CHAPTER 2

Hopf Algebras, Algebraic, Formal, and Quantum Groups

Introduction

In the first chapter we have encountered quantum monoids and studied their role as monoids operating on quantum spaces. The "elements" of quantum monoids operating on quantum spaces should be understood as endomorphisms of the quantum spaces. In the construction of the multiplication for universal quantum monoids of quantum spaces we have seen that this multiplication is essentially the "composition" of endomorphisms.

We are, however, primarily interested in automorphisms and we know that automorphisms should form a group under composition. This chapter is devoted to finding group structures on quantum monoids, i.e. to define and study quantum groups.

This is easy in the commutative situation, i.e. if the representing algebra of a quantum monoid is commutative. Then we can define a morphism that sends elements of the quantum group to their inverses. This will lead us to the notion of affine algebraic groups.

In the noncommutative situation, however, it will turn out that such an inversion morphism (of quantum spaces) does not exist. It will have to be replaced by a more complicated construction. Thus quantum groups will not be groups in the sense of category theory. Still we will be able to perform one of the most important and most basic constructions in group theory, the formation of the group of invertible elements of a monoid. In the case of a quantum monoid acting universally on a quantum space this will lead to the good definition of a quantum automorphism group of the quantum space.

In order to have the appropriate tools for introducing quantum groups we first introduce Hopf algebras which will be the representing algebras of quantum groups. Furthermore we need the notion of a monoid and of a group in a category. We will see, however, that quantum groups are in general not groups in the category of quantum spaces.

We first study the simple cases of affine algebraic groups and of formal groups. They will have Hopf algebras as representing objects and will indeed be groups in reasonable categories. Then we come to quantum groups, and construct quantum automorphism groups of quantum spaces.

At the end of the chapter you should

- know what a Hopf algebra is,

- know what a group in a category is,
- know the definition and examples of affine algebraic groups and formal groups,
- know the definition and examples of quantum groups and be able to construct quantum automorphism groups for small quantum spaces,
- understand why a Hopf algebra is a reasonable representing algebra for a quantum group.

1. Hopf Algebras

The difference between a monoid and a group lies in the existence of an additional map $S : G \ni g \mapsto g^{-1} \in G$ for a group G that allows forming inverses. This map satisfies the equation $S(g)g = 1$ or in a diagrammatic form

$$\begin{array}{ccccc} G & \xrightarrow{\varepsilon} & \{1\} & \xrightarrow{1} & G \\ \downarrow{\scriptstyle \Delta} & & & & \uparrow{\scriptstyle \text{mult}} \\ G \times G & & \xrightarrow{S \times \text{id}} & & G \times G \end{array}$$

We want to carry this property over to a definition of quantum groups. We know already that quantum monoids G are represented by bialgebras H. So an "inverse map" should be a morphism $S : G \longrightarrow G$ with a certain property, if G is to become a quantum group, or an algebra homomorphism $S : H \longrightarrow H$ for the representing bialgebra H of G. We need a slightly more general definition of Hopf algebras. They will then be the representing algebras for quantum groups.

Definition 2.1.1. A *left Hopf algebra* H is a bialgebra H together with a *left antipode* $S : H \longrightarrow H$, i.e. a linear map S such that the following diagram commutes:

$$\begin{array}{ccccc} H & \xrightarrow{\varepsilon} & \mathbb{K} & \xrightarrow{\eta} & H \\ \downarrow{\scriptstyle \Delta} & & & & \uparrow{\scriptstyle \nabla} \\ H \otimes H & & \xrightarrow{S \otimes \text{id}} & & H \otimes H \end{array}$$

Symmetrically we define a *right Hopf algebra* H. A *Hopf algebra* is a left and right Hopf algebra. The map S is called a (left, right, two-sided) *antipode*.

Using the Sweedler notation (A.6.3) the commutative diagram above can also be expressed by the equation

$$\sum S(a_{(1)})a_{(2)} = \eta\varepsilon(a)$$

for all $a \in H$. Observe that we do not require that $S : H \longrightarrow H$ is an algebra homomorphism.

Problem 2.1.1. 1. Let H be a bialgebra and $S \in \text{Hom}(H, H)$. Then S is an antipode for H (and H is a Hopf algebra) iff S is a two sided inverse for id in the algebra $(\text{Hom}(H, H), *, \eta\varepsilon)$ (see A.6.4). In particular S is uniquely determined.

2. Let H be a Hopf algebra. Then S is an antihomomorphism of algebras and coalgebras i.e. S "inverts the order of the multiplication and the comultiplication".

3. Let H and K be Hopf algebras and let $f : H \longrightarrow K$ be a homomorphism of bialgebras. Then $fS_H = S_K f$, i.e. f is compatible with the antipode.

Definition 2.1.2. Because of Problem 2.1.1 3. every homomorphism of bialgebras between Hopf algebras is compatible with the antipodes. So we define a *homomorphism of Hopf algebras* to be a homomorphism of bialgebras. The category of Hopf algebras will be denoted by $\mathbb{K}$-**Hopf**.

Proposition 2.1.3. *Let H be a bialgebra with an algebra generating set X. Let $S : H \longrightarrow H^{op}$ be an algebra homomorphism such that $\sum S(x_{(1)})x_{(2)} = \eta\varepsilon(x)$ for all $x \in X$. Then S is a left antipode of H.*

Proof. Assume $a, b \in H$ such that $\sum S(a_{(1)})a_{(2)} = \eta\varepsilon(a)$ and $\sum S(b_{(1)})b_{(2)} = \eta\varepsilon(b)$. Then

$$\begin{aligned}\sum S((ab)_{(1)})(ab)_{(2)} &= \sum S(a_{(1)}b_{(1)})a_{(2)}b_{(2)} = \sum S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)}\\ &= \sum S(b_{(1)})\eta\varepsilon(a)b_{(2)} = \eta\varepsilon(a)\eta\varepsilon(b) = \eta\varepsilon(ab).\end{aligned}$$

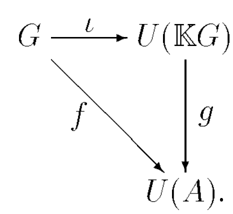
Since every element of H is a finite sum of finite products of elements in X, for which the equality holds, this equality extends to all of H by induction. □

Example 2.1.4. 1. Let V be a vector space and $T(V)$ the tensor algebra over V. We have seen in Problem A.5.6 that $T(V)$ is a bialgebra and that V generates $T(V)$ as an algebra. Define $S : V \longrightarrow T(V)^{op}$ by $S(v) := -v$ for all $v \in V$. By the universal property of the tensor algebra this map extends to an algebra homomorphism $S : T(V) \longrightarrow T(V)^{op}$. Since $\Delta(v) = v \otimes 1 + 1 \otimes v$ we have $\sum S(v_{(1)})v_{(2)} = \nabla(S \otimes 1)\Delta(v) = -v + v = 0 = \eta\varepsilon(v)$ for all $v \in V$, hence $T(V)$ is a Hopf algebra by the preceding proposition.

2. Let V be a vector space and $S(V)$ the symmetric algebra over V (that is commutative). We have seen in Problem A.5.7 that $S(V)$ is a bialgebra and that V generates $S(V)$ as an algebra. Define $S : V \longrightarrow S(V)$ by $S(v) := -v$ for all $v \in V$. S extends to an algebra homomorphism $S : S(V) \longrightarrow S(V)$. Since $\Delta(v) = v \otimes 1 + 1 \otimes v$ we have $\sum S(v_{(1)})v_{(2)} = \nabla(S \otimes 1)\Delta(v) = -v + v = 0 = \eta\varepsilon(v)$ for all $v \in V$, hence $S(V)$ is a Hopf algebra by the preceding proposition.

Example 2.1.5. (**Group Algebras**) For each algebra A we can form the *group of units* $U(A) := \{a \in A | \exists a^{-1} \in A\}$ with the multiplication of A as composition of the group. Then U is a covariant functor $U : \mathbb{K}\text{-}\mathbf{Alg} \longrightarrow \mathbf{Gr}$. This functor leads to the following universal problem.

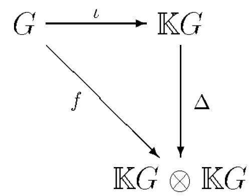
Let G be a group. An algebra $\mathbb{K}G$ together with a group homomorphism $\iota : G \longrightarrow U(\mathbb{K}G)$ is called a (the) *group algebra of* G, if for every algebra A and for every group homomorphism $f : G \longrightarrow U(A)$ there exists a unique homomorphism of algebras $g : \mathbb{K}G \longrightarrow A$ such that the following diagram commutes



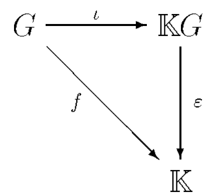
The group algebra $\mathbb{K}G$ is (if it exists) unique up to isomorphism. It is generated as an algebra by the image of G. The map $\iota : G \longrightarrow U(\mathbb{K}G) \subseteq \mathbb{K}G$ is injective and the image of G in $\mathbb{K}G$ is a basis.

The group algebra can be constructed as the free vector space $\mathbb{K}G$ with basis G and the algebra structure of $\mathbb{K}G$ is given by $\mathbb{K}G \otimes \mathbb{K}G \ni g \otimes h \mapsto gh \in \mathbb{K}G$ and the unit $\eta : \mathbb{K} \ni \alpha \mapsto \alpha e \in \mathbb{K}G$.

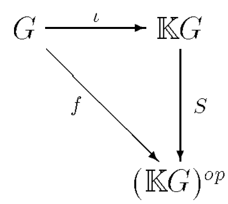
The group algebra $\mathbb{K}G$ is a Hopf algebra. The comultiplication is given by the diagram



with $f(g) := g \otimes g$ which defines a group homomorphism $f : G \longrightarrow U(\mathbb{K}G \otimes \mathbb{K}G)$. The counit is given by



where $f(g) = 1$ for all $g \in G$. One shows easily by using the universal property, that Δ is coassociative and has counit ε. Define an algebra homomorphism $S : \mathbb{K}G \longrightarrow (\mathbb{K}G)^{op}$ by



with $f(g) := g^{-1}$ which is a group homomorphism $f : G \longrightarrow U((\mathbb{K}G)^{op})$. Then Proposition 1.3 shows that $\mathbb{K}G$ is a Hopf algebra.

The preceding example of a Hopf algebra gives rise to the definition of particular elements in arbitrary Hopf algebras, that share certain properties with elements of a group. We will use and study these elements later on in chapter 5.

Definition 2.1.6. Let H be a Hopf algebra. An element $g \in H, g \neq 0$ is called a *group-like element* if

$$\Delta(g) = g \otimes g.$$

Observe that $\varepsilon(g) = 1$ for each group-like element g in a Hopf algebra H. In fact we have $g = \nabla(\varepsilon \otimes 1)\Delta(g) = \varepsilon(g)g \neq 0$ hence $\varepsilon(g) = 1$. If the base ring is not a field then one adds this property to the definition of a group-like element.

Problem 2.1.2. 1. Let $\mathbb{K}$ be a field. Show that an element $x \in \mathbb{K}G$ satisfies $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$ if and only if $x = g \in G$.

2. Show that the group-like elements of a Hopf algebra form a group under multiplication of the Hopf algebra.

Example 2.1.7. (**Universal Enveloping Algebras**) A *Lie algebra* consists of a vector space $\mathfrak{g}$ together with a (linear) multiplication $\mathfrak{g} \otimes \mathfrak{g} \ni x \otimes y \mapsto [x,y] \in \mathfrak{g}$ such that the following laws hold:

$$\begin{aligned} &[x,x] = 0, \\ &[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad \text{(Jacobi identity)}. \end{aligned}$$

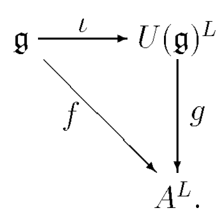
A *homomorphism of Lie algebras* $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a linear map f such that $f([x,y]) = [f(x), f(y)]$. Thus Lie algebras form a category $\mathbb{K}$-**Lie**.

An important example is the Lie algebra associated with an associative algebra (with unit). If A is an algebra then the vector space A with the Lie multiplication

$$[x,y] := xy - yx \tag{1}$$

is a Lie algebra denoted by A^L. This construction of a Lie algebra defines a covariant functor $\text{-}^L : \mathbb{K}\text{-}\mathbf{Alg} \longrightarrow \mathbb{K}\text{-}\mathbf{Lie}$. This functor leads to the following universal problem.

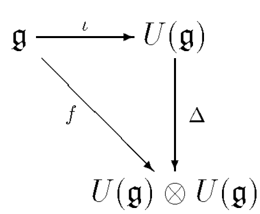
Let $\mathfrak{g}$ be a Lie algebra. An algebra $U(\mathfrak{g})$ together with a Lie algebra homomorphism $\iota : \mathfrak{g} \longrightarrow U(\mathfrak{g})^L$ is called a (the) *universal enveloping algebra of* $\mathfrak{g}$, if for every algebra A and for every Lie algebra homomorphism $f : \mathfrak{g} \longrightarrow A^L$ there exists a unique homomorphism of algebras $g : U(\mathfrak{g}) \longrightarrow A$ such that the following diagram commutes



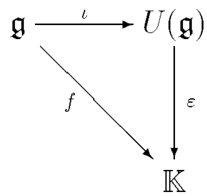
The universal enveloping algebra $U(\mathfrak{g})$ is (if it exists) unique up to isomorphism. It is generated as an algebra by the image of $\mathfrak{g}$.

The universal enveloping algebra can be constructed as $U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x,y])$ where $T(\mathfrak{g}) = \mathbb{K} \oplus \mathfrak{g} \oplus \mathfrak{g} \otimes \mathfrak{g} \ldots$ is the tensor algebra. The map $\iota : \mathfrak{g} \longrightarrow U(\mathfrak{g})^L$ is injective.

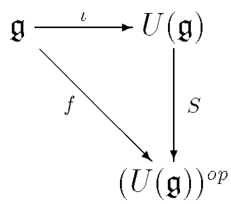
The universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra. The comultiplication is given by the diagram



with $f(x) := x \otimes 1 + 1 \otimes x$ which defines a Lie algebra homomorphism $f : \mathfrak{g} \longrightarrow (U(\mathfrak{g}) \otimes U(\mathfrak{g}))^L$. The counit is given by



with $f(x) = 0$ for all $x \in \mathfrak{g}$. One shows easily by using the universal property, that Δ is coassociative and has counit ε. Define an algebra homomorphism $S : U(\mathfrak{g}) \longrightarrow (U(\mathfrak{g}))^{op}$ by



with $f(x) := -x$ which is a Lie algebra homomorphism $f : \mathfrak{g} \longrightarrow (U(\mathfrak{g})^{op})^L$. Then Proposition 1.3 shows that $U(\mathfrak{g})$ is a Hopf algebra.

(Observe, that the meaning of U in this example and the previous example (group of units, universal enveloping algebra) is totally different, in the first case U can be applied to an algebra and gives a group, in the second case U can be applied to a Lie algebra and gives an algebra.)

The preceding example of a Hopf algebra gives rise to the definition of particular elements in arbitrary Hopf algebras, that share certain properties with elements of a Lie algebra. We will use these elements later on in chapter 5.

Definition 2.1.8. Let H be a Hopf algebra. An element $x \in H$ is called a *primitive element* if

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$

Let $g \in H$ be a group-like element. An element $x \in H$ is called a *skew primitive or g-primitive element* if

$$\Delta(x) = x \otimes 1 + g \otimes x.$$

Problem 2.1.3. Show that the set of primitive elements $P(H) = \{x \in H | \Delta(x) = x \otimes 1 + 1 \otimes x\}$ of a Hopf algebra H is a Lie subalgebra of H^L.

Proposition 2.1.9. *Let H be a Hopf algebra with antipode S. The following are equivalent:*

1. $S^2 = id$.
2. $\sum S(a_{(2)})a_{(1)} = \eta\varepsilon(a)$ for all $a \in H$.
3. $\sum a_{(2)}S(a_{(1)}) = \eta\varepsilon(a)$ for all $a \in H$.

PROOF. Let $S^2 = $id. Then

$$\begin{aligned}\sum S(a_{(2)})a_{(1)} &= S^2(\sum S(a_{(2)})a_{(1)}) = S(\sum S(a_{(1)})S^2(a_{(2)})) \\ &= S(\sum S(a_{(1)})a_{(2)}) = S(\eta\varepsilon(a)) = \eta\varepsilon(a)\end{aligned}$$

by using Problem 2.1.1.

Conversely assume that 2. holds. Then

$$\begin{aligned}S * S^2(a) &= \sum S(a_{(1)}S^2(a_{(2)}) = S(\sum S(a_{(2)})a_{(1)} \\ &= S(\eta\varepsilon(a)) = \eta\varepsilon(a).\end{aligned}$$

Thus S^2 and id are inverses of S in the convolution algebra $\mathrm{Hom}(H, H)$, hence $S^2 = \mathrm{id}$.

Analogously one shows that 1. and 3. are equivalent. □

Corollary 2.1.10. *If H is a commutative Hopf algebra or a cocommutative Hopf algebra with antipode S, then $S^2 =$id.*

Remark 2.1.11. Kaplansky: Ten conjectures on Hopf algebras

In a set of lecture notes on bialgebras based on a course given at Chicago university in 1973, made public in 1975, Kaplansky formulated a set of conjectures on Hopf algebras that have been the aim of intensive research.

1. If C is a Hopf subalgebra of the Hopf algebra B then B is a free left C-module.
   (Yes, if H is finite dimensional [Nichols-Zoeller]; No for infinite dimensional Hopf algebras [Oberst- Schneider]; $B : C$ is not necessarily faithfully flat [Schauenburg])
2. Call a coalgebra C *admissible* if it admits an algebra structure making it a Hopf algebra. The conjecture states that C is admissible if and only if every finite subset of C lies in a finite-dimensional admissible subcoalgebra.
   (Remarks.
   (a) Both implications seem hard.
   (b) There is a corresponding conjecture where "Hopf algebra" is replaced by "bialgebra".
   (c) There is a dual conjecture for locally finite algebras.)
   (No results known.)
3. A Hopf algebra of characteristic 0 has no non-zero central nilpotent elements.
   (First counter example given by [Schmidt-Samoa]. If H is unimodular and not semisimple, e.g. a Drinfel'd double of a not semisimple finite dimensional Hopf algebra, then the integral Λ satisfies $\Lambda \neq 0$, $\Lambda^2 = \varepsilon(\Lambda)\Lambda = 0$ since $D(H)$ is not semisimple, and $a\Lambda = \varepsilon(a)\Lambda = \Lambda\varepsilon(a) = \Lambda a$ since $D(H)$ is unimodular [Sommerhäuser].)
4. (Nichols). Let x be an element in a Hopf algebra H with antipode S. Assume that for any a in H we have

   $$\sum b_i x S(c_i) = \varepsilon(a)x$$

   where $\Delta a = \sum b_i \otimes c_i$. Conjecture: x is in the center of H.
   ($ax = \sum a_{(1)}x\varepsilon(a_{(2)}) = \sum a_{(1)}xS(a_{(2)})a_{(3)}) = \sum \varepsilon(a_{(1)})xa_{(2)} = xa$.)

In the remaining six conjectures H is a finite-dimensional Hopf algebra over an algebraically closed field.

5. If H is semisimple on either side (i.e. either H or the dual H^* is semisimple as an algebra) the square of the antipode is the identity.

   (Yes if char($\mathbb{K}$) = 0 [Larson-Radford], yes if char($\mathbb{K}$) is large [Sommerhäuser])
6. The size of the matrices occurring in any full matrix constituent of H divides the dimension of H.

   (Yes is Hopf algebra is defined over $\mathbb{Z}$ [Larson]; in general not known; work by [Montgomery-Witherspoon], [Zhu], [Gelaki])
7. If H is semisimple on both sides the characteristic does not divide the dimension.

   (Larson-Radford)
8. If the dimension of H is prime then H is commutative and cocommutative.

   (Yes in characteristic 0 [Zhu: 1994])

   Remark. Kac, Larson, and Sweedler have partial results on 5 – 8.

   (Was also proved by [Kac])

In the two final conjectures assume that the characteristic does not divide the dimension of H.

9. The dimension of the radical is the same on both sides.

   (Counterexample by [Nichols]; counterexample in Frobenius- Lusztig kernel of $U_q(sl(2))$ [Schneider])
10. There are only a finite number (up to isomorphism) of Hopf algebras of a given dimension.

    (Yes for semisimple, cosemisimple Hopf algebras: Stefan 1997)

    (Counterexamples: [Andruskiewitsch, Schneider], [Beattie, others] 1997)