xy^i \otimes m_i'$. We apply to this the equation $(1 \otimes \delta)\delta = (\Delta \otimes 1)\delta$ and compare coefficients to get

$$\delta(m_i) = y^i \otimes m_i + xy^{i-1} \otimes m_{i-1}', \quad \delta(m_i') = y^i \otimes m_i'$$

for all $i \in \mathbb{N}_0$ (with $m_{-1}' = 0$). Consequently for each $m_i \in M_i$ there is exactly one $\partial(m_i) = m_{i-1}' \in M$ such that

$$\delta(m_i) = y^i \otimes m_i + xy^{i-1} \otimes \partial(m_i).$$

Since $\delta(m_{i-1}') = y^{i-1} \otimes m_{i-1}'$ for all $i \in \mathbb{N}$ we see that $\partial(m_i) \in M_{i-1}$. So we have defined $\partial : M_i \longrightarrow M_{i-1}$. Furthermore we see from this equation that $\partial^2(m_i) = 0$ for all $i \in \mathbb{N}$. Hence we have obtained a chain complex from (M, δ).

If we apply $(\epsilon \otimes 1)\delta(m) = m$ then we get $m = \sum m_i$ with $m_i \in M_i$ hence $M = \bigoplus_{i \in \mathbb{N}} M_i$. This together with the inverse construction leads to the required equivalence.)

3. A cochain complex has the form

$$M = (M_0 \xrightarrow{\partial_0} M_1 \xrightarrow{\partial_1} M_2 \xrightarrow{\partial_2} \dots)$$

with $\partial_{i+1}\partial_i = 0$. Show that the category $\mathbb{K}$-**Cocomp** of cochain complexes is equivalent to **Comod**-B where B is chosen as in example 5.

Lemma 3.2.5. *Let $\mathcal{C}$ be a monoidal category. Then the following diagrams commute*

$$\begin{tikzcd}
(I\otimes A)\otimes B \arrow[rr, "\alpha"] \arrow[dr, "\lambda(A)\otimes 1_B"'] & & I\otimes(A\otimes B) \arrow[dl, "\lambda(A\otimes B)"] \\
& A\otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A\otimes B)\otimes I \arrow[rr, "\alpha"] \arrow[dr, "\rho(A\otimes B)"'] & & A\otimes(B\otimes I) \arrow[dl, "1_A\otimes\rho(B)"] \\
& A\otimes B &
\end{tikzcd}$$

and we have $\lambda(I) = \rho(I)$.

Proof. First we observe that the identity functor $\mathrm{Id}_{\mathcal{C}}$ and the functor $I\otimes$ - are isomorphic by the natural isomorphism λ. Thus we have $I\otimes f = I\otimes g \Longrightarrow f = g$. In the following diagram

$$\begin{tikzcd}
((I\otimes I)\otimes A)\otimes B \arrow[rr, "\alpha\otimes 1"] \arrow[dr, "(\rho\otimes 1)\otimes 1"] \arrow[ddd, "\alpha"'] & & (I\otimes(I\otimes A))\otimes B \arrow[rr, "\alpha"] \arrow[dl, "(1\otimes\lambda)\otimes 1"] & & I\otimes((I\otimes A)\otimes B) \arrow[dl, "1\otimes(\lambda\otimes 1)"'] \arrow[ddd, "1\otimes\alpha"] \\
& (I\otimes A)\otimes B \arrow[rr, "\alpha"] \arrow[d, "\alpha"] & & I\otimes(A\otimes B) \arrow[d, "1"'] & \\
& I\otimes(A\otimes B) \arrow[rr, "1"] & & I\otimes(A\otimes B) \arrow[dr, "1\otimes\lambda"'] & \\
(I\otimes I)\otimes(A\otimes B) \arrow[ur, "\rho\otimes(1\otimes 1)"] \arrow[rrrr, "\alpha"] & & & & I\otimes(I\otimes(A\otimes B))
\end{tikzcd}$$

all subdiagrams commute except for the right hand trapezoid. Since all morphisms are isomorphisms the right hand trapezoid must commute also. Hence the first diagram of the Lemma commutes.

In a similar way one shows the commutativity of the second diagram.

Furthermore the following diagram commutes

$$\begin{tikzcd}
I\otimes(I\otimes I) \arrow[dr, "1\otimes\rho"'] & & (I\otimes I)\otimes I \arrow[ll, "\alpha"'] \arrow[rr, "\alpha"] \arrow[dl, "\rho"] \arrow[dr, "\rho\otimes 1"'] & & I\otimes(I\otimes I) \arrow[dl, "1\otimes\lambda"] \\
& I\otimes I \arrow[dr, "\rho"'] & & I\otimes I \arrow[dl, "\rho"] & \\
& & I & &
\end{tikzcd}$$

Here the left hand triangle commutes by the previous property. The commutativity of the right hand diagram is given by the axiom. The lower square commutes since ρ is a natural transformation. In particular $\rho(1 \otimes \rho) = \rho(1 \otimes \lambda)$. Since ρ is an isomorphism and $I \otimes \text{-} \cong \mathrm{Id}_{\mathcal{C}}$ we get $\rho = \lambda$. $\square$

Problem 3.2.2. For morphisms $f : I \to M$ and $g : I \to N$ in a monoidal category we define $(f \otimes 1 : N \to M \otimes N) := (f \otimes 1_I)\rho(I)^{-1}$ and $(1 \otimes g : M \to M \otimes N) := (1 \otimes g)\lambda(I)^{-1}$. Show that the diagram

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[d, "1\otimes g"] \\ N \arrow[r, "f\otimes 1"'] & M \otimes N \end{tikzcd}$$

commutes.

Definition 3.2.6. Let $(\mathcal{C}, \otimes)$ and $(\mathcal{D}, \otimes)$ be monoidal categories. A functor

$$\mathcal{F} : \mathcal{C} \to \mathcal{D}$$

together with a natural transformation

$$\xi(M, N) : \mathcal{F}(M) \otimes \mathcal{F}(N) \to \mathcal{F}(M \otimes N)$$

and a morphism

$$\xi_0 : I_{\mathcal{D}} \to \mathcal{F}(I_{\mathcal{C}})$$

is called *weakly monoidal* if the following diagrams commute

$$\begin{tikzcd} (\mathcal{F}(M) \otimes \mathcal{F}(N)) \otimes \mathcal{F}(P) \arrow[r, "\xi\otimes 1"] \arrow[d, "\alpha"'] & \mathcal{F}(M \otimes N) \otimes \mathcal{F}(P) \arrow[r, "\xi"] & \mathcal{F}((M \otimes N) \otimes P) \arrow[d, "\mathcal{F}(\alpha)"] \\ \mathcal{F}(M) \otimes (\mathcal{F}(N) \otimes \mathcal{F}(P)) \arrow[r, "1\otimes\xi"] & \mathcal{F}(M) \otimes \mathcal{F}(N \otimes P) \arrow[r, "\xi"] & \mathcal{F}(M \otimes (N \otimes P)) \end{tikzcd}$$

$$\begin{tikzcd} I \otimes \mathcal{F}(M) \arrow[r, "\xi_0\otimes 1"] \arrow[dr, "\lambda"'] & \mathcal{F}(I) \otimes \mathcal{F}(M) \arrow[r, "\xi"] & \mathcal{F}(I \otimes M) \arrow[dl, "\mathcal{F}(\lambda)"] \\ & \mathcal{F}(M) & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F}(M) \otimes I \arrow[r, "1\otimes\xi_0"] \arrow[dr, "\rho"'] & \mathcal{F}(M) \otimes \mathcal{F}(I) \arrow[r, "\xi"] & \mathcal{F}(M \otimes I) \arrow[dl, "\mathcal{F}(\rho)"] \\ & \mathcal{F}(M). & \end{tikzcd}$$

If, in addition, the morphisms ξ and ξ_0 are isomorphisms then the functor is called a *monoidal functor*. The functor is called a *strict monoidal functor* if ξ and ξ_0 are the identity morphisms.

A natural transformation $\zeta : \mathcal{F} \longrightarrow \mathcal{F}'$ between weakly monoidal functors is called a *monoidal natural transformation* if the diagrams

$$\begin{tikzcd}
\mathcal{F}(M) \otimes \mathcal{F}(N) \arrow[r, "\xi"] \arrow[d, "\zeta\otimes\zeta"'] & \mathcal{F}(M \otimes N) \arrow[d, "\zeta"] \\
\mathcal{F}'(M) \otimes \mathcal{F}'(N) \arrow[r, "\xi'"'] & \mathcal{F}'(M \otimes N)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{F}(I) \arrow[dd, "\zeta"] \\
I \arrow[ur, "\xi_0"] \arrow[dr, "\xi_0'"'] & \\
 & \mathcal{F}'(I)
\end{tikzcd}$$

commute.

We can generalize the notions of an algebra or of a coalgebra in the context of a monoidal category. We define

Definition 3.2.7. Let $\mathcal{C}$ be a monoidal category. An *algebra* or a *monoid* in $\mathcal{C}$ consists of an object A together with a multiplication $\nabla : A \otimes A \longrightarrow A$ that is associative

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes\nabla"] \arrow[d, "\nabla\otimes 1"'] & A \otimes A \arrow[d, "\nabla"] \\
A \otimes A \arrow[r, "\nabla"'] & A
\end{tikzcd}$$

or more precisely

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "\alpha"] \arrow[d, "\nabla\otimes\mathrm{id}"'] & A \otimes (A \otimes A) \arrow[r, "\mathrm{id}\otimes\nabla"] & A \otimes A \arrow[d, "\nabla"] \\
A \otimes A \arrow[rr, "\nabla"] & & A
\end{tikzcd}$$

and has a unit $\eta : I \longrightarrow A$ such that the following diagram commutes

$$\begin{tikzcd}
I \otimes A \cong A \cong A \otimes I \arrow[r, "\mathrm{id}\otimes\eta"] \arrow[d, "\eta\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & A \otimes A \arrow[d, "\nabla"] \\
A \otimes A \arrow[r, "\nabla"'] & A.
\end{tikzcd}$$

Let A and B be algebras in $\mathcal{C}$. A *morphism of algebras* $f : A \longrightarrow B$ is a morphism in $\mathcal{C}$ such that

$$\begin{tikzcd} A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\nabla_A"'] & B \otimes B \arrow[d, "\nabla_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & I \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_B"] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

commute.

Remark 3.2.8. It is obvious that the composition of two morphisms of algebras is again a morphism of algebras. The identity also is a morphism of algebras. Thus we obtain the category $\mathbf{Alg}(\mathcal{C})$ of algebras in $\mathcal{C}$.

Definition 3.2.9. Let $\mathcal{C}$ be a monoidal category. A *coalgebra* or a *comonoid* in $\mathcal{C}$ consists of an object C together with a comultiplication $\Delta : A \longrightarrow A \otimes A$ that is coassociative

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes \mathrm{id}"] \\ C \otimes C \arrow[r, "\mathrm{id} \otimes \Delta"'] & C \otimes C \otimes C \end{tikzcd}$$

or more precisely

$$\begin{tikzcd} C \arrow[rrr, "\Delta"] \arrow[d, "\Delta"] & & & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"'] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"] & (C \otimes C) \otimes C \arrow[rr, "\alpha"] & & C \otimes (C \otimes C) \end{tikzcd}$$

and has a counit $\varepsilon : C \longrightarrow I$ such that the following diagram commutes

$$\begin{tikzcd} C \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] \arrow[drr, "\mathrm{id}"] & & C \otimes C \arrow[d, "\mathrm{id} \otimes \epsilon"] \\ C \otimes C \arrow[r, "\epsilon \otimes \mathrm{id}"'] & I \otimes C \cong C & \cong C \otimes I. \end{tikzcd}$$

Let C and D be coalgebras in $\mathcal{C}$. A *morphism of coalgebras* $f : C \longrightarrow D$ is a morphism in $\mathcal{C}$ such that

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f \otimes f"'] & D \otimes D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon_C"'] & & D \arrow[dl, "\varepsilon_D"] \\ & I & \end{tikzcd}$$

commute.

Remark 3.2.10. It is obvious that the composition of two morphisms of coalgebras is again a morphism of coalgebras. The identity also is a morphism of coalgebras. Thus we obtain the category $\mathbf{Coalg}(\mathcal{C})$ of coalgebras in $\mathcal{C}$.

Remark 3.2.11. Observe that the notions of bialgebra, Hopf algebra, and comodule algebra cannot be generalized to an arbitrary monoidal category since we need to have an algebra structure on the tensor product of two algebras and this requires us to interchange the middle tensor factors. These interchanges or flips are known under the name symmetry, quasisymmetry or braiding and will be discussed later on.