CHAPTER 2

Hopf Algebras, Algebraic, Formal, and Quantum Groups

2. Monoids and Groups in a Category

Before we use Hopf algebras to describe quantum groups and some of the better known groups, such as affine algebraic groups and formal groups, we introduce the concept of a general group (and of a monoid) in an arbitrary category. Usually this concept is defined with respect to a categorical product in the given category. But in some categories there are in general no products. Still, one can define the concept of a group in a very simple fashion. We will start with that definition and then show that it coincides with the usual notion of a group in a category in case that category has finite products.

Definition 2.2.1. Let $\mathcal{C}$ be an arbitrary category. Let $G \in \mathcal{C}$ be an object. We use the notation $G(X) := \mathrm{Mor}_{\mathcal{C}}(X, G)$ for all $X \in \mathcal{C}$, $G(f) := \mathrm{Mor}_{\mathcal{C}}(f, G)$ for all morphisms $f : X \to Y$ in $\mathcal{C}$, and $f(X) := \mathrm{Mor}_{\mathcal{C}}(X, f)$ for all morphisms $f : G \to G'$.

G together with a natural transformation $m : G(\text{-}) \times G(\text{-}) \to G(\text{-})$ is called a *group* (*monoid*) *in the category* $\mathcal{C}$, if the sets $G(X)$ together with the multiplication $m(X) : G(X) \times G(X) \to G(X)$ are groups (monoids) for all $X \in \mathcal{C}$.

Let (G, m) and (G', m') be groups in $\mathcal{C}$. A morphism $f : G \to G'$ in $\mathcal{C}$ is called a *homomorphism of groups* in $\mathcal{C}$, if the diagrams

$$\begin{tikzcd} G(X) \times G(X) \arrow[r, "m(X)"] \arrow[d, "f(X)\times f(X)"'] & G(X) \arrow[d, "f(X)"] \\ G'(X) \times G'(X) \arrow[r, "m'(X)"] & G'(X) \end{tikzcd}$$

commute for all $X \in \mathcal{C}$.

Let (G, m) and (G', m') be monoids in $\mathcal{C}$. A morphism $f : G \to G'$ in $\mathcal{C}$ is called a *homomorphism of monoids* in $\mathcal{C}$, if the diagrams

$$\begin{tikzcd} G(X) \times G(X) \arrow[r, "m(X)"] \arrow[d, "f(X)\times f(X)"'] & G(X) \arrow[d, "f(X)"] \\ G'(X) \times G'(X) \arrow[r, "m'(X)"] & G'(X) \end{tikzcd}$$

and

$$\begin{tikzcd} & \{*\} \arrow[dl, "u"'] \arrow[dr, "u'"] & \\ G(X) \arrow[rr, "f(X)"] & & G'(X) \end{tikzcd}$$

commute for all $X \in \mathcal{C}$.

Problem 2.2.1. 1) If a set Z together with a multiplication $m : Z \times Z \to Z$ is a monoid, then the unit element $e \in Z$ is uniquely determined. If it is a group then also the inverse $i : Z \to Z$ is uniquely determined. Unit element and inverses of groups are preserved by maps that are compatible with the multiplication.

2) Find an example of monoids Y and Z and a map $f : Y \to Z$ with $f(y_1 y_2) = f(y_1)f(y_2)$ for all $y_1, y_2 \in Y$, but $f(e_Y) \neq e_Z$.

3) If (G, m) is a group in $\mathcal{C}$ and $i_X : G(X) \to G(X)$ is the inverse, then i is a natural transformation. The Yoneda Lemma provides a morphism $S : G \to G$ such that $i_X = \mathrm{Mor}_{\mathcal{C}}(X, S) = S(X)$ for all $X \in \mathcal{C}$.

Proposition 2.2.2. *Let $\mathcal{C}$ be a category with finite (categorical) products. An object G in $\mathcal{C}$ carries the structure $m : G(\text{-}) \times G(\text{-}) \to G(\text{-})$ of a group in $\mathcal{C}$ if and only if there are morphisms $m : G \times G \to G$, $u : E \to G$, and $S : G \to G$ such that the diagrams*

$$\begin{tikzcd}
G \times G \times G \arrow[r, "m\times 1"] \arrow[d, "1\times m"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
E \times G \arrow[d, "u\times 1"'] & \cong G \cong \arrow[drr, "1"] & G \times E \arrow[r, "1\times u"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[rrr, "m"] & & & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r] \arrow[d, "\Delta"'] & E \arrow[r, "u"] & G \\
G \times G \arrow[rr, "1\times S", "S\times 1"'] & & G \times G \arrow[u, "m"']
\end{tikzcd}$$

commute where Δ is the morphism defined in A.2. The multiplications are related by $m_X = \mathrm{Mor}_{\mathcal{C}}(X, m) = m(X)$.

An analogous statement holds for monoids.

Proof. The Yoneda Lemma defines a bijection between the set of morphisms $f : X \to Y$ and the set of natural transformations $f(\text{-}) : X(\text{-}) \to Y(\text{-})$ by $f(Z) = \mathrm{Mor}_{\mathcal{C}}(Z, f)$. In particular we have $m_X = \mathrm{Mor}_{\mathcal{C}}(X, m) = m(X)$. The diagram

$$\begin{tikzcd}
G(\text{-}) \times G(\text{-}) \times G(\text{-}) \arrow[r, "m_{\text{-}}\times 1"] \arrow[d, "1\times m_{\text{-}}"'] & G(\text{-}) \times G(\text{-}) \arrow[d, "m_{\text{-}}"] \\
G(\text{-}) \times G(\text{-}) \arrow[r, "m_{\text{-}}"] & G(\text{-})
\end{tikzcd}$$

commutes if and only if $\mathrm{Mor}_{\mathcal{C}}(\text{-}, m(m \times 1)) = \mathrm{Mor}_{\mathcal{C}}(\text{-}, m)(\mathrm{Mor}_{\mathcal{C}}(\text{-}, m) \times 1) = m_{\text{-}}(m_{\text{-}} \times 1) = m_{\text{-}}(1 \times m_{\text{-}}) = \mathrm{Mor}_{\mathcal{C}}(\text{-}, m)(1 \times \mathrm{Mor}_{\mathcal{C}}(\text{-}, m)) = \mathrm{Mor}_{\mathcal{C}}(\text{-}, m(1 \times m))$ if and only if

$m(m \times 1) = m(1 \times m)$ if and only if the diagram

$$\begin{tikzcd} G \times G \times G \arrow[r, "m\times 1"] \arrow[d, "1\times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

commutes. In a similar way one shows the equivalence of the other diagram(s). $\square$

Problem 2.2.2. Let $\mathcal{C}$ be a category with finite products. Show that a morphism $f : G \to G'$ in $\mathcal{C}$ is a homomorphism of groups if and only if

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "f\times f"'] & G \arrow[d, "f"] \\ G' \times G' \arrow[r, "m'"] & G' \end{tikzcd}$$

commutes.

Definition 2.2.3. A *cogroup* (*comonoid*) G in $\mathcal{C}$ is a group (monoid) in $\mathcal{C}^{op}$, i.e. an object $G \in \operatorname{Ob}\mathcal{C} = \operatorname{Ob}\mathcal{C}^{op}$ together with a natural transformation $m(X) : G(X) \times G(X) \to G(X)$ where $G(X) = \operatorname{Mor}_{\mathcal{C}^{op}}(X, G) = \operatorname{Mor}_{\mathcal{C}}(G, X)$, such that $(G(X), m(X))$ is a group (monoid) for each $X \in \mathcal{C}$.

Remark 2.2.4. Let $\mathcal{C}$ be a category with finite (categorical) coproducts. An object G in $\mathcal{C}$ carries the structure $m : G(\text{-}) \times G(\text{-}) \to G(\text{-})$ of a cogroup in $\mathcal{C}$ if and only if there are morphisms $\Delta : G \to G \amalg G$, $\varepsilon : G \to I$, and $S : G \to G$ such that the diagrams

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & G \amalg G \arrow[d, "1\amalg\Delta"] \\ G \amalg G \arrow[r, "\Delta\amalg 1"] & G \amalg G \amalg G \end{tikzcd}
\qquad
\begin{tikzcd} G \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] \arrow[drr, "1"] & & G \amalg G \arrow[d, "1\amalg\varepsilon"] \\ G \amalg G \arrow[r, "\varepsilon\amalg 1"] & I \amalg G \cong G & \cong G \amalg I \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\Delta"'] & I \arrow[r, "\varepsilon"] & G \\ G \amalg G \arrow[rr, "1\amalg S", "S\amalg 1"'] & & G \amalg G \arrow[u, "\nabla"'] \end{tikzcd}$$

commute where ∇ is dual to the morphism Δ defined in A.2. The multiplications are related by $\Delta_X = \operatorname{Mor}_{\mathcal{C}}(\Delta, X) = \Delta(X)$.

Let $\mathcal{C}$ be a category with finite coproducts and let G and G' be cogroups in $\mathcal{C}$. Then a homomorphism of groups $f : G' \to G$ is a morphism $f : G \to G'$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d, "f"'] & G \times G \arrow[d, "f \times f"] \\ G' \arrow[r, "\Delta'"] & G' \times G' \end{tikzcd}$$

commutes. An analogous result holds for comonoids.

Remark 2.2.5. Obviously similar observations and statements can be made for other algebraic structures in a category $\mathcal{C}$. So one can introduce vector spaces and covector spaces, monoids and comonoids, rings and corings in a category $\mathcal{C}$. The structures can be described by morphisms in $\mathcal{C}$ if $\mathcal{C}$ is a category with finite (co-) products.

Problem 2.2.3. Determine the structure of a covector space on a vector space V from the fact that $\mathrm{Hom}(V, W)$ is a vector space for all vector spaces W.

Proposition 2.2.6. *Let $G \in \mathcal{C}$ be a group with multiplication $a * b$, unit e, and inverse a^{-1} in $G(X)$. Then the morphisms $m : G \times G \longrightarrow G$, $u : E \longrightarrow G$, and $S : G \longrightarrow G$ are given by*

$$m = p_1 * p_2, \qquad u = e_E, \qquad S = \mathrm{id}_G^{-1}.$$

Proof. By the Yoneda Lemma A.9.1 these morphisms can be constructed from the natural transformation as follows. Under $\mathrm{Mor}_{\mathcal{C}}(G \times G, G \times G) = G \times G(G \times G) \cong G(G \times G) \times G(G \times G) \xrightarrow{*} G(G \times G) = \mathrm{Mor}_{\mathcal{C}}(G \times G, G)$ the identity $\mathrm{id}_{G \times G} = (p_1, p_2)$ is mapped to $m = p_1 * p_2$. Under $\mathrm{Mor}_{\mathcal{C}}(E, E) = E(E) \longrightarrow G(E) = \mathrm{Mor}_{\mathcal{C}}(E, G)$ the identity of E is mapped to the neutral element $u = e_E$. Under $\mathrm{Mor}_{\mathcal{C}}(G, G) = G(G) \to G(G) = \mathrm{Mor}_{\mathcal{C}}(G, G)$ the identity is mapped to its $*$-inverse $S = \mathrm{id}_G^{-1}$. □

Corollary 2.2.7. *Let $G \in \mathcal{C}$ be a cogroup with multiplication $a * b$, unit e, and inverse a^{-1} in $G(X)$. Then the morphisms $\Delta : G \longrightarrow G \amalg G$, $\varepsilon : G \longrightarrow I$, and $S : G \longrightarrow G$ are given by*

$$\Delta = \iota_1 * \iota_2, \qquad \varepsilon = e_I, \qquad S = \mathrm{id}_G^{-1}.$$