CHAPTER 3

Hopf Algebras, Algebraic, Formal, and Quantum Groups

Introduction

One of the most interesting properties of quantum groups is their representation theory. It has deep applications in theoretical physics. The mathematical side has to distinguish between the representation theory of quantum groups and the representation theory of Hopf algebras. In both cases the particular structure allows to form tensor products of representations such that the category of representations becomes a monoidal (or tensor) category.

The problem we want to study in this chapter is, how much structure of the quantum group or Hopf algebra can be found in the category of representations. We will show that a quantum monoid can be uniquely reconstructed (up to isomorphism) from its representations. The additional structure given by the antipode is itimitely connected with a certain duality of representations. We will also generalize this process of reconstruction.

On the other hand we will show that the process of reconstruction can also be used to obtain the Tambara construction of the universal quantum monoid of a noncommutative geometrical space (from chapter 1.). Thus we will get another perspective for this theorem.

At the end of the chapter you should

- understand representations of Hopf algebras and of quantum groups,
- know the definition and first fundamental properties of monoidal or tensor categories,
- be familiar with the monoidal structure on the category of representations of Hopf algebras and of quantum groups,
- understand why the category of representations contains the full information about the quantum group resp. the Hopf algebra (Theorem of Tannaka-Krein),
- know the process of reconstruction and examples of bialgebras reconstructed from certain diagrams of finite dimensional vector spaces,
- understand better the Tambara construction of a universal algebra for a finite dimensional algebra.

1. Representations of Hopf Algebras

Let A be an algebra over a commutative ring $\mathbb{K}$. Let A-**Mod** be the category of A-modules. An A-module is also called a *representation* of A.

Observe that the action $A \otimes M \to M$ satisfying the module axioms and an algebra homomorphism $A \to \mathrm{End}(M)$ are equivalent descriptions of an A-module structure on the $\mathbb{K}$-module M.

The functor $\mathcal{U} : A\text{-}\mathbf{Mod} \to \mathbb{K}\text{-}\mathbf{Mod}$ with $\mathcal{U}({}_AM) = M$ and $\mathcal{U}(f) = f$ is called the *forgetful functor* or the *underlying functor*.

If B is a bialgebra then a *representation of* B is also defined to be a B-module. It will turn out that the property of being a bialgebra leads to the possibility of building tensor products of representations in a canonical way.

Let C be a coalgebra over a commutative ring $\mathbb{K}$. Let C-**Comod** be the category of C-comodules. A C-comodule is also called a *corepresentation* of C.

The functor $\mathcal{U} : C\text{-}\mathbf{Comod} \to \mathbb{K}\text{-}\mathbf{Mod}$ with $\mathcal{U}({}^CM) = M$ and $\mathcal{U}(f) = f$ is called the *forgetful functor* or the *underlying functor*.

If B is a bialgebra then a *corepresentation of* B is also defined to be a B-comodule. It will turn out that the property of being a bialgebra leads to the possibility of building tensor products of corepresentations in a canonical way.

Usually representations of a ring are considered to be modules over the given ring. The role of comodules certainly arises in the context of coalgebras. But it is not quite clear what the good definition of a representation of a quantum group or its representing Hopf algebra is.

For this purpose consider representations M of an ordinary group G. Assume for the simplicity of the argument that G is finite. Representations of G are vector spaces together with a group action $G \times M \to M$. Equivalently they are vector spaces together with a group homomorphism $G \to \mathrm{Aut}(M)$ or modules over the group algebra: $\mathbb{K}[G] \otimes M \to M$. In the situation of quantum groups we consider the representing Hopf algebra H as algebra of functions on the quantum group G.

Then the algebra of functions on G is the Hopf algebra $\mathbb{K}^G$, the dual of the group algebra $\mathbb{K}[G]$. An easy exercise shows that the module structure $\mathbb{K}[G] \otimes M \to M$ translates to the structure of a comodule $M \to \mathbb{K}^G \otimes M$ and conversely. (Observe that G is finite.) So we should define representations of a quantum group as comodules over the representing Hopf algebra.

Definition 3.1.1. Let G be a quantum group with representing Hopf algebra H. A *representation* of G is a comodule over the representing Hopf algebra H.

From this definition we obtain immediately that we may form tensor products of representations of quantum groups since the representing algebra is a bialgebra.

We come now to the canonical construction of tensor products of (co-)representations.

Lemma 3.1.2. *Let B be a bialgebra. Let $M, N \in B$-**Mod** be two B-modules. Then $M \otimes N$ is a B-module by the action $b(m \otimes n) = \sum b_{(1)} m \otimes b_{(2)} n$. If $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$ are homomorphisms of B-modules in B-**Mod** then $f \otimes g : M \otimes N \longrightarrow M' \otimes N'$ is a homomorphism of B-modules.*

PROOF. We have homomorphisms of $\mathbb{K}$-algebras $\alpha : B \longrightarrow \mathrm{End}(M)$ and $\beta : B \longrightarrow \mathrm{End}(N)$ defining the B-module structure on M and N. Thus we get a homomorphism of algebras $\mathrm{can}(\alpha \otimes \beta)\Delta : B \longrightarrow B \otimes B \longrightarrow \mathrm{End}(M) \otimes \mathrm{End}(N) \longrightarrow \mathrm{End}(M \otimes N)$. Thus $M \otimes N$ is a B-module. The structure is $b(m \otimes n) = \mathrm{can}(\alpha \otimes \beta)(\sum b_{(1)}) \otimes b_{(2)})(m \otimes n) = \mathrm{can}(\sum \alpha(b_{(1)}) \otimes \beta(b_{(2)}))(m \otimes n) = \sum \alpha(b_{(1)})(m) \otimes \beta(b_{(2)})(n) = \sum b_{(1)} m \otimes b_{(2)} n$.

Furthermore we have $1(m \otimes n) = 1m \otimes 1m = m \otimes n$.

If f, g are homomorphisms of B-modules, then we have $(f \otimes g)(b(m \otimes n)) = (f \otimes g)(\sum b_{(1)} m \otimes b_{(2)} n) = \sum f(b_{(1)} m) \otimes g(b_{(2)} n) = \sum b_{(1)} f(m) \otimes b_{(2)} g(n) = b(f(m) \otimes g(n)) = b(f \otimes g)(m \otimes n)$. Thus $f \otimes g$ is a homomorphism of B-modules. ☐

Corollary 3.1.3. *Let B be a bialgebra. Then $\otimes : B$-**Mod** $\times$ B-**Mod** $\longrightarrow B$-**Mod** with $\otimes(M, N) = M \otimes N$ and $\otimes(f, g) = f \otimes g$ is a functor.*

PROOF. The following are obvious from the ordinary properties of the tensor product over $\mathbb{K}$. $1_M \otimes 1_N = 1_{M \otimes N}$ and $(f \otimes g)(f' \otimes g') = ff' \otimes gg'$ for $M, N, f, f', g, g' \in B$-**Mod**. ☐

Lemma 3.1.4. *Let B be a bialgebra. Let $M, N \in B$-**Comod** be two B-comodules. Then $M \otimes N$ is a B-comodule by the coaction $\delta_{M \otimes N}(m \otimes n) = \sum m_{(1)} n_{(1)} \otimes m_{(M)} \otimes n_{(N)}$.*

*If $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$ are homomorphisms of B-comodules in B-**Comod** then $f \otimes g : M \otimes N \longrightarrow M' \otimes N'$ is a homomorphism of B-comodules.*

PROOF. The coaction on $M \otimes N$ may also be described by $(\nabla_B \otimes 1_M \otimes 1_N)(1_B \otimes \tau \otimes 1_N)(\delta_M \otimes \delta_N) : M \otimes N \longrightarrow B \otimes M \otimes B \otimes N \longrightarrow B \otimes B \otimes M \otimes N \longrightarrow B \otimes M \otimes N$. Although a diagrammatic proof of the coassociativity of the coaction and the law of the counit is quite involved it allows generalization so we give it here.

Consider the next diagram.

Square (1) commutes since M and N are comodules.

Squares (2) and (3) commute since $\tau : M \otimes N \longrightarrow N \otimes M$ for $\mathbb{K}$-modules M and N is a natural transformation.

Square (4) represents an interesting property of τ namely

$$\begin{gathered}(1 \otimes 1 \otimes \tau)(\tau_{B \otimes M, B} \otimes 1) = (1 \otimes 1 \otimes \tau)(\tau \otimes 1 \otimes 1)(1 \otimes \tau \otimes 1) = \\ (\tau \otimes 1 \otimes 1)(1 \otimes 1 \otimes \tau)(1 \otimes \tau \otimes 1) = (\tau \otimes 1 \otimes 1)(1 \otimes \tau_{M, B \otimes B})\end{gathered}$$

that uses the fact that $(1 \otimes g)(f \otimes 1) = (f \otimes 1)(1 \otimes g)$ holds and that $\tau_{B \otimes M, B} = (\tau \otimes 1)(1 \otimes \tau)$ and $\tau_{M, B \otimes B} = (1 \otimes \tau)(\tau \otimes 1)$.

Square (5) and (6) commute by the properties of the tensor product.

Square (7) commutes since B is a bialgebra.

$$\begin{tikzcd}
M \otimes N \arrow[r, "\delta\otimes\delta"] \arrow[d, "\delta\otimes\delta"] \arrow[dr, phantom, "(1)"] & B \otimes M \otimes B \otimes N \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "\Delta\otimes 1\otimes\Delta\otimes 1"] \arrow[dr, phantom, "(2)"] & B \otimes B \otimes M \otimes N \arrow[r, "\nabla\otimes 1\otimes 1"] \arrow[d, "\Delta\otimes\Delta\otimes 1\otimes 1"] & B \otimes M \otimes N \arrow[ddd, "\Delta\otimes 1\otimes 1"] \\
B \otimes M \otimes B \otimes N \arrow[r, "1\otimes\delta\otimes 1\otimes\delta"] \arrow[d, "1\otimes\tau\otimes 1"] \arrow[dr, phantom, "(3)"] & B \otimes B \otimes M \otimes B \otimes B \otimes N \arrow[r, "1^2\otimes\tau_{M,B\otimes B}\otimes 1"] \arrow[d, "1\otimes\tau_{B\otimes M,B}\otimes 1\otimes 1"] \arrow[dr, phantom, "(4)"] & B \otimes B \otimes B \otimes B \otimes M \otimes N \arrow[d, "1\otimes\tau\otimes 1\otimes 1\otimes 1"] \arrow[r, phantom, "(7)"] & \\
B \otimes B \otimes M \otimes N \arrow[r, "1\otimes 1\otimes\delta\otimes\delta"] \arrow[d, "\nabla\otimes 1\otimes 1"] \arrow[dr, phantom, "(5)"] & B \otimes B \otimes B \otimes M \otimes B \otimes N \arrow[r, "1\otimes 1\otimes 1\otimes\tau\otimes 1"] \arrow[d, "\nabla\otimes 1\otimes 1\otimes 1\otimes 1"] \arrow[dr, phantom, "(6)"] & B \otimes B \otimes B \otimes B \otimes M \otimes N \arrow[d, "\nabla\otimes 1\otimes 1\otimes 1\otimes 1"] & \\
B \otimes M \otimes N \arrow[r, "1\otimes\delta\otimes\delta"] & B \otimes B \otimes M \otimes B \otimes N \arrow[r, "1\otimes 1\otimes\tau\otimes 1"] & B \otimes B \otimes B \otimes M \otimes N \arrow[r, "1\otimes\nabla\otimes 1\otimes 1"] & B \otimes B \otimes M \otimes N
\end{tikzcd}$$

The law of the counit is

$$\begin{tikzcd}
M \otimes N \arrow[r, "\delta\otimes\delta"] \arrow[dr, "1"] & B \otimes M \otimes B \otimes N \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "\varepsilon\otimes 1\otimes\varepsilon\otimes 1"] & B \otimes B \otimes M \otimes N \arrow[r, "\nabla\otimes 1\otimes 1"] \arrow[d, "\varepsilon\otimes\varepsilon\otimes 1\otimes 1"] & B \otimes M \otimes N \arrow[d, "\varepsilon\otimes 1\otimes 1"] \\
& M \otimes N \arrow[r, "1"] & M \otimes N \arrow[r, "1"] & M \otimes N
\end{tikzcd}$$

where the last square commutes since ε is a homomorphism of algebras.

Now let f and g be homomorphisms of B-comodules. Then the diagram

$$\begin{tikzcd}
M \otimes N \arrow[r, "\delta\otimes\delta"] \arrow[d, "f\otimes g"] & B \otimes M \otimes B \otimes N \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "1\otimes f\otimes 1\otimes g"] & B \otimes B \otimes M \otimes N \arrow[r, "\nabla\otimes 1\otimes 1"] \arrow[d, "1\otimes 1\otimes f\otimes g"] & B \otimes M \otimes N \arrow[d, "1\otimes f\otimes g"] \\
M' \otimes N' \arrow[r, "\delta\otimes\delta"] & B \otimes M' \otimes B \otimes N' \arrow[r, "1\otimes\tau\otimes 1"] & B \otimes B \otimes M' \otimes N' \arrow[r, "\nabla\otimes 1\otimes 1"] & B \otimes M' \otimes N'
\end{tikzcd}$$

commutes. Thus $f \otimes g$ is a homomorphism of B-comodules. $\square$

Corollary 3.1.5. *Let B be a bialgebra. Then $\otimes : B$-**Comod** $\times$ B-**Comod** $\to$ B-**Comod** with $\otimes(M, N) = M \otimes N$ and $\otimes(f, g) = f \otimes g$ is a functor.*

Proposition 3.1.6. *Let B be a bialgebra. Then the tensor product $\otimes : B$-**Mod** $\times$ B-**Mod** $\to B$-**Mod** satisfies the following properties:*

1. *The associativity isomorphism $\alpha : (M_1 \otimes M_2) \otimes M_3 \to M_1 \otimes (M_2 \otimes M_3)$ with $\alpha((m \otimes n) \otimes p) = m \otimes (n \otimes p)$ is a natural transformation from the functor*

*$\otimes \circ (\otimes \times \mathrm{Id})$ to the functor $\otimes \circ (\mathrm{Id} \times \otimes)$ in the variables M_1, M_2, and M_3 in B-**Mod**.*

2. *The counit isomorphisms $\lambda : \mathbb{K} \otimes M \to M$ with $\lambda(\kappa \otimes m) = \kappa m$ and $\rho : M \otimes \mathbb{K} \to M$ with $\rho(m \otimes \kappa) = \kappa m$ are natural transformations in the variable M in B-**Mod** from the functor $\mathbb{K} \otimes$ - resp. - $\otimes \mathbb{K}$ to the identity functor* Id.
3. *The following diagrams of natural transformations are commutative*

$$\begin{tikzcd}
((M_1 \otimes M_2) \otimes M_3) \otimes M_4 \arrow[r, "\alpha(M_1,M_2,M_3)\otimes 1"] \arrow[d, "\alpha(M_1 \otimes M_2, M_3, M_4)"] & (M_1 \otimes (M_2 \otimes M_3)) \otimes M_4 \arrow[r, "\alpha(M_1, M_2 \otimes M_3, M_4)"] & M_1 \otimes ((M_2 \otimes M_3) \otimes M_4) \arrow[d, "1 \otimes \alpha(M_2,M_3,M_4)"'] \\
(M_1 \otimes M_2) \otimes (M_3 \otimes M_4) \arrow[rr, "\alpha(M_1, M_2, M_3 \otimes M_4)"] & & M_1 \otimes (M_2 \otimes (M_3 \otimes M_4))
\end{tikzcd}$$

$$\begin{tikzcd}
(M_1 \otimes \mathbb{K}) \otimes M_2 \arrow[rr, "\alpha(M_1,\mathbb{K},M_2)"] \arrow[dr, "\rho(M_1)\otimes 1"'] & & M_1 \otimes (\mathbb{K} \otimes M_2) \arrow[dl, "1 \otimes \lambda(M_2)"] \\
& M_1 \otimes M_2 &
\end{tikzcd}$$

PROOF. The homomorphisms α, λ, and ρ are already defined in the category $\mathbb{K}$-**Mod** and satisfy the claimed properties. So we have to show, that these are homomorphisms in B-**Mod** and that $\mathbb{K}$ is a B-module. $\mathbb{K}$ is a B-module by $\varepsilon \otimes 1_{\mathbb{K}} : B \otimes \mathbb{K} \to \mathbb{K}$. The easy verification uses the coassociativity and the counital property of B. $\square$

Similarly we get

Proposition 3.1.7. *Let B be a bialgebra. Then the tensor product*

$$\otimes : B\text{-}\mathbf{Comod} \times B\text{-}\mathbf{Comod} \to B\text{-}\mathbf{Comod}$$

satisfies the following properties:

1. *The associativity isomorphism $\alpha : (M_1 \otimes M_2) \otimes M_3 \to M_1 \otimes (M_2 \otimes M_3)$ with $\alpha((m \otimes n) \otimes p) = m \otimes (n \otimes p)$ is a natural transformation from the functor $\otimes \circ (\otimes \times \mathrm{Id})$ to the functor $\otimes \circ (\mathrm{Id} \times \otimes)$ in the variables M_1, M_2, and M_3 in B-**Comod**.*
2. *The counit isomorphisms $\lambda : \mathbb{K} \otimes M \to M$ with $\lambda(\kappa \otimes m) = \kappa m$ and $\rho : M \otimes \mathbb{K} \to M$ with $\rho(m \otimes \kappa) = \kappa m$ are natural transformations in the variable M in B-**Comod** from the functor $\mathbb{K} \otimes$ - resp. - $\otimes \mathbb{K}$ to the identity functor* Id.
3. *The following diagrams of natural transformations are commutative*

$$\begin{tikzcd}
((M_1 \otimes M_2) \otimes M_3) \otimes M_4 \arrow[r, "\alpha(M_1,M_2,M_3)\otimes 1"] \arrow[d, "\alpha(M_1 \otimes M_2, M_3, M_4)"] & (M_1 \otimes (M_2 \otimes M_3)) \otimes M_4 \arrow[r, "\alpha(M_1, M_2 \otimes M_3, M_4)"] & M_1 \otimes ((M_2 \otimes M_3) \otimes M_4) \arrow[d, "1 \otimes \alpha(M_2,M_3,M_4)"'] \\
(M_1 \otimes M_2) \otimes (M_3 \otimes M_4) \arrow[rr, "\alpha(M_1, M_2, M_3 \otimes M_4)"] & & M_1 \otimes (M_2 \otimes (M_3 \otimes M_4))
\end{tikzcd}$$

$$\begin{tikzcd}
(M_1 \otimes \mathbb{K}) \otimes M_2 \arrow[rr, "\alpha(M_1,\mathbb{K},M_2)"] \arrow[dr, "\rho(M_1)\otimes 1"'] & & M_1 \otimes (\mathbb{K} \otimes M_2) \arrow[dl, "1\otimes\lambda(M_2)"] \\
& M_1 \otimes M_2 &
\end{tikzcd}$$

Remark 3.1.8. We now get some simple properties of the underlying functors $\mathcal{U} : B$-**Mod** $\to \mathbb{K}$-**Mod** resp. $\mathcal{U} : B$-**Comod** $\to \mathbb{K}$-**Mod** that are easily verified.

$$\begin{aligned}
&\mathcal{U}(M \otimes N) = \mathcal{U}(M) \otimes \mathcal{U}(N),\\
&\mathcal{U}(f \otimes g) = f \otimes g,\\
&\mathcal{U}(\mathbb{K}) = \mathbb{K},\\
&\mathcal{U}(\alpha) = \alpha,\ \mathcal{U}(\lambda) = \lambda,\ \mathcal{U}(\rho) = \rho.
\end{aligned}$$

Problem 3.1.1. We have seen that in representation theory and in corepresentation theory of quantum groups such as $\mathbb{K}G$, $U(\mathfrak{g})$, $SL_q(2)$, $U_q(sl(2))$ the ordinary tensor product (in $\mathbb{K}$-**Mod**) of two (co-)reprensentations is in a canonical way again a (co-)reprensentation. For two $\mathbb{K}G$-modules M and N the structure is $g(m\otimes n) = gm\otimes gn$ for $g \in G$. For $U(\mathfrak{g})$-modules it is $g(m\otimes n) = gm\otimes n+m\otimes gn$ for $g \in \mathfrak{g}$. For $U_q(sl(2))$-modules it is $E(m \otimes n) = m \otimes En + Em \otimes Kn$, $F(m \otimes n) = K^{-1}m \otimes Fn + Fm \otimes n$, $K(m \otimes n) = Km \otimes Kn$.

Remark 3.1.9. Let A and B be algebras over a commutative ring $\mathbb{K}$. Let $f : A \to B$ be a homomorphism of algebras. Then we have a functor $\mathcal{U}_f : B$-**Mod** $\to$ A-**Mod** with $\mathcal{U}_f({}_BM) = {}_AM$ and $\mathcal{U}_f(g) = g$ where $am := f(a)m$ for $a \in A$ and $m \in M$. The functor $\mathcal{U}_f$ is also called *forgetful* or *underlying functor*.

The action of A on a B-module M can also be seen as the homomorphism $A \to B \to \mathrm{End}(M)$.

We denote the underlying functors previously discussed by

$$\mathcal{U}_A : A\text{-}\mathbf{Mod} \to \mathbb{K}\text{-}\mathbf{Mod} \text{ resp. } \mathcal{U}_B : B\text{-}\mathbf{Mod} \to \mathbb{K}\text{-}\mathbf{Mod}.$$

Proposition 3.1.10. *Let $f : B \to C$ be a homomorphism of bialgebras. Then $\mathcal{U}_f$ satisfies the following properties:*

$$\begin{aligned}
&\mathcal{U}_f(M \otimes N) = \mathcal{U}_f(M) \otimes \mathcal{U}_f(N),\\
&\mathcal{U}_f(g \otimes h) = g \otimes h,\\
&\mathcal{U}_f(\mathbb{K}) = \mathbb{K},\\
&\mathcal{U}_f(\alpha) = \alpha,\ \mathcal{U}_f(\lambda) = \lambda,\ \mathcal{U}_f(\rho) = \rho,\\
&\mathcal{U}_B\mathcal{U}_f(M) = \mathcal{U}_C(M),\\
&\mathcal{U}_B\mathcal{U}_f(g) = \mathcal{U}_C(g).
\end{aligned}$$

PROOF. This is clear since the underlying $\mathbb{K}$-modules and the $\mathbb{K}$-linear maps stay unchanged. The only thing to check is that U_f generates the correct B-module structure on the tensor product. For $U_f(M \otimes N) = M \otimes N$ we have $b(m \otimes n) = f(b)(m \otimes n) = \sum f(b)_{(1)}m \otimes f(b)_{(2)}n = \sum f(b_{(1)})m \otimes f(b_{(2)})n = \sum b_{(1)}m \otimes b_{(2)}n$. $\square$

Remark 3.1.11. Let C and D be coalgebras over a commutative ring $\mathbb{K}$. Let $f : C \longrightarrow D$ be a homomorphism of coalgebras. Then we have a functor $\mathcal{U}_f : C$-**Comod** $\rightarrow D$-**Comod** with $\mathcal{U}_f({}^C M) = {}^D M$ and $\mathcal{U}_f(g) = g$ where $\delta_D = (f \otimes 1)\delta_C : M \rightarrow C \otimes M \longrightarrow D \otimes M$. Again the functor $\mathcal{U}_f$ is called *forgetful* or *underlying functor*.

We denote the underlying functors previously discussed by

$$\mathcal{U}_C : C\text{-}\mathbf{Comod} \rightarrow \mathbb{K}\text{-}\mathbf{Mod} \text{ resp. } \mathcal{U}_D : D\text{-}\mathbf{Comod} \rightarrow \mathbb{K}\text{-}\mathbf{Mod}.$$

Proposition 3.1.12. *Let $f : B \longrightarrow C$ be a homomorphism of bialgebras. Then $\mathcal{U}_f : C$-**Comod** $\longrightarrow D$-**Comod** satisfies the following properties:*

$$\begin{aligned}
&\mathcal{U}_f(M \otimes N) = \mathcal{U}_f(M) \otimes \mathcal{U}_f(N),\\
&\mathcal{U}_f(g \otimes h) = g \otimes h,\\
&\mathcal{U}_f(\mathbb{K}) = \mathbb{K},\\
&\mathcal{U}_f(\alpha) = \alpha,\ \mathcal{U}_f(\lambda) = \lambda,\ \mathcal{U}_f(\rho) = \rho,\\
&\mathcal{U}_C\mathcal{U}_f(M) = \mathcal{U}_B(M),\\
&\mathcal{U}_C\mathcal{U}_f(g) = \mathcal{U}_B(g).
\end{aligned}$$

Proof. We leave the proof to the reader. □

Proposition 3.1.13. *Let H be a Hopf algebra. Let M and N be be H-modules. Then* Hom(M, N)*, the set $\mathbb{K}$-linear maps from M to N, becomes an H-module by $(hf)(m) = \sum h_{(1)} f(S(h_{(2)}m)$. This structure makes*

$$\mathrm{Hom} : H\text{-}\mathbf{Mod} \times H\text{-}\mathbf{Mod} \longrightarrow H\text{-}\mathbf{Mod}$$

a functor contravariant in the first variable and covariant in the second variable.

Proof. The main part to be proved is that the action $H \otimes \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(M, N)$ satisfies the associativity law. Let $f \in \mathrm{Hom}(M, N)$, $h, k \in H$, and $m \in M$. Then $((hk)f)(m) = \sum(hk)_{(1)} f(S((hk)_{(2)}) = \sum h_{(1)}k_{(1)} f(S(k_{(2)})S(h_{(2)})m) = \sum h_{(1)}(kf)(S(h_{(2)})m) = (h(kf))(m)$.

We leave the proof of the other properties, in particular the functorial properties, to the reader. □

Corollary 3.1.14. *Let M be an H-module. Then the dual $\mathbb{K}$-module $M^* =$ Hom$(M, \mathbb{K})$ becomes an H-module by $(hf)(m) = f(S(h)m)$.*

Proof. The space $\mathbb{K}$ is an H-module via $\varepsilon : H \longrightarrow \mathbb{K}$. Hence we have $(hf)(m) = \sum h_{(1)} f(S(h_{(2)}m) = \sum \varepsilon(h_{(1)}) f(S(h_{(2)}m) = f(S(h)m)$. □