CHAPTER 2

Hopf Algebras, Algebraic, Formal, and Quantum Groups

5. Quantum Groups

Definition 2.5.1. (Drinfel'd) A *quantum group* is a noncommutative noncocommutative Hopf algebra.

Remark 2.5.2. We shall consider all Hopf algebras as quantum groups. Observe, however, that the commutative Hopf algebras may be considered as affine algebraic groups and that the cocommutative Hopf algebras may be considered as formal groups. Their property as a quantum space or as a quantum monoid will play some role. But often the (possibly nonexisting) dual Hopf algebra will have the geometrical meaning. The following examples $\mathbb{SL}_q(2)$ and $\mathbb{GL}_q(2)$ will have a geometrical meaning.

Example 2.5.3. The smallest proper quantum group, i.e. the smallest noncommutative noncocommutative Hopf algebra, is the 4-dimensional algebra

$$H_4 := \mathbb{K}\langle g, x\rangle/(g^2 - 1, x^2, xg + gx)$$

which was first described by M. Sweedler. The coalgebra structure is given by

$$\begin{array}{cc} \Delta(g) = g \otimes g, & \Delta(x) = x \otimes 1 + g \otimes x, \\ \varepsilon(g) = 1, & \varepsilon(x) = 0, \\ S(g) = g^{-1}(= g), & S(x) = -gx. \end{array}$$

Since it is finite dimensional its linear dual H_4^* is also a noncommutative noncocommutative Hopf algebra. It is isomorphic as a Hopf algebra to H_4. In fact H_4 is up to isomorphism the only noncommutative noncocommutative Hopf algebra of dimension 4.

Example 2.5.4. The affine algebraic group $\mathbb{SL}(n) : \mathbb{K}\text{-}\mathbf{cAlg} \to \mathbf{Gr}$ defined by $\mathbb{SL}(n)(A)$, the group of $n \times n$-matrices with coefficients in the commutative algebra A and with determinant 1, is represented by the algebra $\mathcal{O}(\mathbb{SL}(n)) = SL(n) = \mathbb{K}[x_{ij}]/(\det(x_{ij}) - 1)$ i.e.

$$\mathbb{SL}(n)(A) \cong \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x_{ij}]/(\det(x_{ij}) - 1), A).$$

Since $\mathbb{SL}(n)(A)$ has a group structure by the multiplication of matrices, the representing commutative algebra has a Hopf algebra structure with the diagonal $\Delta = \iota_1 * \iota_2$ hence

$$\Delta(x_{ik}) = \sum x_{ij} \otimes x_{jk},$$

the counit $\varepsilon(x_{ij}) = \delta_{ij}$ and the antipode $S(x_{ij}) = adj(X)_{ij}$ where $adj(X)$ is the adjoint matrix of $X = (x_{ij})$. We leave the verification of these facts to the reader.

We consider $\mathbb{SL}(n) \subseteq \mathcal{M}_n = \mathbb{A}^{n^2}$ as a subspace of the n^2-dimensional affine space.

Example 2.5.5. Let $M_q(2) = \mathbb{K}\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle / I$ as in 1.3.6 with I the ideal generated by

$$ab - q^{-1}ba, ac - q^{-1}ca, bd - q^{-1}db, cd - q^{-1}dc, (ad - q^{-1}bc) - (da - qcb), bc - cb.$$

We first define the *quantum determinant* $\det_q = ad - q^{-1}bc = da - qcb$ in $M_q(2)$. It is a central element. To show this, it suffices to show that $\det_q$ commutes with the generators a, b, c, d:

$$(ad - q^{-1}bc)a = a(da - qbc), \quad (ad - q^{-1}bc)b = b(ad - q^{-1}bc),$$
$$(ad - q^{-1}bc)c = c(ad - q^{-1}bc), \quad (da - qbc)d = d(ad - q^{-1}bc).$$

We can form the quantum determinant of an arbitrary quantum matrix in A by

$$\det_q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} := a'd' - q^{-1}b'c' = d'a' - qc'b' = \varphi(\det_q)$$

if $\varphi : M_q(2) \longrightarrow A$ is the algebra homomorphism associated with the quantum matrix $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$.

Given two commuting quantum 2×2-matrices $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$. The quantum determinant preserves the product, since

$$\begin{aligned}
&\det_q(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}) = \det_q \begin{pmatrix} a'a'' + b'c'' & a'b'' + b'd'' \\ c'a'' + d'c'' & c'b'' + d'd'' \end{pmatrix} \\
&= (a'a'' + b'c'')(c'b'' + d'd'') - q^{-1}(a'b'' + b'd'')(c'a'' + d'c'') \\
&= a'c'a''b'' + b'c'c''b'' + a'd'a''d'' + b'd'c''d'' \\
&\quad -q^{-1}(a'c'b''a'' + b'c'd''a'' + a'd'b''c'' + b'd'd''c'') \\
&= b'c'c''b'' + a'd'a''d'' - q^{-1}b'c'd''a'' - q^{-1}a'd'b''c'' \\
&= b'c'c''b'' + a'd'a''d'' - q^{-1}b'c'd''a'' - q^{-1}a'd'b''c'' \\
&\quad -q^{-1}b'c'(a''d'' - d''a'' - q^{-1}b''c'' + qc''b'') \\
&= a'd'a''d'' - q^{-1}a'd'b''c'' - q^{-1}b'c'(a''d'' - q^{-1}b''c'') \\
&= (a'd' - q^{-1}b'c')(a''d'' - q^{-1}b''c'') \\
&= \det_q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \det_q \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}.
\end{aligned} \tag{1}$$

In particular we have $\Delta(\det_q) = \det_q \otimes \det_q$ and $\varepsilon(\det_q) = 1$. The quantum determinant is a group like element (see 2.1.6).

Now we define an algebra

$$SL_q(2) := M_q(2)/(\det_q - 1).$$

The algebra $SL_q(2)$ represents the functor

$$\mathbb{SL}_q(2)(A) = \{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}_q(2)(A) | \det_q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = 1\}.$$

There is a surjective homomorphism of algebras $M_q(2) \longrightarrow SL_q(2)$ and $\mathbb{SL}_q(2)$ is a subfunctor of $\mathcal{M}_q(2)$.

Let X, Y be commuting quantum matrices satisfying $\det_q(X) = 1 = \det_q(Y)$. Since $\det_q(X)\det_q(Y) = \det_q(XY)$ for commuting quantum matrices we get

$\det_q(XY) = 1$, hence $\mathbb{SL}_q(2)$ is a quantum submonoid of $\mathcal{M}_q(2)$ and $SL_q(2)$ is a bialgebra with diagonal

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and

$$\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

To show that $SL_q(2)$ has an antipode we first define a homomorphism of algebras $T : M_q(2) \longrightarrow M_q(2)^{op}$ by

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

We check that $T : \mathbb{K}\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle \longrightarrow M_q(2)^{op}$ vanishes on the ideal I.

$$T(ab - q^{-1}ba) = T(b)T(a) - q^{-1}T(a)T(b) = -qbd + q^{-1}qdb = 0.$$

We leave the check of the other defining relations to the reader. Furthermore T restricts to a homomorphism of algebras $S : SL_q(2) \longrightarrow SL_q(2)^{op}$ since $T(\det_q) = T(ad - q^{-1}bc) = T(d)T(a) - q^{-1}T(c)T(b) = ad - q^{-1}(-q^{-1}c)(-qb) = \det_q$ hence $T(\det_q - 1) = \det_q - 1 = 0$ in $SL_q(2)$.

One verifies easily that S satisfies $\sum S(x_{(1)})x_{(2)} = \varepsilon(x)$ for all given generators of $SL_q(2)$, hence S is a left antipode by 2.1.3. Symmetrically S is a right antipode. Thus the bialgebra $SL_q(2)$ is a Hopf algebra or a quantum group.

Example 2.5.6. The affine algebraic group $\mathbb{GL}(n) : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbf{Gr}$ defined by $\mathbb{GL}(n)(A)$, the group of invertible $n \times n$-matrices with coefficients in the commutative algebra A, is represented by the algebra $\mathcal{O}(\mathbb{GL}(n)) = GL(n) = \mathbb{K}[x_{ij}, t]/(\det(x_{ij})t - 1)$ i.e.

$$\mathbb{GL}(n)(A) \cong \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x_{ij}, t]/(\det(x_{ij})t - 1), A)).$$

Since $\mathbb{GL}(n)(A)$ has a group structure by the multiplication of matrices, the representing commutative algebra has a Hopf algebra structure with the diagonal $\Delta = \iota_1 * \iota_2$ hence

$$\Delta(x_{ik}) = \sum x_{ij} \otimes x_{jk},$$

the counit $\varepsilon(x_{ij}) = \delta_{ij}$ and the antipode $S(x_{ij}) = t \cdot adj(X)_{ij}$ where $adj(X)$ is the adjoint matrix of $X = (x_{ij})$. We leave the verification of these facts from linear algebra to the reader. The diagonal applied to t gives

$$\Delta(t) = t \otimes t.$$

Hence $t (= \det(X)^{-1})$ is a grouplike element in $GL(n)$. This reflects the rule $\det(AB) = \det(A)\det(B)$ hence $\det(AB)^{-1} = \det(A)^{-1}\det(B)^{-1}$.

Example 2.5.7. Let $M_q(2)$ be as in the example 2.5.5. We define

$$GL_q(2) := M_q(2)[t]/J$$

with J generated by the elements $t \cdot (ad - q^{-1}bc) - 1$. The algebra $GL_q(2)$ represents the functor

$$\mathbb{GL}_q(2)(A) = \{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}_q(2)(A) | \det_q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \text{ invertible in } A\}.$$

In fact there is a canonical homomorphism of algebras $M_q(2) \longrightarrow GL_q(2)$. A homomorphism of algebras $\varphi : M_q(2) \longrightarrow A$ has a unique continuation to $GL_q(2)$ iff $\det_q(\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix})$ is invertible:

$$\begin{tikzcd} M_q(2) \arrow[r] \arrow[dr] & M_q(2)[t] \arrow[r] \arrow[d] & G_q(2) \arrow[dl] \\ & A & \end{tikzcd}$$

with $t \mapsto \det_q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^{-1}$. Thus $\mathbb{GL}_q(2)(A)$ is a subset of $\mathcal{M}_q(2)(A)$. Observe that $M_q(2) \longrightarrow GL_q(2)$ is *not* surjective.

Since the quantum determinant preserves products and the product of invertible elements is again invertible we get $\mathbb{GL}_q(2)$ is a quantum submonoid of $\mathcal{M}_q(2)$, hence $\Delta : GL_q(2) \longrightarrow GL_q(2) \otimes GL_q(2)$ with $\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\Delta(t) = t \otimes t$.

We construct the antipode for $GL_q(2)$. We define $T : M_q(2)[t] \longrightarrow M_q(2)[t]^{op}$ by

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} := t \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} \qquad \text{and} \qquad T(t) := \det_q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - q^{-1}bc.$$

As in 2.5.5 T defines a homomorphism of algebras. We obtain an induced homomorphism of algebras $S : GL_q(2) \longrightarrow GL_q(2)^{op}$ or a $GL_q(2)^{op}$-point in $GL_q(2)$ since $S(t(ad - q^{-1}bc) - 1) = (S(d)S(a) - q^{-1}S(c)S(b))S(t) - S(1) = (t^2 ad - q^{-1}t^2 cb)(ad - q^{-1}bc) - 1 = t^2(ad - q^{-1}bc)^2 - 1 = 0$.

Since S satisfies $\sum S(x_{(1)})x_{(2)} = \varepsilon(x)$ for all given generators, S is a left antipode by 2.1.3. Symmetrically S is a right antipode. Thus the bialgebra $GL_q(2)$ is a Hopf algebra or a quantum group.

Example 2.5.8. Let $sl(2)$ be the 3-dimensional vector space generated by the matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $sl(2)$ is a subspace of the algebra $M(2)$ of 2×2-matrices over $\mathbb{K}$. We easily verify $[X, Y] = XY - YX = H$, $[H, X] = HX - XH = 2X$, and $[H, Y] = HY - YH = -2Y$, so that $sl(2)$ becomes a Lie subalgebra of $M(2)^L$, which is the Lie algebra of matrices of trace zero. The universal enveloping algebra $U(sl(2))$ is a Hopf algebra generated as an algebra by the elements X, Y, H with the relations

$$[X,Y] = H, \quad [H,X] = 2X, \quad [H,Y] = -2Y.$$

As a consequence of the Poincaré-Birkhoff-Witt Theorem (that we don't prove) the Hopf algebra $U(sl(2))$ has the basis $\{X^iY^jH^k | i,j,k \in \mathbb{N}\}$. Furthermore one can prove that all finite dimensional $U(sl(2))$-modules are semisimple.

Example 2.5.9. We define the so-called q-deformed version $U_q(sl(2))$ of $U(sl(2))$ for any $q \in \mathbb{K}$, $q \neq 1, -1$ and q invertible. Let $U_q(sl(2))$ be the algebra generated by the elements E, F, K, K' with the relations

$$\begin{gathered} KK' = K'K = 1, \\ KEK' = q^2E, \qquad KFK' = q^{-2}F, \\ EF - FE = \frac{K - K'}{q - q^{-1}}. \end{gathered}$$

Since K' is the inverse of K in $U_q(sl(2))$ we write $K^{-1} = K'$. The representation theory of this algebra is fundamentally different depending on whether q is a root of unity or not.

We show that $U_q(sl(2))$ is a Hopf algebra or quantum group. We define

$$\begin{gathered} \Delta(E) = 1 \otimes E + E \otimes K, \qquad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \\ \Delta(K) = K \otimes K, \\ \varepsilon(E) = \varepsilon(F) = 0, \qquad \varepsilon(K) = 1, \\ S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}. \end{gathered}$$

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First we show that Δ can be expanded in a unique way to an algebra homomorphism $\Delta : U_q(sl(2)) \longrightarrow U_q(sl(2)) \otimes U_q(sl(2))$. Write $U_q(sl(2))$ as the residue class algebra $\mathbb{K}\langle E, F, K, K^{-1}\rangle / I$ where I is generated by

$$\begin{gathered} KK^{-1} - 1, \qquad K^{-1}K - 1, \\ KEK^{-1} - q^2E, \qquad KFK^{-1} - q^{-2}F, \\ EF - FE - \frac{K - K^{-1}}{q - q^{-1}}. \end{gathered}$$

Since K^{-1} must be mapped to the inverse of $\Delta(K)$ we must have $\Delta(K^{-1}) = K^{-1} \otimes K^{-1}$. Now Δ can be expanded in a unique way to the free algebra $\Delta : K\langle E, F, K, K^{-1}\rangle \longrightarrow U_q(sl(2)) \otimes U_q(sl(2))$. We have $\Delta(KK^{-1}) = \Delta(K)\Delta(K^{-1}) = 1$ and similarly $\Delta(K^{-1}K) = 1$. Furthermore we have $\Delta(KEK^{-1}) = \Delta(K)\Delta(E)\Delta(K^{-1}) = (K \otimes K)$ $(1 \otimes E + E \otimes K)(K^{-1} \otimes K^{-1}) = KK^{-1} \otimes KEK^{-1} + KEK^{-1} \otimes K^2K^{-1} = q^2(1 \otimes E +$

$E \otimes K) = q^2\Delta(E) = \Delta(q^2E)$ and similarly $\Delta(KFK^{-1}) = \Delta(q^{-2}F)$. Finally we have

$$\begin{aligned}\Delta(EF-FE) &= (1\otimes E + E\otimes K)(K'\otimes F + F\otimes 1)\\ &\quad -(K'\otimes F + F\otimes 1)(1\otimes E + E\otimes K)\\ &= K'\otimes EF + F\otimes E + EK'\otimes KF + EF\otimes K\\ &\quad -K'\otimes FE - K'E\otimes FK - F\otimes E - FE\otimes K\\ &= K'\otimes(EF-FE) + (EF-FE)\otimes K\\ &= \frac{K'\otimes(K-K') + (K-K')\otimes K}{q-q^{-1}}\\ &= \Delta\left(\frac{K-K'}{q-q^{-1}}\right)\end{aligned}$$

hence Δ vanishes on I and can be factorized through a unique algebra homomorphism

$$\Delta : U_q(sl(2)) \longrightarrow U_q(sl(2)) \otimes U_q(sl(2)).$$

In a similar way, actually much simpler, one gets an algebra homomorphism

$$\varepsilon : U_q(sl(2)) \longrightarrow \mathbb{K}.$$

To check that Δ is coassociative it suffices to check this for the generators of the algebra. We have $(\Delta\otimes 1)\Delta(E) = (\Delta\otimes 1)(1\otimes E + E\otimes K) = 1\otimes 1\otimes E + 1\otimes E\otimes K + E\otimes K\otimes K = (1\otimes\Delta)(1\otimes E + E\otimes K) = (1\otimes\Delta)\Delta(E)$. Similarly we get $(\Delta\otimes 1)\Delta(F) = (1\otimes\Delta)\Delta(F)$. For K the claim is obvious. The counit axiom is easily checked on the generators.

Now we show that S is an antipode for $U_q(sl(2))$. First define $S : \mathbb{K}\langle E, F, K, K^{-1}\rangle \to U_q(sl(2))^{op}$ by the definition of S on the generators. We have

$$\begin{gathered}S(KK^{-1}) = 1 = S(K^{-1}K),\\ S(KEK^{-1}) = -KEK^{-1}K^{-1} = -q^2EK^{-1} = S(q^2E),\\ S(KFK^{-1}) = -KKFK^{-1} = -q^{-2}KF = S(q^{-2}F),\\ S(EF-FE) = KFEK^{-1} - EK^{-1}KF = KFK^{-1}KEK - EF\\ = \frac{K^{-1}-K}{q-q^{-1}} = S\left(\frac{K-K^{-1}}{q-q^{-1}}\right).\end{gathered}$$

So S defines a homomorphism of algebras $S : U_q(sl(2)) \longrightarrow U_q(sl(2))$. Since S satisfies $\sum S(x_{(1)})x_{(2)} = \varepsilon(x)$ for all given generators, S is a left antipode by 2.1.3. Symmetrically S is a right antipode. Thus the bialgebra $U_q(sl(2))$ is a Hopf algebra or a quantum group.

This quantum group is of central interest in theoretical physics. Its representation theory is well understood. If q is not a root of unity then the finite dimensional $U_q(sl(2))$-modules are semisimple. Many more properties can be found in [Kassel: Quantum Groups].