CHAPTER 2

Hopf Algebras, Algebraic, Formal, and Quantum Groups

3. Affine Algebraic Groups

We apply the preceding considerations to the categories $\mathbb{K}$-**cAlg** and $\mathbb{K}$-**cCoalg**.

Consider $\mathbb{K}$-**cAlg**, the category of commutative $\mathbb{K}$-algebras. Let $A, B \in \mathbb{K}$-**cAlg**. Then $A \otimes B$ is again a commutative $\mathbb{K}$-algebra with componentwise multiplication. In fact this holds also for non-commutative $\mathbb{K}$-algebras (A.5.3), but in $\mathbb{K}$-**cAlg** we have

Proposition 2.3.1. *The tensor product in* $\mathbb{K}$-**cAlg** *is the (categorical) coproduct.*

Proof. Let $f \in \mathbb{K}\text{-}\mathbf{cAlg}(A, Z), g \in \mathbb{K}\text{-}\mathbf{cAlg}(B, Z)$. The map $[f, g] : A \otimes B \longrightarrow Z$ defined by $[f, g](a \otimes b) := f(a)g(b)$ is the unique algebra homomorphism such that $[f, g](a \otimes 1) = f(a)$ and $[f, g](1 \otimes b) = g(b)$ or such that the diagram

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[dr, "f"'] & A \otimes B \arrow[d, "{[f,g]}"] & B \arrow[l, "\iota_B"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

commutes, where $\iota_A(a) = a \otimes 1$ and $\iota_B(b) = 1 \otimes b$ are algebra homomorphisms. $\square$

So the category $\mathbb{K}$-**cAlg** has finite coproducts and also an initial object $\mathbb{K}$.

A more general property of the tensor product of arbitrary algebras was already considered in 1.2.13.

Observe that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\iota_1"] \arrow[dr, "1_A"'] & A \otimes A \arrow[d, "\nabla"] & A \arrow[l, "\iota_2"'] \arrow[dl, "1_A"] \\ & A & \end{tikzcd}$$

where ∇ is the multiplication of the algebra and by the diagram the codiagonal of the coproduct.

Definition 2.3.2. An *affine algebraic group* is a group in the category of commutative geometric spaces.

By the duality between the categories of commutative geometric spaces and commutative algebras, an affine algebraic group is represented by a cogroup in the category of $\mathbb{K}$-**cAlg** of commutative algebras.

For an arbitrary affine algebraic group H we get by Corollary 2.2.7

$$\Delta = \iota_1 * \iota_2 \in \mathbb{K}\text{-}\mathbf{cAlg}(H, H \otimes H),$$

$$\varepsilon = e \in \mathbb{K}\text{-}\mathbf{cAlg}(H, \mathbb{K}), \qquad \text{and } S = (\mathrm{id})^{-1} \in \mathbb{K}\text{-}\mathbf{cAlg}(H, H).$$

These maps and Corollary 2.2.7 lead to

Proposition 2.3.3. *Let* $H \in \mathbb{K}$-**cAlg**. *H is a representing object for a functor* $\mathbb{K}$-**cAlg** $\longrightarrow$ **Gr** *if and only if H is a Hopf algebra.*

Proof. Both statements are equivalent to the existence of morphisms in $\mathbb{K}$-**cAlg**

$$\Delta : H \to H \otimes H \quad \varepsilon : H \to \mathbb{K} \quad S : H \to H$$

such that the following diagrams commute

(coassociativity)

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & H \otimes H \arrow[d, "\Delta\otimes 1"] \\ H \otimes H \arrow[r, "1\otimes\Delta"] & H \otimes H \otimes H \end{tikzcd}$$

(counit)

$$\begin{tikzcd} H \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "1"] & & H \otimes H \arrow[d, "1\otimes\varepsilon"] \\ H \otimes H \arrow[r, "\varepsilon\otimes 1"] & \mathbb{K} \otimes H \cong H & \cong H \otimes \mathbb{K} \end{tikzcd}$$

(coinverse)

$$\begin{tikzcd} H \arrow[r, "\epsilon"] \arrow[d, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & H \\ H \otimes H \arrow[rr, "S\otimes\mathrm{id}", "\mathrm{id}\otimes S"'] & & H \otimes H \arrow[u, "\nabla"'] \end{tikzcd}$$

□

This Proposition says two things. First of all each commutative Hopf algebra H defines a functor $\mathbb{K}$-$\mathbf{cAlg}(H,\text{-}) : \mathbb{K}$-$\mathbf{cAlg} \to \mathbf{Set}$ that factors through the category of groups or simply a functor $\mathbb{K}$-$\mathbf{cAlg}(H,\text{-}) : \mathbb{K}$-$\mathbf{cAlg} \to \mathbf{Gr}$. Secondly each representable functor $\mathbb{K}$-$\mathbf{cAlg}(H,\text{-}) : \mathbb{K}$-$\mathbf{cAlg} \to \mathbf{Set}$ that factors through the category of groups is represented by a commutative Hopf algebra.

Corollary 2.3.4. *An algebra $H \in \mathbb{K}$-**cAlg** represents an affine algebraic group if and only if H is a commutative Hopf algebra.*

The category of commutative Hopf algebras is dual to the category of affine algebraic groups.

In the following lemmas we consider functors represented by commutative algebras. They define functors on the category $\mathbb{K}$-**cAlg** as well as more generally on $\mathbb{K}$-**Alg**. We first study the functors and the representing algebras. Then we use them to construct commutative Hopf algebras.

Lemma 2.3.5. *The functor $\mathbb{G}_a : \mathbb{K}$-$\mathbf{Alg} \to \mathbf{Ab}$ defined by $\mathbb{G}_a(A) := A^+$, the underlying additive group of the algebra A, is a representable functor represented by the algebra $\mathbb{K}[x]$ the polynomial ring in one variable x.*

Proof. $\mathbb{G}_a$ is an underlying functor that forgets the multiplicative structure of the algebra and only preserves the additive group of the algebra. We have to determine natural isomorphisms (natural in A) $\mathbb{G}_a(A) \cong \mathbb{K}$-$\mathbf{Alg}(\mathbb{K}[x], A)$. Each element $a \in A^+$ is mapped to the homomorphism of algebras $a_* : \mathbb{K}[x] \ni p(x) \mapsto p(a) \in A$. This is a homomorphism of algebras since $a_*(p(x) + q(x)) = p(a) + q(a) = a_*(p(x)) + a_*(q(x))$

and $a_*(p(x)q(x)) = p(a)q(a) = a_*(p(x))a_*(q(x))$. Another reason to see this is that $\mathbb{K}[x]$ is the free (commutative) $\mathbb{K}$-algebra over $\{x\}$ i.e. since each map $\{x\} \to A$ can be uniquely extended to a homomorphism of algebras $\mathbb{K}[x] \to A$. The map $A \ni a \mapsto a_* \in \mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x], A)$ is bijective with the inverse map $\mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x], A) \ni f \mapsto f(x) \in A$. Finally this map is natural in A since

$$\begin{tikzcd} A \arrow[r, "-_*"] \arrow[d, "g"'] & \mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x], A) \arrow[d, "\text{K-Alg}(K[x],g)"] \\ B \arrow[r, "-_*"'] & \mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x], B) \end{tikzcd}$$

commutes for all $g \in \mathbb{K}\text{-}\mathbf{Alg}(A, B)$. □

Remark 2.3.6. Since A^+ has the structure of an additive group the sets of homomorphisms of algebras $\mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x], A)$ are also additive groups.

Lemma 2.3.7. *The functor $\mathbb{G}_m = U : \mathbb{K}\text{-}\mathbf{Alg} \to \mathbf{Gr}$ defined by $\mathbb{G}_m(A) := U(A)$, the underlying multiplicative group of units of the algebra A, is a representable functor represented by the algebra $\mathbb{K}[x, x^{-1}] = \mathbb{K}[x, y]/(xy-1)$ the ring of Laurent polynomials in one variable x.*

Proof. We have to determine natural isomorphisms (natural in A) $\mathbb{G}_m(A) \cong \mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x, x^{-1}], A)$. Each element $a \in \mathbb{G}_m(A)$ is mapped to the homomorphism of algebras $a_* := (\mathbb{K}[x, x^{-1}] \ni x \mapsto a \in A)$. This defines a unique homomorphism of algebras since each homomorphism of algebras f from $\mathbb{K}[x, x^{-1}] = \mathbb{K}[x, y]/(xy - 1)$ to A is completely determined by the images of x and of y but in addition the images have to satisfy $f(x)f(y) = 1$, i.e. $f(x)$ must be invertible and $f(y)$ must be the inverse to $f(x)$. This mapping is bijective. Furthermore it is natural in A since

$$\begin{tikzcd} A \arrow[r, "-_*"] \arrow[d, "g"'] & \mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x, x^{-1}], A) \arrow[d, "\text{K-Alg}(K[x,x^{-1}],g)"] \\ B \arrow[r, "-_*"'] & \mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x, x^{-1}], B) \end{tikzcd}$$

for all $g \in \mathbb{K}\text{-}\mathbf{Alg}(A, B)$ commute. □

Remark 2.3.8. Since $U(A)$ has the structure of a (multiplicative) group the sets $\mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x, x^{-1}], A)$ are also groups.

Lemma 2.3.9. *The functor $\mathbb{M}_n : \mathbb{K}\text{-}\mathbf{Alg} \to \mathbb{K}\text{-}\mathbf{Alg}$ with $\mathbb{M}_n(A)$ the algebra of $n \times n$-matrices with entries in A is representable by the algebra $\mathbb{K}\langle x_{11}, x_{12}, \ldots, x_{nn}\rangle$, the non commutative polynomialring in the variables x_{ij}.*

Proof. The polynomial ring $\mathbb{K}\langle x_{ij}\rangle$ is free over the set $\{x_{ij}\}$ in the category of (non commutative) algebras, i.e. for each algebra and for each map $f : \{x_{ij}\} \to A$

there exists a unique homomorphism of algebras $g : \mathbb{K}\langle x_{11}, x_{12}, \dots, x_{nn}\rangle \to A$ such that the diagram

$$\begin{tikzcd} \{x_{ij}\} \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{K}\langle x_{ij}\rangle \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes. So each matrix in $M_n(A)$ defines a unique a homomorphism of algebras $\mathbb{K}\langle x_{11}, x_{12}, \dots, x_{nn}\rangle \to A$ and conversely. $\square$

Example 2.3.10. 1. The affine algebraic group called *additive group*

$$\mathbb{G}_a : \mathbb{K}\text{-}\mathbf{cAlg} \to \mathbf{Ab}$$

with $\mathbb{G}_a(A) := A^+$ from Lemma 2.3.5 is represented by the Hopf algebra $\mathbb{K}[x]$. We determine comultiplication, counit, and antipode.

By Corollary 2.2.7 the comultiplication is $\Delta = \iota_1 * \iota_2 \in \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x], \mathbb{K}[x] \otimes \mathbb{K}[x]) \cong \mathbb{G}_a(\mathbb{K}[x] \otimes \mathbb{K}[x])$. Hence

$$\Delta(x) = \iota_1(x) + \iota_2(x) = x \otimes 1 + 1 \otimes x.$$

The counit is $\varepsilon = e_{\mathbb{K}} = 0 \in \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x], \mathbb{K}) \cong \mathbb{G}_a(\mathbb{K})$ hence

$$\varepsilon(x) = 0.$$

The antipode is $S = \mathrm{id}_{\mathbb{K}[x]}^{-1} \in \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x], \mathbb{K}[x]) \cong \mathbb{G}_a(\mathbb{K}[x])$ hence

$$S(x) = -x.$$

2. The affine algebraic group called *multiplicative group*

$$\mathbb{G}_m : \mathbb{K}\text{-}\mathbf{cAlg} \to \mathbf{Ab}$$

with $\mathbb{G}_m(A) := A^* = U(A)$ from Lemma 2.3.7 is represented by the Hopf algebra $\mathbb{K}[x, x^{-1}] = \mathbb{K}[x, y]/(xy - 1)$. We determine comultiplication, counit, and antipode.

By Corollary 2.2.7 the comultiplication is

$$\Delta = \iota_1 * \iota_2 \in \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x, x^{-1}], \mathbb{K}[x, x^{-1}] \otimes \mathbb{K}[x, x^{-1}]) \cong \mathbb{G}_m(\mathbb{K}[x, x^{-1}] \otimes \mathbb{K}[x, x^{-1}]).$$

Hence

$$\Delta(x) = \iota_1(x) \cdot \iota_2(x) = x \otimes x.$$

The counit is $\varepsilon = e_{\mathbb{K}} = 1 \in \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x, x^{-1}], \mathbb{K}) \cong \mathbb{G}_m(\mathbb{K})$ hence

$$\varepsilon(x) = 1.$$

The antipode is $S = \mathrm{id}_{\mathbb{K}[x,x^{-1}]}^{-1} \in \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x, x^{-1}], \mathbb{K}[x, x^{-1}]) \cong \mathbb{G}_a(\mathbb{K}[x, x^{-1}])$ hence

$$S(x) = x^{-1}.$$

3. The affine algebraic group called *additive matrix group*

$$\mathbb{M}_n^+ : \mathbb{K}\text{-}\mathbf{cAlg} \to \mathbf{Ab},$$

with $\mathbb{M}_n^+(A)$ the additive group of $n\times n$-matrices with coefficients in A is represented by the commutative algebra $M_n^+ = \mathbb{K}[x_{ij}|1 \le i,j \le n]$ (Lemma 2.3.9). This algebra must be a Hopf algebra.

The comultiplication is $\Delta = \iota_1 * \iota_2 \in \mathbb{K}\text{-}\mathbf{cAlg}(M_n^+, M_n^+\otimes M_n^+) \cong \mathbb{M}_n^+(M_n^+\otimes M_n^+)$. Hence

$$\Delta(x_{ij}) = \iota_1(x_{ij}) + \iota_2(x_{ij}) = x_{ij}\otimes 1 + 1\otimes x_{ij}.$$

The counit is $\varepsilon = e_{\mathbb{K}} = (0) \in \mathbb{K}\text{-}\mathbf{cAlg}(M_n^+, \mathbb{K}) \cong \mathbb{M}_n^+(\mathbb{K})$ hence

$$\varepsilon(x_{ij}) = 0.$$

The antipode is $S = \mathrm{id}_{M_n^+}^{-1} \in \mathbb{K}\text{-}\mathbf{cAlg}(M_n^+, M_n^+) \cong \mathbb{M}_n^+(M_n^+)$ hence

$$S(x_{ij}) = -x_{ij}.$$

4. The matrix algebra $\mathbb{M}_n(A)$ also has a noncommutative multiplication, the matrix multiplication, defining a monoid structure $\mathbb{M}_n^\times(A)$. Thus $\mathbb{K}[x_{ij}]$ carries another coalgebra structure which defines a bialgebra $M_n^\times = \mathbb{K}[x_{ij}]$. Obviously there is no antipode.

The comultiplication is $\Delta = \iota_1 * \iota_2 \in \mathbb{K}\text{-}\mathbf{cAlg}(M_n^\times, M_n^\times\otimes M_n^\times) \cong \mathbb{M}_n^\times(M_n^\times\otimes M_n^\times)$. Hence $\Delta((x_{ij})) = \iota_1((x_{ij}))\cdot\iota_2((x_{ij})) = (x_{ij})\otimes(x_{ij})$ or

$$\Delta(x_{ik}) = \sum_j x_{ij}\otimes x_{jk}.$$

The counit is $\varepsilon = e_{\mathbb{K}} = E \in \mathbb{K}\text{-}\mathbf{cAlg}(M_n^\times, \mathbb{K}) \cong \mathbb{M}_n^\times(\mathbb{K})$ hence

$$\varepsilon(x_{ij}) = \delta_{ij}.$$

5. Let $\mathbb{K}$ be a field of characteristic p. The algebra $H = \mathbb{K}[x]/(x^p)$ carries the structure of a Hopf algebra with $\Delta(x) = x\otimes 1 + 1\otimes x$, $\varepsilon(x) = 0$, and $S(x) = -x$. To show that Δ is well defined we have to show $\Delta(x)^p = 0$. But this is obvious by the rules for computing p-th powers in characteristic p. We have $(x\otimes 1 + 1\otimes x)^p = x^p\otimes 1 + 1\otimes x^p = 0$.

Thus the algebra H represents an affine algebraic group:

$$\alpha_p(A) := \mathbb{K}\text{-}\mathbf{cAlg}(H, A) \cong \{a\in A | a^p = 0\}.$$

The group multiplication is the addition of p-nilpotent elements. So we have the *group of p-nilpotent elements.*

6. The algebra $H = \mathbb{K}[x]/(x^n - 1)$ is a Hopf algebra with the comultiplication $\Delta(x) = x\otimes x$, the counit $\varepsilon(x) = 1$, and the antipode $S(x) = x^{n-1}$. These maps are well defined since we have for example $\Delta(x)^n = (x\otimes x)^n = x^n\otimes x^n = 1\otimes 1$. Observe that this Hopf algebra is isomorphic to the group algebra $\mathbb{K}C_n$ of the cyclic group C_n of order n.

Thus the algebra H represents an affine algebraic group:

$$\mu_n(A) := \mathbb{K}\text{-}\mathbf{cAlg}(H, A) \cong \{a\in A | a^n = 1\},$$

that is the *group of n-th roots of unity*. The group multiplication is the ordinary multiplication of roots of unity.

7. The linear groups or matrix groups $\mathbb{GL}(n)(A)$, $\mathbb{SL}(n)(A)$ and other such groups are further examples of affine algebraic groups. We will discuss them in the section on quantum groups.

Problem 2.3.1. 1. The construction of the general linear group

$$\mathbb{GL}(n)(A) = \{(a_{ij}) \in \mathbb{M}_n(A) | (a_{ij}) \text{ invertible}\}$$

defines an affine algebraic group. Describe the representing Hopf algebra.

2. The special linear group $\mathbb{SL}(n)(A)$ is an affine algebraic group. What is the representing Hopf algebra?

3. The real unit circle $\mathbb{S}^1(\mathbb{R})$ carry the structure of a group by the addition of angles. Is it possible to make $\mathbb{S}^1$ with the affine algebra $\mathbb{K}[c, s]/(s^2 + c^2 - 1)$ into an affine algebraic group? (Hint: How can you add two points (x_1, y_1) and (x_2, y_2) on the unit circle, such that you get the addition of the associated angles?)