CHAPTER 2

Hopf Algebras, Algebraic, Formal, and Quantum Groups

6. Quantum Automorphism Groups

Lemma 2.6.1. *The category* $\mathbb{K}$-**Alg** *of* $\mathbb{K}$*-algebras has arbitrary coproducts.*

Proof. This is a well known fact from universal algebra. In fact all equationally defined algebraic categories are complete and cocomplete. We indicate the construction of the coproduct of a family $(A_i|i \in I)$ of $\mathbb{K}$-algebras.

Define $\coprod_{i\in I} A_i := T(\bigoplus_{i\in I} A_i)/L$ where T denotes the tensor algebra and where L is the two sided ideal in $T(\bigoplus_{i\in I} A_i)$ generated by the set

$$J := \{\iota j_k(x_k y_k) - \iota(j_k(x_k))\iota(j_k(y_k)), 1_{T(\bigoplus A_i)} - \iota j_k(1_{A_k})|x_k, y_k \in A_k, k \in I\}.$$

Then one checks easily for a family of algebra homomorphisms $(f_k : A_k \to B|k \in I)$ that the following diagram gives the required universal property

$$\begin{tikzcd}
A_k \arrow[r, "j_k"] \arrow[rrrd, "f_k"'] & \bigoplus A_i \arrow[r, "\iota"] \arrow[rrd, "f"'] & T(\bigoplus A_i) \arrow[r, "\nu"] \arrow[rd, "f'"'] & T(\bigoplus A_i)/L \arrow[d, "\bar{f}"] \\
 & & & B
\end{tikzcd}$$

$\square$

Corollary 2.6.2. *The category of bialgebras has finite coproducts.*

Proof. The coproduct $\coprod B_i$ of bialgebras $(B_i|i \in I)$ in $\mathbb{K}$-**Alg** is an algebra. For the diagonal and the counit we obtain the following commutative diagrams

$$\begin{tikzcd}
B_k \arrow[r, "j_k"] \arrow[d, "\Delta_k"'] & \coprod B_i \arrow[d, "\exists_1 \Delta"] \\
B_k \otimes B_k \arrow[r, "j_k \otimes j_k"'] & \coprod B_i \otimes \coprod B_i
\end{tikzcd}$$

$$\begin{tikzcd}
B_k \arrow[r, "j_k"] \arrow[rd, "\varepsilon_k"'] & \coprod B_i \arrow[d, "\exists_1 \varepsilon"] \\
 & \mathbb{K}
\end{tikzcd}$$

since in both cases $\coprod B_i$ is a coproduct in $\mathbb{K}$-**Alg**. Then it is easy to show that these homomorphisms define a bialgebra structure on $\coprod B_i$ and that $\coprod B_i$ satisfies the universal property for bialgebras. $\square$

Theorem 2.6.3. *Let B be a bialgebra. Then there exists a Hopf algebra $H(B)$ and a homomorphism of bialgebras $\iota : B \to H(B)$ such that for every Hopf algebra H and for every homomorphism of bialgebras $f : B \to H$ there is a unique* *homomorphism of Hopf algebras $g : H(B) \to H$ such that the diagram*

$$\begin{tikzcd} B \arrow[r, "\iota"] \arrow[dr, "f"'] & H(B) \arrow[d, "g"] \\ & H \end{tikzcd}$$

commutes.

Proof. Define a sequence of bialgebras $(B_i|i \in \mathbb{N})$ by

$$\begin{aligned} &B_0 := B, \\ &B_{i+1} := B_i^{opcop}, i \in \mathbb{N}. \end{aligned}$$

Let B' be the coproduct of the family $(B_i|i \in \mathbb{N})$ with injections $\iota_i : B_i \to B'$. Because B' is a coproduct of bialgebras there is a unique homomorphism of bialgebras $S' : B' \to B'^{opcop}$ such that the diagrams

$$\begin{tikzcd} B_i \arrow[r, "\iota_i"] \arrow[d, "\mathrm{id}"'] & B' \arrow[d, "S'"] \\ B_{i+1}^{opcop} \arrow[r, "\iota_{i+1}"'] & B'^{opcop} \end{tikzcd}$$

commute.

Now let I be the two sided ideal in B' generated by

$$\{(S' * 1 - u\varepsilon)(x_i), (1 * S' - u\varepsilon)(x_i) | x_i \in \iota_i(B_i), i \in \mathbb{N}\}.$$

I is a coideal, i.e. $\varepsilon_{B'}(I) = 0$ and $\Delta_{B'}(I) \subseteq I \otimes B' + B' \otimes I$.

Since $\varepsilon_{B'}$ and $\Delta_{B'}$ are homomorphisms of algebras it suffices to check this for the generating elements of I. Let $x \in B_i$ be given. Then we have $\varepsilon((1 * S')\iota_i(x)) = \varepsilon(\nabla(1 \otimes S')\Delta\iota_i(x)) = \nabla_{\mathbb{K}}(\varepsilon \otimes \varepsilon S')(\iota_i \otimes \iota_i)\Delta_i(x) = (\varepsilon\iota_i \otimes \varepsilon\iota_i)\Delta_i(x) = \varepsilon_i(x) = \varepsilon(u\varepsilon\iota_i(x))$. Symmetrically we have $\varepsilon((S' * 1)\iota_i(x)) = \varepsilon(u\varepsilon\iota_i(x))$. Furthermore we have

$$\begin{aligned} &\Delta((1 * S')\iota_i(x)) \\ &= \Delta\nabla(1 \otimes S')\Delta\iota_i(x) \\ &= (\nabla \otimes \nabla)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta)(1 \otimes S')(\iota_i \otimes \iota_i)\Delta_i(x) \\ &= (\nabla \otimes \nabla)(1 \otimes \tau \otimes 1)(\Delta \otimes \tau(S' \otimes S')\Delta)(\iota_i \otimes \iota_i)\Delta_i(x) \\ &= \sum(\nabla \otimes \nabla)(1 \otimes \tau \otimes 1)(\iota_i(x_{(1)}) \otimes \iota_i(x_{(2)}) \otimes S'\iota_i(x_{(4)}) \otimes S'\iota_i(x_{(3)})) \\ &= \sum \iota_i(x_{(1)})S'\iota_i(x_{(4)}) \otimes \iota_i(x_{(2)})S'\iota_i(x_{(3)}) \\ &= \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes (1 * S')\iota_i(x_{(2)}). \end{aligned}$$

Hence we have

$$\begin{aligned}
\Delta((1 * S' - u\varepsilon)\iota_i(x)) \\
&= \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes (1 * S')\iota_i(x_{(2)}) - \Delta u\varepsilon\iota_i(x) \\
&= \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes ((1 * S') - u\varepsilon)\iota_i(x_{(2)}) \\
&\qquad + \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes u\varepsilon\iota_i(x_{(2)}) - \Delta u\varepsilon\iota_i(x) \\
&= \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes (1 * S' - u\varepsilon)\iota_i(x_{(2)}) \\
&\qquad + \sum \iota_i(x_{(1)})S'\iota_i(x_{(2)}) \otimes 1_{B'} - u\varepsilon\iota_i(x) \otimes 1_{B'} \\
&= \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes (1 * S' - u\varepsilon)\iota_i(x_{(2)}) \\
&\qquad + (1 * S' - u\varepsilon)\iota_i(x) \otimes 1_{B'} \\
&\in B' \otimes I + I \otimes B'.
\end{aligned}$$

Thus I is a coideal and a biideal of B'.

Now let $H(B) := B'/I$ and let $\nu : B' \to H(B)$ be the residue class homomorphism. We show that $H(B)$ is a bialgebra and ν is a homomorphism of bialgebras. $H(B)$ is an algebra and ν is a homomorphism of algebras since I is a two sided ideal. Since $I \subseteq \mathrm{Ker}(\varepsilon)$ there is a unique factorization

$$\begin{tikzcd}
B' \arrow[r, "\nu"] \arrow[dr, "\varepsilon'"'] & B'/I \arrow[d, "\varepsilon"] \\
& \mathbb{K}
\end{tikzcd}$$

where $\varepsilon : B'/I \to \mathbb{K}$ is a homomorphism of algebras. Since $\Delta(I) \subseteq B' \otimes I + I \otimes B' \subseteq \mathrm{Ker}(\nu \otimes \nu : B' \otimes B' \to B'/I \otimes B'/I)$ and thus $I \subseteq \mathrm{Ker}(\Delta(\nu \otimes \nu))$ we have a unique factorization

$$\begin{tikzcd}
B' \arrow[r, "\nu"] \arrow[d, "\Delta_{B'}"'] & B'/I \arrow[d, "\Delta"] \\
B' \otimes B' \arrow[r, "\nu\otimes\nu"'] & B'/I \otimes B'/I
\end{tikzcd}$$

by an algebra homomorphism $\Delta : B'/I \to B'/I \otimes B'/I$. Now it is easy to verify that B'/I becomes a bialgebra and ν a bialgebra homomorphism.

We show that the map $\nu S' : B' \to B'/I$ can be factorized through B'/I in the commutative diagram

$$\begin{tikzcd}
B' \arrow[r, "\nu"] \arrow[dr, "\nu S'"'] & B'/I \arrow[d, "S"] \\
& B'/I
\end{tikzcd}$$

This holds if $I \subseteq \mathrm{Ker}(\nu S')$. Since $\mathrm{Ker}(\nu) = I$ it suffices to show $S'(I) \subseteq I$. We have

$$\begin{aligned}
&S'((S' * 1)\iota_i(x)) = \\
&= \nabla\tau({S'}^2\iota_i \otimes S'\iota_i)\Delta_i(x) \\
&= \nabla\tau(S' \otimes 1)(\iota_{i+1} \otimes \iota_{i+1})\Delta_i(x) \\
&= \nabla(1 \otimes S')(\iota_{i+1} \otimes \iota_{i+1})\tau\Delta_i(x) \\
&= \nabla(1 \otimes S')(\iota_{i+1} \otimes \iota_{i+1})\Delta_{i+1}(x) \\
&= (1 * S')\iota_{i+1}(x)
\end{aligned}$$

and

$$S'(u\varepsilon\iota_i(x)) = S'(1)\varepsilon_i(x) = S'(1)\varepsilon_{i+1}(x) = S'(u\varepsilon\iota_{i+1}(x))$$

hence we get

$$S'((S' * 1 - u\varepsilon)\iota_i(x)) = (1 * S' - u\varepsilon)\iota_{i+1}(x) \in I.$$

This shows $S'(I) \subseteq I$. So there is a unique homomorphism of bialgebras $S : H(B) \to H(B)^{opcop}$ such that the diagram

$$\begin{tikzcd}
B' \arrow[r, "\nu"] \arrow[d, "S'"'] & H(B) \arrow[d, "S"] \\
{B'}^{opcop} \arrow[r, "\nu"'] & H(B)^{opcop}
\end{tikzcd}$$

commutes.

Now we show that $H(B)$ is a Hopf algebra with antipode S. By Proposition 2.1.3 it suffices to test on generators of $H(B)$ hence on images $\nu\iota_i(x)$ of elements $x \in B_i$. We have

$$\begin{aligned}
(1 * S)\nu\iota_i(x) = \nabla(\nu \otimes S\nu)\Delta\iota_i(x) = \nabla(\nu \otimes \nu)(1 \otimes S')\Delta\iota_i(x) = \\
= \nu(1 * S')\iota_i(x) = \nu u\varepsilon\iota_i(x) = u\varepsilon\nu\iota_i(x).
\end{aligned}$$

By Proposition 2.1.3 S is an antipode for $H(B)$.

We prove now that $H(B)$ together with $\iota := \nu\iota_0 : B \to H(B)$ is a free Hopf algebra over B. Let H be a Hopf algebra and let $f : B \to H$ be a homomorphism of bialgebras. We will show that there is a unique homomorphism $\bar{f} : H(B) \to H$ such that

$$\begin{tikzcd}
B \arrow[r, "\iota"] \arrow[dr, "f"'] & H(B) \arrow[d, "\bar{f}"] \\
& H
\end{tikzcd}$$

commutes.

We define a family of homomorphisms of bialgebras $f_i : B_i \to H$ by

$$\begin{aligned}
&f_0 := f, \\
&f_{i+1} := S_H f_i, i \in \mathbb{N}.
\end{aligned}$$

We have in particular $f_i = S_H^i f$ for all $i \in \mathbb{N}$. Thus there is a unique homomorphism of bialgebras $f' : B' = \coprod B_i \longrightarrow H$ such that $f'\iota_i = f_i$ for all $i \in \mathbb{N}$.

We show that $f'(I) = 0$. Let $x \in B_i$. Then

$$\begin{aligned} f'((1 * S')\iota_i(x)) &= f'(\nabla(1 \otimes S')(\iota_i \otimes \iota_i)\Delta_i(x)) \\ &= \sum f'\iota_i(x_{(1)}) f'S'\iota_i(x_{(2)}) \\ &= \sum f'\iota_i(x_{(1)}) f'\iota_{i+1}(x_{(2)}) \\ &= \sum f_i(x_{(1)}) f_{i+1}(x_{(2)}) \\ &= \sum f_i(x_{(1)}) S f_i(x_{(2)}) \\ &= (1 * S) f_i(x) = u\varepsilon f_i(x) = u\varepsilon_i(x) \\ &= f'(u\varepsilon\iota_i(x)). \end{aligned}$$

This together with the symmetric statement gives $f'(I) = 0$. Hence there is a unique factorization through a homomorphism of algebras $\bar{f} : H(B) \longrightarrow H$ such that $f' = \bar{f}\nu$.

The homomorphism $\bar{f} : H(B) \longrightarrow H$ is a homomorphism of bialgebras since the diagram

$$\begin{tikzcd} B' \arrow[rr, bend left, "f'"] \arrow[r, "\nu"'] \arrow[d, "\Delta"'] & B'/I \arrow[r, "\bar{f}"'] \arrow[d, "\Delta'"] & H \arrow[d, "\Delta_H"] \\ B' \otimes B' \arrow[r, "\nu\otimes\nu"] \arrow[rr, bend right, "f'\otimes f'"'] & B'/I \otimes B'/I \arrow[r, "\bar{f}\otimes\bar{f}"] & H \otimes H \end{tikzcd}$$

commutes with the possible exception of the right hand square $\Delta\bar{f}$ and $(\bar{f}\otimes\bar{f})\Delta'$. But ν is surjective so also the last square commutes. Similarly we get $\varepsilon_H \bar{f} = \varepsilon_{H(B)}$. Thus $\bar{f}$ is a homomorphism of bialgebras and hence a homomorphism of Hopf algebras. □

Remark 2.6.4. In chapter 1 we have constructed universal bialgebras $M(A)$ with coaction $\delta : A \longrightarrow M(A) \otimes A$ for certain algebras A (see 1.3.12). This induces a homomorphism of algebras

$$\delta' : A \longrightarrow H(M(A)) \otimes A$$

such that A is a comodule-algebra over the Hopf algebra $H(M(A))$. If H is a Hopf algebra and A is an H-comodule algebra by $\partial : A \longrightarrow H \otimes A$ then there is a unique homomorphism of bialgebras $f : M(A) \longrightarrow H$ such that

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "\partial"'] & M(A) \otimes A \arrow[d, "f\otimes 1"] \\ & H \otimes A \end{tikzcd}$$

commutes. Since the $f : M(A) \longrightarrow H$ factorizes uniquely through $\bar{f} : H(M(A)) \longrightarrow H$ we get a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\delta'"] \arrow[dr, "\partial"'] & H(M(A)) \otimes A \arrow[d, "\bar{f}\otimes 1"] \\ & H \otimes A \end{tikzcd}$$

with a unique homomorphism of Hopf algebras $\bar{f} : H(M(A)) \longrightarrow H$.

This proof depends only on the existence of a universal algebra $M(A)$ for the algebra A. Hence we have

Corollary 2.6.5. *Let $\mathcal{X}$ be a quantum space with universal quantum space (and quantum monoid) $\mathcal{M}(\mathcal{X})$. Then there is a unique (up to isomorphism) quantum group $\mathcal{H}(\mathcal{M}(\mathcal{X}))$ acting universally on $\mathcal{X}$.*

This quantum group $\mathcal{H}(\mathcal{M}(\mathcal{X}))$ can be considered as the "quantum subgroup of invertible elements" of $\mathcal{M}(\mathcal{X})$ or the quantum group of "quantum automorphisms" of $\mathcal{X}$.