CHAPTER 2

Hopf Algebras, Algebraic, Formal, and Quantum Groups

Introduction

In the first chapter we have encountered quantum monoids and studied their role as monoids operating on quantum spaces. The "elements" of quantum monoids operating on quantum spaces should be understood as endomorphisms of the quantum spaces. In the construction of the multiplication for universal quantum monoids of quantum spaces we have seen that this multiplication is essentially the "composition" of endomorphisms.

We are, however, primarily interested in automorphisms and we know that automorphisms should form a group under composition. This chapter is devoted to finding group structures on quantum monoids, i.e. to define and study quantum groups.

This is easy in the commutative situation, i.e. if the representing algebra of a quantum monoid is commutative. Then we can define a morphism that sends elements of the quantum group to their inverses. This will lead us to the notion of affine algebraic groups.

In the noncommutative situation, however, it will turn out that such an inversion morphism (of quantum spaces) does not exist. It will have to be replaced by a more complicated construction. Thus quantum groups will not be groups in the sense of category theory. Still we will be able to perform one of the most important and most basic constructions in group theory, the formation of the group of invertible elements of a monoid. In the case of a quantum monoid acting universally on a quantum space this will lead to the good definition of a quantum automorphism group of the quantum space.

In order to have the appropriate tools for introducing quantum groups we first introduce Hopf algebras which will be the representing algebras of quantum groups. Furthermore we need the notion of a monoid and of a group in a category. We will see, however, that quantum groups are in general not groups in the category of quantum spaces.

We first study the simple cases of affine algebraic groups and of formal groups. They will have Hopf algebras as representing objects and will indeed be groups in reasonable categories. Then we come to quantum groups, and construct quantum automorphism groups of quantum spaces.

At the end of the chapter you should

- know what a Hopf algebra is,

- know what a group in a category is,
- know the definition and examples of affine algebraic groups and formal groups,
- know the definition and examples of quantum groups and be able to construct quantum automorphism groups for small quantum spaces,
- understand why a Hopf algebra is a reasonable representing algebra for a quantum group.

1. Hopf Algebras

The difference between a monoid and a group lies in the existence of an additional map $S : G \ni g \mapsto g^{-1} \in G$ for a group G that allows forming inverses. This map satisfies the equation $S(g)g = 1$ or in a diagrammatic form

$$\begin{tikzcd} G \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \{1\} \arrow[r, "1"] & G \\ G \times G \arrow[rr, "S\times\mathrm{id}"] & & G \times G \arrow[u, "\mathrm{mult}"'] \end{tikzcd}$$

We want to carry this property over to a definition of quantum groups. We know already that quantum monoids G are represented by bialgebras H. So an "inverse map" should be a morphism $S : G \to G$ with a certain property, if G is to become a quantum group, or an algebra homomorphism $S : H \to H$ for the representing bialgebra H of G. We need a slightly more general definition of Hopf algebras. They will then be the representing algebras for quantum groups.

Definition 2.1.1. A *left Hopf algebra* H is a bialgebra H together with a *left antipode* $S : H \to H$, i.e. a linear map S such that the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & H \\ H \otimes H \arrow[rr, "S\otimes\mathrm{id}"] & & H \otimes H \arrow[u, "\nabla"'] \end{tikzcd}$$

Symmetrically we define a *right Hopf algebra* H. A *Hopf algebra* is a left and right Hopf algebra. The map S is called a (left, right, two-sided) *antipode*.

Using the Sweedler notation (A.6.3) the commutative diagram above can also be expressed by the equation

$$\sum S(a_{(1)})a_{(2)} = \eta\varepsilon(a)$$

for all $a \in H$. Observe that we do not require that $S : H \to H$ is an algebra homomorphism.

Problem 2.1.1. 1. Let H be a bialgebra and $S \in \mathrm{Hom}(H, H)$. Then S is an antipode for H (and H is a Hopf algebra) iff S is a two sided inverse for id in the algebra $(\mathrm{Hom}(H, H), *, \eta\varepsilon)$ (see A.6.4). In particular S is uniquely determined.

2. Let H be a Hopf algebra. Then S is an antihomomorphism of algebras and coalgebras i.e. S "inverts the order of the multiplication and the comultiplication".

3. Let H and K be Hopf algebras and let $f : H \to K$ be a homomorphism of bialgebras. Then $fS_H = S_K f$, i.e. f is compatible with the antipode.

Definition 2.1.2. Because of Problem 2.1.1 3. every homomorphism of bialgebras between Hopf algebras is compatible with the antipodes. So we define a *homomorphism of Hopf algebras* to be a homomorphism of bialgebras. The category of Hopf algebras will be denoted by $\mathbb{K}$-**Hopf**.

Proposition 2.1.3. *Let H be a bialgebra with an algebra generating set X. Let $S : H \longrightarrow H^{op}$ be an algebra homomorphism such that $\sum S(x_{(1)})x_{(2)} = \eta\varepsilon(x)$ for all $x \in X$. Then S is a left antipode of H.*

PROOF. Assume $a, b \in H$ such that $\sum S(a_{(1)})a_{(2)} = \eta\varepsilon(a)$ and $\sum S(b_{(1)})b_{(2)} = \eta\varepsilon(b)$. Then

$$\begin{aligned}\sum S((ab)_{(1)})(ab)_{(2)} &= \sum S(a_{(1)}b_{(1)})a_{(2)}b_{(2)} = \sum S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)}\\ &= \sum S(b_{(1)})\eta\varepsilon(a)b_{(2)} = \eta\varepsilon(a)\eta\varepsilon(b) = \eta\varepsilon(ab).\end{aligned}$$

Since every element of H is a finite sum of finite products of elements in X, for which the equality holds, this equality extends to all of H by induction. □

Example 2.1.4. 1. Let V be a vector space and $T(V)$ the tensor algebra over V. We have seen in Problem A.5.6 that $T(V)$ is a bialgebra and that V generates $T(V)$ as an algebra. Define $S : V \longrightarrow T(V)^{op}$ by $S(v) := -v$ for all $v \in V$. By the universal property of the tensor algebra this map extends to an algebra homomorphism $S : T(V) \longrightarrow T(V)^{op}$. Since $\Delta(v) = v \otimes 1 + 1 \otimes v$ we have $\sum S(v_{(1)})v_{(2)} = \nabla(S \otimes 1)\Delta(v) = -v + v = 0 = \eta\varepsilon(v)$ for all $v \in V$, hence $T(V)$ is a Hopf algebra by the preceding proposition.

2. Let V be a vector space and $S(V)$ the symmetric algebra over V (that is commutative). We have seen in Problem A.5.7 that $S(V)$ is a bialgebra and that V generates $S(V)$ as an algebra. Define $S : V \longrightarrow S(V)$ by $S(v) := -v$ for all $v \in V$. S extends to an algebra homomorphism $S : S(V) \longrightarrow S(V)$. Since $\Delta(v) = v \otimes 1 + 1 \otimes v$ we have $\sum S(v_{(1)})v_{(2)} = \nabla(S \otimes 1)\Delta(v) = -v + v = 0 = \eta\varepsilon(v)$ for all $v \in V$, hence $S(V)$ is a Hopf algebra by the preceding proposition.

Example 2.1.5. (**Group Algebras**) For each algebra A we can form the *group of units* $U(A) := \{a \in A | \exists a^{-1} \in A\}$ with the multiplication of A as composition of the group. Then U is a covariant functor $U : \mathbb{K}\text{-}\mathbf{Alg} \longrightarrow \mathbf{Gr}$. This functor leads to the following universal problem.

Let G be a group. An algebra $\mathbb{K}G$ together with a group homomorphism $\iota : G \longrightarrow U(\mathbb{K}G)$ is called a (the) *group algebra of* G, if for every algebra A and for every group homomorphism $f : G \longrightarrow U(A)$ there exists a unique homomorphism of algebras $g : \mathbb{K}G \longrightarrow A$ such that the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[dr, "f"'] & U(\mathbb{K}G) \arrow[d, "g"] \\ & U(A). \end{tikzcd}$$

The group algebra $\mathbb{K}G$ is (if it exists) unique up to isomorphism. It is generated as an algebra by the image of G. The map $\iota : G \longrightarrow U(\mathbb{K}G) \subseteq \mathbb{K}G$ is injective and the image of G in $\mathbb{K}G$ is a basis.

The group algebra can be constructed as the free vector space $\mathbb{K}G$ with basis G and the algebra structure of $\mathbb{K}G$ is given by $\mathbb{K}G \otimes \mathbb{K}G \ni g \otimes h \mapsto gh \in \mathbb{K}G$ and the unit $\eta : \mathbb{K} \ni \alpha \mapsto \alpha e \in \mathbb{K}G$.

The group algebra $\mathbb{K}G$ is a Hopf algebra. The comultiplication is given by the diagram

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{K}G \arrow[d, "\Delta"] \\ & \mathbb{K}G \otimes \mathbb{K}G \end{tikzcd}$$

with $f(g) := g \otimes g$ which defines a group homomorphism $f : G \longrightarrow U(\mathbb{K}G \otimes \mathbb{K}G)$. The counit is given by

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{K}G \arrow[d, "\varepsilon"] \\ & \mathbb{K} \end{tikzcd}$$

where $f(g) = 1$ for all $g \in G$. One shows easily by using the universal property, that Δ is coassociative and has counit ε. Define an algebra homomorphism $S : \mathbb{K}G \longrightarrow (\mathbb{K}G)^{op}$ by

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{K}G \arrow[d, "S"] \\ & (\mathbb{K}G)^{op} \end{tikzcd}$$

with $f(g) := g^{-1}$ which is a group homomorphism $f : G \longrightarrow U((\mathbb{K}G)^{op})$. Then Proposition 1.3 shows that $\mathbb{K}G$ is a Hopf algebra.

The preceding example of a Hopf algebra gives rise to the definition of particular elements in arbitrary Hopf algebras, that share certain properties with elements of a group. We will use and study these elements later on in chapter 5.

Definition 2.1.6. Let H be a Hopf algebra. An element $g \in H, g \neq 0$ is called a *group-like element* if

$$\Delta(g) = g \otimes g.$$

Observe that $\varepsilon(g) = 1$ for each group-like element g in a Hopf algebra H. In fact we have $g = \nabla(\varepsilon \otimes 1)\Delta(g) = \varepsilon(g)g \neq 0$ hence $\varepsilon(g) = 1$. If the base ring is not a field then one adds this property to the definition of a group-like element.

Problem 2.1.2. 1. Let $\mathbb{K}$ be a field. Show that an element $x \in \mathbb{K}G$ satisfies $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$ if and only if $x = g \in G$.

2. Show that the group-like elements of a Hopf algebra form a group under multiplication of the Hopf algebra.

Example 2.1.7. **(Universal Enveloping Algebras)** A *Lie algebra* consists of a vector space $\mathfrak{g}$ together with a (linear) multiplication $\mathfrak{g} \otimes \mathfrak{g} \ni x \otimes y \mapsto [x, y] \in \mathfrak{g}$ such that the following laws hold:

$$\begin{aligned} &[x, x] = 0, \\ &[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \text{(Jacobi identity).} \end{aligned}$$

A *homomorphism of Lie algebras* $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a linear map f such that $f([x, y]) = [f(x), f(y)]$. Thus Lie algebras form a category $\mathbb{K}$-**Lie**.

An important example is the Lie algebra associated with an associative algebra (with unit). If A is an algebra then the vector space A with the Lie multiplication

$$[x, y] := xy - yx \tag{1}$$

is a Lie algebra denoted by A^L. This construction of a Lie algebra defines a covariant functor $\text{-}^L : \mathbb{K}\text{-}\mathbf{Alg} \longrightarrow \mathbb{K}\text{-}\mathbf{Lie}$. This functor leads to the following universal problem.

Let $\mathfrak{g}$ be a Lie algebra. An algebra $U(\mathfrak{g})$ together with a Lie algebra homomorphism $\iota : \mathfrak{g} \longrightarrow U(\mathfrak{g})^L$ is called a (the) *universal enveloping algebra of* $\mathfrak{g}$, if for every algebra A and for every Lie algebra homomorphism $f : \mathfrak{g} \longrightarrow A^L$ there exists a unique homomorphism of algebras $g : U(\mathfrak{g}) \longrightarrow A$ such that the following diagram commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "f"'] & U(\mathfrak{g})^L \arrow[d, "g"] \\ & A^L. \end{tikzcd}$$

The universal enveloping algebra $U(\mathfrak{g})$ is (if it exists) unique up to isomorphism. It is generated as an algebra by the image of $\mathfrak{g}$.

The universal enveloping algebra can be constructed as $U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y])$ where $T(\mathfrak{g}) = \mathbb{K} \oplus \mathfrak{g} \oplus \mathfrak{g} \otimes \mathfrak{g} \ldots$ is the tensor algebra. The map $\iota : \mathfrak{g} \longrightarrow U(\mathfrak{g})^L$ is injective.

The universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra. The comultiplication is given by the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "f"'] & U(\mathfrak{g}) \arrow[d, "\Delta"] \\ & U(\mathfrak{g}) \otimes U(\mathfrak{g}) \end{tikzcd}$$

with $f(x) := x \otimes 1 + 1 \otimes x$ which defines a Lie algebra homomorphism $f : \mathfrak{g} \to (U(\mathfrak{g}) \otimes U(\mathfrak{g}))^L$. The counit is given by

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "f"'] & U(\mathfrak{g}) \arrow[d, "\varepsilon"] \\ & \mathbb{K} \end{tikzcd}$$

with $f(x) = 0$ for all $x \in \mathfrak{g}$. One shows easily by using the universal property, that Δ is coassociative and has counit ε. Define an algebra homomorphism $S : U(\mathfrak{g}) \to (U(\mathfrak{g}))^{op}$ by

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "f"'] & U(\mathfrak{g}) \arrow[d, "S"] \\ & (U(\mathfrak{g}))^{op} \end{tikzcd}$$

with $f(x) := -x$ which is a Lie algebra homomorphism $f : \mathfrak{g} \to (U(\mathfrak{g})^{op})^L$. Then Proposition 1.3 shows that $U(\mathfrak{g})$ is a Hopf algebra.

(Observe, that the meaning of U in this example and the previous example (group of units, universal enveloping algebra) is totally different, in the first case U can be applied to an algebra and gives a group, in the second case U can be applied to a Lie algebra and gives an algebra.)

The preceding example of a Hopf algebra gives rise to the definition of particular elements in arbitrary Hopf algebras, that share certain properties with elements of a Lie algebra. We will use these elements later on in chapter 5.

Definition 2.1.8. Let H be a Hopf algebra. An element $x \in H$ is called a *primitive element* if

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$

Let $g \in H$ be a group-like element. An element $x \in H$ is called a *skew primitive or g-primitive element* if

$$\Delta(x) = x \otimes 1 + g \otimes x.$$

Problem 2.1.3. Show that the set of primitive elements $P(H) = \{x \in H | \Delta(x) = x \otimes 1 + 1 \otimes x\}$ of a Hopf algebra H is a Lie subalgebra of H^L.

Proposition 2.1.9. *Let H be a Hopf algebra with antipode S. The following are equivalent:*

1. $S^2 = id$.
2. $\sum S(a_{(2)})a_{(1)} = \eta\varepsilon(a)$ for all $a \in H$.
3. $\sum a_{(2)}S(a_{(1)}) = \eta\varepsilon(a)$ for all $a \in H$.

PROOF. Let $S^2 =$id. Then

$$\begin{aligned}\sum S(a_{(2)})a_{(1)} &= S^2(\sum S(a_{(2)})a_{(1)}) = S(\sum S(a_{(1)})S^2(a_{(2)}))\\ &= S(\sum S(a_{(1)})a_{(2)}) = S(\eta\varepsilon(a)) = \eta\varepsilon(a)\end{aligned}$$

by using Problem 2.1.1.

Conversely assume that 2. holds. Then

$$\begin{aligned}S * S^2(a) &= \sum S(a_{(1)}S^2(a_{(2)}) = S(\sum S(a_{(2)})a_{(1)}\\ &= S(\eta\varepsilon(a)) = \eta\varepsilon(a).\end{aligned}$$

Thus S^2 and id are inverses of S in the convolution algebra $\mathrm{Hom}(H, H)$, hence $S^2 = \mathrm{id}$.

Analogously one shows that 1. and 3. are equivalent. □

Corollary 2.1.10. *If H is a commutative Hopf algebra or a cocommutative Hopf algebra with antipode S, then $S^2 =$id.*

Remark 2.1.11. Kaplansky: Ten conjectures on Hopf algebras

In a set of lecture notes on bialgebras based on a course given at Chicago university in 1973, made public in 1975, Kaplansky formulated a set of conjectures on Hopf algebras that have been the aim of intensive research.

1. If C is a Hopf subalgebra of the Hopf algebra B then B is a free left C-module.
   (Yes, if H is finite dimensional [Nichols-Zoeller]; No for infinite dimensional Hopf algebras [Oberst- Schneider]; $B : C$ is not necessarily faithfully flat [Schauenburg])
2. Call a coalgebra C *admissible* if it admits an algebra structure making it a Hopf algebra. The conjecture states that C is admissible if and only if every finite subset of C lies in a finite-dimensional admissible subcoalgebra.
   (Remarks.
   (a) Both implications seem hard.
   (b) There is a corresponding conjecture where "Hopf algebra" is replaced by "bialgebra".
   (c) There is a dual conjecture for locally finite algebras.)
   (No results known.)
3. A Hopf algebra of characteristic 0 has no non-zero central nilpotent elements.
   (First counter example given by [Schmidt-Samoa]. If H is unimodular and not semisimple, e.g. a Drinfel'd double of a not semisimple finite dimensional Hopf algebra, then the integral Λ satisfies $\Lambda \neq 0$, $\Lambda^2 = \varepsilon(\Lambda)\Lambda = 0$ since $D(H)$ is not semisimple, and $a\Lambda = \varepsilon(a)\Lambda = \Lambda\varepsilon(a) = \Lambda a$ since $D(H)$ is unimodular [Sommerhäuser].)
4. (Nichols). Let x be an element in a Hopf algebra H with antipode S. Assume that for any a in H we have

   $$\sum b_i x S(c_i) = \varepsilon(a)x$$

   where $\Delta a = \sum b_i \otimes c_i$. Conjecture: x is in the center of H.
   ($ax = \sum a_{(1)}x\varepsilon(a_{(2)}) = \sum a_{(1)}xS(a_{(2)})a_{(3)}) = \sum \varepsilon(a_{(1)})xa_{(2)} = xa$.)

In the remaining six conjectures H is a finite-dimensional Hopf algebra over an algebraically closed field.

5. If H is semisimple on either side (i.e. either H or the dual H^* is semisimple as an algebra) the square of the antipode is the identity.

   (Yes if $\mathrm{char}(\mathbb{K}) = 0$ [Larson-Radford], yes if $\mathrm{char}(\mathbb{K})$ is large [Sommerhäuser])
6. The size of the matrices occurring in any full matrix constituent of H divides the dimension of H.

   (Yes is Hopf algebra is defined over $\mathbb{Z}$ [Larson]; in general not known; work by [Montgomery-Witherspoon], [Zhu], [Gelaki])
7. If H is semisimple on both sides the characteristic does not divide the dimension.

   (Larson-Radford)
8. If the dimension of H is prime then H is commutative and cocommutative.

   (Yes in characteristic 0 [Zhu: 1994])

   Remark. Kac, Larson, and Sweedler have partial results on 5 – 8.

   (Was also proved by [Kac])

   In the two final conjectures assume that the characteristic does not divide the dimension of H.
9. The dimension of the radical is the same on both sides.

   (Counterexample by [Nichols]; counterexample in Frobenius- Lusztig kernel of $U_q(sl(2))$ [Schneider])
10. There are only a finite number (up to isomorphism) of Hopf algebras of a given dimension.

    (Yes for semisimple, cosemisimple Hopf algebras: Stefan 1997)

    (Counterexamples: [Andruskiewitsch, Schneider], [Beattie, others] 1997)

2. Monoids and Groups in a Category

Before we use Hopf algebras to describe quantum groups and some of the better known groups, such as affine algebraic groups and formal groups, we introduce the concept of a general group (and of a monoid) in an arbitrary category. Usually this concept is defined with respect to a categorical product in the given category. But in some categories there are in general no products. Still, one can define the concept of a group in a very simple fashion. We will start with that definition and then show that it coincides with the usual notion of a group in a category in case that category has finite products.

Definition 2.2.1. Let $\mathcal{C}$ be an arbitrary category. Let $G \in \mathcal{C}$ be an object. We use the notation $G(X) := \mathrm{Mor}_{\mathcal{C}}(X, G)$ for all $X \in \mathcal{C}$, $G(f) := \mathrm{Mor}_{\mathcal{C}}(f, G)$ for all morphisms $f : X \to Y$ in $\mathcal{C}$, and $f(X) := \mathrm{Mor}_{\mathcal{C}}(X, f)$ for all morphisms $f : G \to G'$.

G together with a natural transformation $m : G(\text{-}) \times G(\text{-}) \to G(\text{-})$ is called a *group* (*monoid*) *in the category* $\mathcal{C}$, if the sets $G(X)$ together with the multiplication $m(X) : G(X) \times G(X) \to G(X)$ are groups (monoids) for all $X \in \mathcal{C}$.

Let (G, m) and (G', m') be groups in $\mathcal{C}$. A morphism $f : G \to G'$ in $\mathcal{C}$ is called a *homomorphism of groups* in $\mathcal{C}$, if the diagrams

$$\begin{tikzcd} G(X) \times G(X) \arrow[r, "m(X)"] \arrow[d, "f(X)\times f(X)"'] & G(X) \arrow[d, "f(X)"] \\ G'(X) \times G'(X) \arrow[r, "m'(X)"] & G'(X) \end{tikzcd}$$

commute for all $X \in \mathcal{C}$.

Let (G, m) and (G', m') be monoids in $\mathcal{C}$. A morphism $f : G \to G'$ in $\mathcal{C}$ is called a *homomorphism of monoids* in $\mathcal{C}$, if the diagrams

$$\begin{tikzcd} G(X) \times G(X) \arrow[r, "m(X)"] \arrow[d, "f(X)\times f(X)"'] & G(X) \arrow[d, "f(X)"] \\ G'(X) \times G'(X) \arrow[r, "m'(X)"] & G'(X) \end{tikzcd}$$

and

$$\begin{tikzcd} & \{*\} \arrow[dl, "u"'] \arrow[dr, "u'"] & \\ G(X) \arrow[rr, "f(X)"] & & G'(X) \end{tikzcd}$$

commute for all $X \in \mathcal{C}$.

Problem 2.2.4. 1) If a set Z together with a multiplication $m : Z \times Z \to Z$ is a monoid, then the unit element $e \in Z$ is uniquely determined. If it is a group then also the inverse $i : Z \longrightarrow Z$ is uniquely determined. Unit element and inverses of groups are preserved by maps that are compatible with the multiplication.

2) Find an example of monoids Y and Z and a map $f : Y \longrightarrow Z$ with $f(y_1 y_2) = f(y_1)f(y_2)$ for all $y_1, y_2 \in Y$, but $f(e_Y) \neq e_Z$.

3) If (G, m) is a group in $\mathcal{C}$ and $i_X : G(X) \longrightarrow G(X)$ is the inverse, then i is a natural transformation. The Yoneda Lemma provides a morphism $S : G \longrightarrow G$ such that $i_X = \mathrm{Mor}_{\mathcal{C}}(X, S) = S(X)$ for all $X \in \mathcal{C}$.

Proposition 2.2.2. *Let $\mathcal{C}$ be a category with finite (categorical) products. An object G in $\mathcal{C}$ carries the structure $m : G(\text{-}) \times G(\text{-}) \longrightarrow G(\text{-})$ of a group in $\mathcal{C}$ if and only if there are morphisms $m : G \times G \longrightarrow G$, $u : E \longrightarrow G$, and $S : G \longrightarrow G$ such that the diagrams*

$$\begin{tikzcd}
G \times G \times G \arrow[r, "m \times 1"] \arrow[d, "1 \times m"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
E \times G \cong G \cong G \times E \arrow[r, "1 \times u"] \arrow[d, "u \times 1"'] \arrow[dr, "1"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r] \arrow[d, "\Delta"'] & E \arrow[r, "u"] & G \\
G \times G \arrow[rr, "1 \times S"', "S \times 1"] & & G \times G \arrow[u, "m"']
\end{tikzcd}$$

commute where Δ is the morphism defined in A.2. The multiplications are related by $m_X = \mathrm{Mor}_{\mathcal{C}}(X, m) = m(X)$.

An analogous statement holds for monoids.

Proof. The Yoneda Lemma defines a bijection between the set of morphisms $f : X \longrightarrow Y$ and the set of natural transformations $f(\text{-}) : X(\text{-}) \longrightarrow Y(\text{-})$ by $f(Z) = \mathrm{Mor}_{\mathcal{C}}(Z, f)$. In particular we have $m_X = \mathrm{Mor}_{\mathcal{C}}(X, m) = m(X)$. The diagram

$$\begin{tikzcd}
G(\text{-}) \times G(\text{-}) \times G(\text{-}) \arrow[r, "m_- \times 1"] \arrow[d, "1 \times m_-"'] & G(\text{-}) \times G(\text{-}) \arrow[d, "m_-"] \\
G(\text{-}) \times G(\text{-}) \arrow[r, "m_-"] & G(\text{-})
\end{tikzcd}$$

commutes if and only if $\mathrm{Mor}_{\mathcal{C}}(\text{-}, m(m \times 1)) = \mathrm{Mor}_{\mathcal{C}}(\text{-}, m)(\mathrm{Mor}_{\mathcal{C}}(\text{-}, m) \times 1) = m_-(m_- \times 1) = m_-(1 \times m_-) = \mathrm{Mor}_{\mathcal{C}}(\text{-}, m)(1 \times \mathrm{Mor}_{\mathcal{C}}(\text{-}, m)) = \mathrm{Mor}_{\mathcal{C}}(\text{-}, m(1 \times m))$ if and only if

$m(m \times 1) = m(1 \times m)$ if and only if the diagram

$$\begin{tikzcd} G \times G \times G \arrow[r, "m\times 1"] \arrow[d, "1\times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

commutes. In a similar way one shows the equivalence of the other diagram(s). $\square$

Problem 2.2.5. Let $\mathcal{C}$ be a category with finite products. Show that a morphism $f : G \to G'$ in $\mathcal{C}$ is a homomorphism of groups if and only if

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "f\times f"'] & G \arrow[d, "f"] \\ G' \times G' \arrow[r, "m'"] & G' \end{tikzcd}$$

commutes.

Definition 2.2.3. A *cogroup* (*comonoid*) G in $\mathcal{C}$ is a group (monoid) in $\mathcal{C}^{op}$, i.e. an object $G \in \operatorname{Ob}\mathcal{C} = \operatorname{Ob}\mathcal{C}^{op}$ together with a natural transformation $m(X) : G(X) \times G(X) \to G(X)$ where $G(X) = \operatorname{Mor}_{\mathcal{C}^{op}}(X, G) = \operatorname{Mor}_{\mathcal{C}}(G, X)$, such that $(G(X), m(X))$ is a group (monoid) for each $X \in \mathcal{C}$.

Remark 2.2.4. Let $\mathcal{C}$ be a category with finite (categorical) coproducts. An object G in $\mathcal{C}$ carries the structure $m : G(\text{-}) \times G(\text{-}) \to G(\text{-})$ of a cogroup in $\mathcal{C}$ if and only if there are morphisms $\Delta : G \to G \amalg G$, $\varepsilon : G \to I$, and $S : G \to G$ such that the diagrams

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & G \amalg G \arrow[d, "1\amalg\Delta"] \\ G \amalg G \arrow[r, "\Delta\amalg 1"] & G \amalg G \amalg G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] \arrow[drr, "1"] & & G \amalg G \arrow[d, "1\amalg\varepsilon"] \\ G \amalg G \arrow[r, "\varepsilon\amalg 1"] & I \amalg G \cong G \cong & G \amalg I \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\Delta"'] & I \arrow[r, "\varepsilon"] & G \\ G \amalg G \arrow[rr, "1\amalg S"', "S\amalg 1"] & & G \amalg G \arrow[u, "\nabla"'] \end{tikzcd}$$

commute where ∇ is dual to the morphism Δ defined in A.2. The multiplications are related by $\Delta_X = \operatorname{Mor}_{\mathcal{C}}(\Delta, X) = \Delta(X)$.

Let $\mathcal{C}$ be a category with finite coproducts and let G and G' be cogroups in $\mathcal{C}$. Then a homomorphism of groups $f : G' \to G$ is a morphism $f : G \to G'$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d, "f"'] & G \times G \arrow[d, "f \times f"] \\ G' \arrow[r, "\Delta'"] & G' \times G' \end{tikzcd}$$

commutes. An analogous result holds for comonoids.

Remark 2.2.5. Obviously similar observations and statements can be made for other algebraic structures in a category $\mathcal{C}$. So one can introduce vector spaces and covector spaces, monoids and comonoids, rings and corings in a category $\mathcal{C}$. The structures can be described by morphisms in $\mathcal{C}$ if $\mathcal{C}$ is a category with finite (co-) products.

Problem 2.2.6. Determine the structure of a covector space on a vector space V from the fact that $\mathrm{Hom}(V, W)$ is a vector space for all vector spaces W.

Proposition 2.2.6. *Let $G \in \mathcal{C}$ be a group with multiplication $a * b$, unit e, and inverse a^{-1} in $G(X)$. Then the morphisms $m : G \times G \longrightarrow G$, $u : E \longrightarrow G$, and $S : G \longrightarrow G$ are given by*

$$m = p_1 * p_2, \qquad u = e_E, \qquad S = \mathrm{id}_G^{-1}.$$

Proof. By the Yoneda Lemma A.9.1 these morphisms can be constructed from the natural transformation as follows. Under $\mathrm{Mor}_{\mathcal{C}}(G \times G, G \times G) = G \times G(G \times G) \cong G(G \times G) \times G(G \times G) \xrightarrow{*} G(G \times G) = \mathrm{Mor}_{\mathcal{C}}(G \times G, G)$ the identity $\mathrm{id}_{G \times G} = (p_1, p_2)$ is mapped to $m = p_1 * p_2$. Under $\mathrm{Mor}_{\mathcal{C}}(E, E) = E(E) \longrightarrow G(E) = \mathrm{Mor}_{\mathcal{C}}(E, G)$ the identity of E is mapped to the neutral element $u = e_E$. Under $\mathrm{Mor}_{\mathcal{C}}(G, G) = G(G) \to G(G) = \mathrm{Mor}_{\mathcal{C}}(G, G)$ the identity is mapped to its $*$-inverse $S = \mathrm{id}_G^{-1}$. $\square$

Corollary 2.2.7. *Let $G \in \mathcal{C}$ be a cogroup with multiplication $a * b$, unit e, and inverse a^{-1} in $G(X)$. Then the morphisms $\Delta : G \longrightarrow G \amalg G$, $\varepsilon : G \longrightarrow I$, and $S : G \longrightarrow G$ are given by*

$$\Delta = \iota_1 * \iota_2, \qquad \varepsilon = e_I, \qquad S = \mathrm{id}_G^{-1}.$$

3. Affine Algebraic Groups

We apply the preceding considerations to the categories $\mathbb{K}$-**cAlg** and $\mathbb{K}$-**cCoalg**.

Consider $\mathbb{K}$-**cAlg**, the category of commutative $\mathbb{K}$-algebras. Let $A, B \in \mathbb{K}$-**cAlg**. Then $A \otimes B$ is again a commutative $\mathbb{K}$-algebra with componentwise multiplication. In fact this holds also for non-commutative $\mathbb{K}$-algebras (A.5.3), but in $\mathbb{K}$-**cAlg** we have

Proposition 2.3.1. *The tensor product in* $\mathbb{K}$-**cAlg** *is the (categorical) coproduct.*

Proof. Let $f \in \mathbb{K}\text{-}\mathbf{cAlg}(A, Z), g \in \mathbb{K}\text{-}\mathbf{cAlg}(B, Z)$. The map $[f, g] : A \otimes B \longrightarrow Z$ defined by $[f, g](a \otimes b) := f(a)g(b)$ is the unique algebra homomorphism such that $[f, g](a \otimes 1) = f(a)$ and $[f, g](1 \otimes b) = g(b)$ or such that the diagram

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[dr, "f"'] & A \otimes B \arrow[d, "{[f,g]}"] & B \arrow[l, "\iota_B"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

commutes, where $\iota_A(a) = a \otimes 1$ and $\iota_B(b) = 1 \otimes b$ are algebra homomorphisms. $\square$

So the category $\mathbb{K}$-**cAlg** has finite coproducts and also an initial object $\mathbb{K}$.

A more general property of the tensor product of arbitrary algebras was already considered in 1.2.13.

Observe that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\iota_1"] \arrow[dr, "1_A"'] & A \otimes A \arrow[d, "\nabla"] & A \arrow[l, "\iota_2"'] \arrow[dl, "1_A"] \\ & A & \end{tikzcd}$$

where ∇ is the multiplication of the algebra and by the diagram the codiagonal of the coproduct.

Definition 2.3.2. An *affine algebraic group* is a group in the category of commutative geometric spaces.

By the duality between the categories of commutative geometric spaces and commutative algebras, an affine algebraic group is represented by a cogroup in the category of $\mathbb{K}$-**cAlg** of commutative algebras.

For an arbitrary affine algebraic group H we get by Corollary 2.2.7

$$\Delta = \iota_1 * \iota_2 \in \mathbb{K}\text{-}\mathbf{cAlg}(H, H \otimes H),$$

$$\varepsilon = e \in \mathbb{K}\text{-}\mathbf{cAlg}(H, \mathbb{K}), \qquad \text{and } S = (\mathrm{id})^{-1} \in \mathbb{K}\text{-}\mathbf{cAlg}(H, H).$$

These maps and Corollary 2.2.7 lead to

Proposition 2.3.3. *Let* $H \in \mathbb{K}$-**cAlg**. *H is a representing object for a functor* $\mathbb{K}$-**cAlg** $\longrightarrow$ **Gr** *if and only if H is a Hopf algebra.*

Proof. Both statements are equivalent to the existence of morphisms in $\mathbb{K}$-**cAlg**

$$\Delta : H \to H \otimes H \quad \varepsilon : H \to \mathbb{K} \quad S : H \to H$$

such that the following diagrams commute

(coassociativity)
$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & H \otimes H \arrow[d, "\Delta \otimes 1"] \\ H \otimes H \arrow[r, "1 \otimes \Delta"] & H \otimes H \otimes H \end{tikzcd}$$

(counit)
$$\begin{tikzcd} H \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "1"] & & H \otimes H \arrow[d, "1 \otimes \varepsilon"] \\ H \otimes H \arrow[r, "\varepsilon \otimes 1"] & \mathbb{K} \otimes H \cong H \cong & H \otimes \mathbb{K} \end{tikzcd}$$

(coinverse)
$$\begin{tikzcd} H \arrow[r, "\epsilon"] \arrow[d, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & H \\ H \otimes H \arrow[rr, "S \otimes \mathrm{id}", "\mathrm{id} \otimes S"'] & & H \otimes H \arrow[u, "\nabla"'] \end{tikzcd}$$

□

This Proposition says two things. First of all each commutative Hopf algebra H defines a functor $\mathbb{K}$-$\mathbf{cAlg}(H, \text{-}) : \mathbb{K}$-$\mathbf{cAlg} \to \mathbf{Set}$ that factors through the category of groups or simply a functor $\mathbb{K}$-$\mathbf{cAlg}(H, \text{-}) : \mathbb{K}$-$\mathbf{cAlg} \to \mathbf{Gr}$. Secondly each representable functor $\mathbb{K}$-$\mathbf{cAlg}(H, \text{-}) : \mathbb{K}$-$\mathbf{cAlg} \to \mathbf{Set}$ that factors through the category of groups is represented by a commutative Hopf algebra.

Corollary 2.3.4. *An algebra $H \in \mathbb{K}$-**cAlg** represents an affine algebraic group if and only if H is a commutative Hopf algebra.*

The category of commutative Hopf algebras is dual to the category of affine algebraic groups.

In the following lemmas we consider functors represented by commutative algebras. They define functors on the category $\mathbb{K}$-**cAlg** as well as more generally on $\mathbb{K}$-**Alg**. We first study the functors and the representing algebras. Then we use them to construct commutative Hopf algebras.

Lemma 2.3.5. *The functor $\mathbb{G}_a : \mathbb{K}$-$\mathbf{Alg} \to \mathbf{Ab}$ defined by $\mathbb{G}_a(A) := A^+$, the underlying additive group of the algebra A, is a representable functor represented by the algebra $\mathbb{K}[x]$ the polynomial ring in one variable x.*

Proof. $\mathbb{G}_a$ is an underlying functor that forgets the multiplicative structure of the algebra and only preserves the additive group of the algebra. We have to determine natural isomorphisms (natural in A) $\mathbb{G}_a(A) \cong \mathbb{K}$-$\mathbf{Alg}(\mathbb{K}[x], A)$. Each element $a \in A^+$ is mapped to the homomorphism of algebras $a_* : \mathbb{K}[x] \ni p(x) \mapsto p(a) \in A$. This is a homomorphism of algebras since $a_*(p(x) + q(x)) = p(a) + q(a) = a_*(p(x)) + a_*(q(x))$

and $a_*(p(x)q(x)) = p(a)q(a) = a_*(p(x))a_*(q(x))$. Another reason to see this is that $\mathbb{K}[x]$ is the free (commutative) $\mathbb{K}$-algebra over $\{x\}$ i.e. since each map $\{x\} \to A$ can be uniquely extended to a homomorphism of algebras $\mathbb{K}[x] \to A$. The map $A \ni a \mapsto a_* \in \mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x], A)$ is bijective with the inverse map $\mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x], A) \ni f \mapsto f(x) \in A$. Finally this map is natural in A since

$$\begin{tikzcd} A \arrow[r, "\_{*}"] \arrow[d, "g"'] & \mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x], A) \arrow[d, "\text{K-Alg}(K[x],g)"] \\ B \arrow[r, "\_{*}"'] & \mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x], B) \end{tikzcd}$$

commutes for all $g \in \mathbb{K}\text{-}\mathbf{Alg}(A, B)$. □

Remark 2.3.6. Since A^+ has the structure of an additive group the sets of homomorphisms of algebras $\mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x], A)$ are also additive groups.

Lemma 2.3.7. *The functor $\mathbb{G}_m = U : \mathbb{K}\text{-}\mathbf{Alg} \to \mathbf{Gr}$ defined by $\mathbb{G}_m(A) := U(A)$, the underlying multiplicative group of units of the algebra A, is a representable functor represented by the algebra $\mathbb{K}[x, x^{-1}] = \mathbb{K}[x, y]/(xy-1)$ the ring of Laurent polynomials in one variable x.*

Proof. We have to determine natural isomorphisms (natural in A) $\mathbb{G}_m(A) \cong \mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x, x^{-1}], A)$. Each element $a \in \mathbb{G}_m(A)$ is mapped to the homomorphism of algebras $a_* := (\mathbb{K}[x, x^{-1}] \ni x \mapsto a \in A)$. This defines a unique homomorphism of algebras since each homomorphism of algebras f from $\mathbb{K}[x, x^{-1}] = \mathbb{K}[x, y]/(xy - 1)$ to A is completely determined by the images of x and of y but in addition the images have to satisfy $f(x)f(y) = 1$, i.e. $f(x)$ must be invertible and $f(y)$ must be the inverse to $f(x)$. This mapping is bijective. Furthermore it is natural in A since

$$\begin{tikzcd} A \arrow[r, "\_{*}"] \arrow[d, "g"'] & \mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x, x^{-1}], A) \arrow[d, "\text{K-Alg}(K[x,x^{-1}],g)"] \\ B \arrow[r, "\_{*}"'] & \mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x, x^{-1}], B) \end{tikzcd}$$

for all $g \in \mathbb{K}\text{-}\mathbf{Alg}(A, B)$ commute. □

Remark 2.3.8. Since $U(A)$ has the structure of a (multiplicative) group the sets $\mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x, x^{-1}], A)$ are also groups.

Lemma 2.3.9. *The functor $\mathbb{M}_n : \mathbb{K}\text{-}\mathbf{Alg} \to \mathbb{K}\text{-}\mathbf{Alg}$ with $\mathbb{M}_n(A)$ the algebra of $n \times n$-matrices with entries in A is representable by the algebra $\mathbb{K}\langle x_{11}, x_{12}, \ldots, x_{nn}\rangle$, the non commutative polynomialring in the variables x_{ij}.*

Proof. The polynomial ring $\mathbb{K}\langle x_{ij}\rangle$ is free over the set $\{x_{ij}\}$ in the category of (non commutative) algebras, i.e. for each algebra and for each map $f : \{x_{ij}\} \to A$

there exists a unique homomorphism of algebras $g : \mathbb{K}\langle x_{11}, x_{12}, \ldots, x_{nn}\rangle \to A$ such that the diagram

$$\begin{tikzcd} \{x_{ij}\} \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{K}\langle x_{ij}\rangle \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes. So each matrix in $M_n(A)$ defines a unique a homomorphism of algebras $\mathbb{K}\langle x_{11}, x_{12}, \ldots, x_{nn}\rangle \to A$ and conversely. $\square$

Example 2.3.10. 1. The affine algebraic group called *additive group*

$$\mathbb{G}_a : \mathbb{K}\text{-}\mathbf{cAlg} \to \mathbf{Ab}$$

with $\mathbb{G}_a(A) := A^+$ from Lemma 2.3.5 is represented by the Hopf algebra $\mathbb{K}[x]$. We determine comultiplication, counit, and antipode.

By Corollary 2.2.7 the comultiplication is $\Delta = \iota_1 * \iota_2 \in \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x], \mathbb{K}[x] \otimes \mathbb{K}[x]) \cong \mathbb{G}_a(\mathbb{K}[x] \otimes \mathbb{K}[x])$. Hence

$$\Delta(x) = \iota_1(x) + \iota_2(x) = x \otimes 1 + 1 \otimes x.$$

The counit is $\varepsilon = e_{\mathbb{K}} = 0 \in \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x], \mathbb{K}) \cong \mathbb{G}_a(\mathbb{K})$ hence

$$\varepsilon(x) = 0.$$

The antipode is $S = \mathrm{id}_{\mathbb{K}[x]}^{-1} \in \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x], \mathbb{K}[x]) \cong \mathbb{G}_a(\mathbb{K}[x])$ hence

$$S(x) = -x.$$

2. The affine algebraic group called *multiplicative group*

$$\mathbb{G}_m : \mathbb{K}\text{-}\mathbf{cAlg} \to \mathbf{Ab}$$

with $\mathbb{G}_m(A) := A^* = U(A)$ from Lemma 2.3.7 is represented by the Hopf algebra $\mathbb{K}[x, x^{-1}] = \mathbb{K}[x, y]/(xy - 1)$. We determine comultiplication, counit, and antipode.

By Corollary 2.2.7 the comultiplication is

$$\Delta = \iota_1 * \iota_2 \in \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x, x^{-1}], \mathbb{K}[x, x^{-1}] \otimes \mathbb{K}[x, x^{-1}]) \cong \mathbb{G}_m(\mathbb{K}[x, x^{-1}] \otimes \mathbb{K}[x, x^{-1}]).$$

Hence

$$\Delta(x) = \iota_1(x) \cdot \iota_2(x) = x \otimes x.$$

The counit is $\varepsilon = e_{\mathbb{K}} = 1 \in \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x, x^{-1}], \mathbb{K}) \cong \mathbb{G}_m(\mathbb{K})$ hence

$$\varepsilon(x) = 1.$$

The antipode is $S = \mathrm{id}_{\mathbb{K}[x,x^{-1}]}^{-1} \in \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x, x^{-1}], \mathbb{K}[x, x^{-1}]) \cong \mathbb{G}_a(\mathbb{K}[x, x^{-1}])$ hence

$$S(x) = x^{-1}.$$

3. The affine algebraic group called *additive matrix group*

$$\mathbb{M}_n^+ : \mathbb{K}\text{-}\mathbf{cAlg} \to \mathbf{Ab},$$

with $\mathbb{M}_n^+(A)$ the additive group of $n \times n$-matrices with coefficients in A is represented by the commutative algebra $M_n^+ = \mathbb{K}[x_{ij} | 1 \le i, j \le n]$ (Lemma 2.3.9). This algebra must be a Hopf algebra.

The comultiplication is $\Delta = \iota_1 * \iota_2 \in \mathbb{K}\text{-}\mathbf{cAlg}(M_n^+, M_n^+ \otimes M_n^+) \cong \mathbb{M}_n^+(M_n^+ \otimes M_n^+)$. Hence

$$\Delta(x_{ij}) = \iota_1(x_{ij}) + \iota_2(x_{ij}) = x_{ij} \otimes 1 + 1 \otimes x_{ij}.$$

The counit is $\varepsilon = e_{\mathbb{K}} = (0) \in \mathbb{K}\text{-}\mathbf{cAlg}(M_n^+, \mathbb{K}) \cong \mathbb{M}_n^+(\mathbb{K})$ hence

$$\varepsilon(x_{ij}) = 0.$$

The antipode is $S = \mathrm{id}_{M_n^+}^{-1} \in \mathbb{K}\text{-}\mathbf{cAlg}(M_n^+, M_n^+) \cong \mathbb{M}_n^+(M_n^+)$ hence

$$S(x_{ij}) = -x_{ij}.$$

4. The matrix algebra $\mathbb{M}_n(A)$ also has a noncommutative multiplication, the matrix multiplication, defining a monoid structure $\mathbb{M}_n^\times(A)$. Thus $\mathbb{K}[x_{ij}]$ carries another coalgebra structure which defines a bialgebra $M_n^\times = \mathbb{K}[x_{ij}]$. Obviously there is no antipode.

The comultiplication is $\Delta = \iota_1 * \iota_2 \in \mathbb{K}\text{-}\mathbf{cAlg}(M_n^\times, M_n^\times \otimes M_n^\times) \cong \mathbb{M}_n^\times(M_n^\times \otimes M_n^\times)$. Hence $\Delta((x_{ij})) = \iota_1((x_{ij})) \cdot \iota_2((x_{ij})) = (x_{ij}) \otimes (x_{ij})$ or

$$\Delta(x_{ik}) = \sum_j x_{ij} \otimes x_{jk}.$$

The counit is $\varepsilon = e_{\mathbb{K}} = E \in \mathbb{K}\text{-}\mathbf{cAlg}(M_n^\times, \mathbb{K}) \cong \mathbb{M}_n^\times(\mathbb{K})$ hence

$$\varepsilon(x_{ij}) = \delta_{ij}.$$

5. Let $\mathbb{K}$ be a field of characteristic p. The algebra $H = \mathbb{K}[x]/(x^p)$ carries the structure of a Hopf algebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$, and $S(x) = -x$. To show that Δ is well defined we have to show $\Delta(x)^p = 0$. But this is obvious by the rules for computing p-th powers in characteristic p. We have $(x \otimes 1 + 1 \otimes x)^p = x^p \otimes 1 + 1 \otimes x^p = 0$.

Thus the algebra H represents an affine algebraic group:

$$\alpha_p(A) := \mathbb{K}\text{-}\mathbf{cAlg}(H, A) \cong \{a \in A | a^p = 0\}.$$

The group multiplication is the addition of p-nilpotent elements. So we have the *group of p-nilpotent elements.*

6. The algebra $H = \mathbb{K}[x]/(x^n - 1)$ is a Hopf algebra with the comultiplication $\Delta(x) = x \otimes x$, the counit $\varepsilon(x) = 1$, and the antipode $S(x) = x^{n-1}$. These maps are well defined since we have for example $\Delta(x)^n = (x \otimes x)^n = x^n \otimes x^n = 1 \otimes 1$. Observe that this Hopf algebra is isomorphic to the group algebra $\mathbb{K}C_n$ of the cyclic group C_n of order n.

Thus the algebra H represents an affine algebraic group:

$$\mu_n(A) := \mathbb{K}\text{-}\mathbf{cAlg}(H, A) \cong \{a \in A | a^n = 1\},$$

that is the *group of n-th roots of unity*. The group multiplication is the ordinary multiplication of roots of unity.

7. The linear groups or matrix groups $\mathbb{GL}(n)(A)$, $\mathbb{SL}(n)(A)$ and other such groups are further examples of affine algebraic groups. We will discuss them in the section on quantum groups.

Problem 2.3.7. 1. The construction of the general linear group

$$\mathbb{GL}(n)(A) = \{(a_{ij}) \in \mathbb{M}_n(A) | (a_{ij}) \text{ invertible}\}$$

defines an affine algebraic group. Describe the representing Hopf algebra.

2. The special linear group $\mathbb{SL}(n)(A)$ is an affine algebraic group. What is the representing Hopf algebra?

3. The real unit circle $\mathbb{S}^1(\mathbb{R})$ carry the structure of a group by the addition of angles. Is it possible to make $\mathbb{S}^1$ with the affine algebra $\mathbb{K}[c, s]/(s^2 + c^2 - 1)$ into an affine algebraic group? (Hint: How can you add two points (x_1, y_1) and (x_2, y_2) on the unit circle, such that you get the addition of the associated angles?)

4. Formal Groups

Consider now $\mathbb{K}$-**cCoalg** the category of cocommutative $\mathbb{K}$-coalgebras. Let $C, D \in \mathbb{K}$-**cCoalg**. Then $C \otimes D$ is again a cocommutative $\mathbb{K}$-coalgebra by Problem A.11.4. In fact this holds also for non-commutative $\mathbb{K}$-algebras, but in $\mathbb{K}$-**cCoalg** we have

Proposition 2.4.1. *The tensor product in* $\mathbb{K}$-**cCoalg** *is the (categorical) product.*

Proof. Let $f \in \mathbb{K}$-**cCoalg**$(Z, C), g \in \mathbb{K}$-**cCoalg**(Z, D). The map $(f, g) : Z \to C \otimes D$ defined by $(f, g)(z) := \sum f(z_{(1)}) \otimes g(z_{(2)})$ is the unique homomorphism of coalgebras such that $(1 \otimes \varepsilon_D)(f, g)(z) = f(z)$ and $(\varepsilon_C \otimes 1)(f, g)(z) = g(z)$ or such that the diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, "(f,g)"] \arrow[dr, "g"] & \\ C & C \otimes D \arrow[l, "p_C"'] \arrow[r, "p_D"] & D \end{tikzcd}$$

commutes, where $p_C(c \otimes d) = (1 \otimes \varepsilon)(c \otimes d) = c\varepsilon(d)$ and $p_D(c \otimes d) = (\varepsilon \otimes 1)(c \otimes d) = \varepsilon(c)d$ are homomorphisms of coalgebras. $\square$

So the category $\mathbb{K}$-**cCoalg** has finite products and also a final object $\mathbb{K}$.

Definition 2.4.2. A *formal group* is a group in the category of $\mathbb{K}$-**cCoalg** of cocommutative coalgebras.

A formal group G defines a contravariant representable functor from $\mathbb{K}$-**cCoalg** to **Gr**, the category of groups.

Proposition 2.4.3. *Let* $H \in \mathbb{K}$-**cCoalg**. H *a represents a formal group if and only if there are given morphisms in* $\mathbb{K}$-**cCoalg**

$$\nabla : H \otimes H \to H, \quad u : \mathbb{K} \to H, \quad S : H \to H$$

such that the following diagrams commute

(associativity)
$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "1\otimes\nabla"] \arrow[d, "\nabla\otimes 1"'] & H \otimes H \arrow[d, "\nabla"] \\ H \otimes H \arrow[r, "\nabla"] & H \end{tikzcd}$$

(unit)
$$\begin{tikzcd} \mathbb{K} \otimes H \cong H \cong H \otimes \mathbb{K} \arrow[r, "u\otimes 1"] \arrow[d, "1\otimes u"'] \arrow[dr, "1"] & H \otimes H \arrow[d, "\nabla"] \\ H \otimes H \arrow[r, "\nabla"] & H \end{tikzcd}$$

(inverse)
$$\begin{tikzcd} H \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & H \\ H \otimes H \arrow[rr, "S\otimes\mathrm{id}", "\mathrm{id}\otimes S"'] & & H \otimes H \arrow[u, "\nabla"'] \end{tikzcd}$$

PROOF. For an arbitrary formal group H we get $\nabla = p_1 * p_2 \in \mathbb{K}\text{-}\mathbf{cCoalg}(H \otimes H, H)$, $u = e \in \mathbb{K}\text{-}\mathbf{cCoalg}(\mathbb{K}, H)$, and $S = (\mathrm{id})^{-1} \in \mathbb{K}\text{-}\mathbf{cCoalg}(H, H)$. These maps, the Yoneda Lemma and Remark 2.2.6 lead to the proof of the proposition. □

Remark 2.4.4. In particular the representing object $(H, \nabla, u, \Delta, \varepsilon, S)$ of a formal group G is a cocommutative Hopf algebra and every such Hopf algebra represents a formal group. Hence the category of formal groups is equivalent to the category of cocommutative Hopf algebras.

Corollary 2.4.5. *A coalgebra $H \in \mathbb{K}\text{-}\mathbf{cCoalg}$ represents a formal group if and only if H is a cocommutative Hopf algebra.*
The category of cocommutative Hopf algebras is equivalent to the category of formal groups.

Corollary 2.4.6. *The following categories are equivalent:*
1. *The category of commutative, cocommutative Hopf algebras.*
2. *The category of commutative formal groups.*
3. *The dual of the category of commutative affine algebraic groups.*

Example 2.4.7. 1. Group algebras $\mathbb{K}G$ are formal groups.
2. Universal enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$ are formal groups.
3. The tensor algebra $T(V)$ and the symmetric algebra $S(V)$ are formal groups.
4. Let C be a cocommutative coalgebra and G be a group. Then the group $\mathbb{K}G(C) = \mathbb{K}\text{-}\mathbf{cCoalg}(C, \mathbb{K}G)$ is isomorphic to the set of families $(h_g^*|g \in G)$ of decompositions of the unit of C^* into a sum of orthogonal idempotents $h_g^* \in C^*$ that are locally finite.

   To see this embed $\mathbb{K}\text{-}\mathbf{cCoalg}(C, \mathbb{K}G) \subseteq \mathrm{Hom}(C, \mathbb{K}G)$ and embedthe set $\mathrm{Hom}(C, \mathbb{K}G)$ into the set $(C^*)^G$ of G-families of elements in the algebra C^* by $h \mapsto (h_g^*)$ with $h(c) = \sum_{g \in G} h_g^*(c) g$. The linear map h is a homomorphism of coalgebras iff $(h \otimes h)\Delta = \Delta h$ and $\varepsilon h = \varepsilon$ iff $\sum h(c_{(1)}) \otimes h(c_{(2)}) = \sum h(c)_{(1)} \otimes h(c)_{(2)}$ and $\varepsilon(h(c)) = \varepsilon(c)$ for all $c \in C$ iff $\sum h_g^*(c_{(1)}) g \otimes h_l^*(c_{(2)}) l = \sum h_g^*(c) g \otimes g$ and $\sum h_g^*(c) = \varepsilon(c)$ iff $\sum h_g^*(c_{(1)}) h_l^*(c_{(2)}) = \delta_{gl} h_g^*(c)$ and $\sum h_g^* = \varepsilon$ iff $h_g^* * h_l^* = \delta_{gl} h_g^*$ and $\sum h_g^* = 1_{C^*}$. Furthermore the families must be locally finite, i.e. for each $c \in C$ only finitely many of them give non-zero values.
5. Let C be a cocommutative coalgebra and $\mathbb{K}[x]$ be the Hopf algebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$ (the symmetric algebra of the one dimensional vector space $\mathbb{K}x$). We embed as before $\mathbb{K}\text{-}\mathbf{cCoalg}(C, \mathbb{K}[x]) \subseteq \mathrm{Hom}(C, \mathbb{K}[x]) = (C^*)^{\{\mathbf{N}_0\}}$, the set of locally finite $\mathbf{N}_0$-families in C^* by $h(c) = \sum_{i=0}^\infty h_i^*(c) x^i$. The map h is a homomorphism of coalgebras iff $\Delta(h(c)) = \sum h_i^*(c)(x \otimes 1 + 1 \otimes x)^i = \sum h_i^*(c)\binom{i}{l} x^l \otimes x^{i-l} = (h \otimes h)\Delta(c) = \sum h_i^*(c_{(1)}) h_j^*(c_{(2)}) x^i \otimes x^j$ and $\varepsilon(\sum h_i^*(c) x^i) = \varepsilon(c)$ iff $h_i^* * h_j^* = \binom{i+j}{i} h_{i+j}^*$ and $h_0^* = \varepsilon = 1_{C^*}$.

   Now let $\mathbb{K}$ be a field of characteristic zero. Let $p_i := h_i^*/i!$. Then the conditions simplify to $p_i p_j = p_{i+j}$ and $p_0 = 1$. Hence the series for h is completely determined by the term $p := p_1$ since $p_n = p_1^n$. Since the series must be locally finite we get that for each $c \in C$ there must be an $n \in \mathbf{N}_0$ such that $p^m(c) = 0$ for all $m \geq n$. Hence the element p is *topologically nilpotent* and

$$\mathbb{K}\text{-}\mathbf{cCoalg}(C, \mathbb{K}[x]) \cong \mathrm{rad}_t(C^*)$$

the radial of topologically nilpotent elements of C^*.

It is easy to see that $\mathrm{rad}_t(C^*)$ is a group under addition and that this group structure coincides with the one on $\mathbb{K}\text{-}\mathbf{cCoalg}(C, \mathbb{K}[x])$.

Remark 2.4.8. Let H be a finite dimensional Hopf algebra. Then by A.6.6 and A.6.8 we get that H^* is an algebra and a coalgebra. The commutative diagrams defining the bialgebra property and the antipode can be transferred easily, so H^* is again a Hopf algebra. Hence the functor $\text{-}^* : \mathbf{vec} \to \mathbf{vec}$ from finite dimensional vector spaces to itself induces a duality $\text{-}^* : \mathbb{K}\text{-}\mathbf{hopfalg} \to \mathbb{K}\text{-}\mathbf{hopfalg}$ from the category of finite dimensional Hopf algebras to itself.

An affine algebraic group is called *finite* if the representing Hopf algebra is finite dimensional. A formal group is called *finite* if the representing Hopf algebra is finite dimensional.

Thus the category of finite affine algebraic groups is equivalent to the category of finite formal groups.

The category of finite commutative affine algebraic groups is self dual. The category of finite commutative affine algebraic groups is equivalent (and dual) to the category of finite commutative formal groups.

5. Quantum Groups

Definition 2.5.1. (Drinfel'd) A *quantum group* is a noncommutative noncocommutative Hopf algebra.

Remark 2.5.2. We shall consider all Hopf algebras as quantum groups. Observe, however, that the commutative Hopf algebras may be considered as affine algebraic groups and that the cocommutative Hopf algebras may be considered as formal groups. Their property as a quantum space or as a quantum monoid will play some role. But often the (possibly nonexisting) dual Hopf algebra will have the geometrical meaning. The following examples $\mathbb{SL}_q(2)$ and $\mathbb{GL}_q(2)$ will have a geometrical meaning.

Example 2.5.3. The smallest proper quantum group, i.e. the smallest noncommutative noncocommutative Hopf algebra, is the 4-dimensional algebra

$$H_4 := \mathbb{K}\langle g, x\rangle/(g^2 - 1, x^2, xg + gx)$$

which was first described by M. Sweedler. The coalgebra structure is given by

$$\begin{array}{cc} \Delta(g) = g \otimes g, & \Delta(x) = x \otimes 1 + g \otimes x, \\ \varepsilon(g) = 1, & \varepsilon(x) = 0, \\ S(g) = g^{-1}(= g), & S(x) = -gx. \end{array}$$

Since it is finite dimensional its linear dual H_4^* is also a noncommutative noncocommutative Hopf algebra. It is isomorphic as a Hopf algebra to H_4. In fact H_4 is up to isomorphism the only noncommutative noncocommutative Hopf algebra of dimension 4.

Example 2.5.4. The affine algebraic group $\mathbb{SL}(n) : \mathbb{K}\text{-}\mathbf{cAlg} \to \mathbf{Gr}$ defined by $\mathbb{SL}(n)(A)$, the group of $n \times n$-matrices with coefficients in the commutative algebra A and with determinant 1, is represented by the algebra $\mathcal{O}(\mathbb{SL}(n)) = SL(n) = \mathbb{K}[x_{ij}]/(\det(x_{ij}) - 1)$ i.e.

$$\mathbb{SL}(n)(A) \cong \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x_{ij}]/(\det(x_{ij}) - 1), A).$$

Since $\mathbb{SL}(n)(A)$ has a group structure by the multiplication of matrices, the representing commutative algebra has a Hopf algebra structure with the diagonal $\Delta = \iota_1 * \iota_2$ hence

$$\Delta(x_{ik}) = \sum x_{ij} \otimes x_{jk},$$

the counit $\varepsilon(x_{ij}) = \delta_{ij}$ and the antipode $S(x_{ij}) = adj(X)_{ij}$ where $adj(X)$ is the adjoint matrix of $X = (x_{ij})$. We leave the verification of these facts to the reader.

We consider $\mathbb{SL}(n) \subseteq \mathcal{M}_n = \mathbb{A}^{n^2}$ as a subspace of the n^2-dimensional affine space.

Example 2.5.5. Let $M_q(2) = \mathbb{K}\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle / I$ as in 1.3.6 with I the ideal generated by

$$ab - q^{-1}ba, ac - q^{-1}ca, bd - q^{-1}db, cd - q^{-1}dc, (ad - q^{-1}bc) - (da - qcb), bc - cb.$$

We first define the *quantum determinant* $\det_q = ad - q^{-1}bc = da - qcb$ in $M_q(2)$. It is a central element. To show this, it suffices to show that $\det_q$ commutes with the generators a, b, c, d:

$$\begin{array}{cc} (ad - q^{-1}bc)a = a(da - qbc), & (ad - q^{-1}bc)b = b(ad - q^{-1}bc), \\ (ad - q^{-1}bc)c = c(ad - q^{-1}bc), & (da - qbc)d = d(ad - q^{-1}bc). \end{array}$$

We can form the quantum determinant of an arbitrary quantum matrix in A by

$$\det_q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} := a'd' - q^{-1}b'c' = d'a' - qc'b' = \varphi(\det_q)$$

if $\varphi : M_q(2) \longrightarrow A$ is the algebra homomorphism associated with the quantum matrix $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$.

Given two commuting quantum 2×2-matrices $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$. The quantum determinant preserves the product, since

$$\begin{aligned} \det_q(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}) &= \det_q \begin{pmatrix} a'a'' + b'c'' & a'b'' + b'd'' \\ c'a'' + d'c'' & c'b'' + d'd'' \end{pmatrix} \\ &= (a'a'' + b'c'')(c'b'' + d'd'') - q^{-1}(a'b'' + b'd'')(c'a'' + d'c'') \\ &= a'c'a''b'' + b'c'c''b'' + a'd'a''d'' + b'd'c''d'' \\ &\quad -q^{-1}(a'c'b''a'' + b'c'd''a'' + a'd'b''c'' + b'd'd''c'') \\ &= b'c'c''b'' + a'd'a''d'' - q^{-1}b'c'd''a'' - q^{-1}a'd'b''c'' \\ &= b'c'c''b'' + a'd'a''d'' - q^{-1}b'c'd''a'' - q^{-1}a'd'b''c'' \\ &\quad -q^{-1}b'c'(a''d'' - d''a'' - q^{-1}b''c'' + qc''b'') \\ &= a'd'a''d'' - q^{-1}a'd'b''c'' - q^{-1}b'c'(a''d'' - q^{-1}b''c'') \\ &= (a'd' - q^{-1}b'c')(a''d'' - q^{-1}b''c'') \\ &= \det_q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \det_q \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}. \end{aligned} \tag{2}$$

In particular we have $\Delta(\det_q) = \det_q \otimes \det_q$ and $\varepsilon(\det_q) = 1$. The quantum determinant is a group like element (see 2.1.6).

Now we define an algebra

$$SL_q(2) := M_q(2)/(\det_q - 1).$$

The algebra $SL_q(2)$ represents the functor

$$\mathbb{SL}_q(2)(A) = \{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}_q(2)(A) | \det_q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = 1\}.$$

There is a surjective homomorphism of algebras $M_q(2) \longrightarrow SL_q(2)$ and $\mathbb{SL}_q(2)$ is a subfunctor of $\mathcal{M}_q(2)$.

Let X, Y be commuting quantum matrices satisfying $\det_q(X) = 1 = \det_q(Y)$. Since $\det_q(X)\det_q(Y) = \det_q(XY)$ for commuting quantum matrices we get

$\det_q(XY) = 1$, hence $\mathbb{SL}_q(2)$ is a quantum submonoid of $\mathcal{M}_q(2)$ and $SL_q(2)$ is a bialgebra with diagonal

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and

$$\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

To show that $SL_q(2)$ has an antipode we first define a homomorphism of algebras $T : M_q(2) \longrightarrow M_q(2)^{op}$ by

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

We check that $T : \mathbb{K}\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle \longrightarrow M_q(2)^{op}$ vanishes on the ideal I.

$$T(ab - q^{-1}ba) = T(b)T(a) - q^{-1}T(a)T(b) = -qbd + q^{-1}qdb = 0.$$

We leave the check of the other defining relations to the reader. Furthermore T restricts to a homomorphism of algebras $S : SL_q(2) \longrightarrow SL_q(2)^{op}$ since $T(\det_q) = T(ad - q^{-1}bc) = T(d)T(a) - q^{-1}T(c)T(b) = ad - q^{-1}(-q^{-1}c)(-qb) = \det_q$ hence $T(\det_q - 1) = \det_q - 1 = 0$ in $SL_q(2)$.

One verifies easily that S satisfies $\sum S(x_{(1)})x_{(2)} = \varepsilon(x)$ for all given generators of $SL_q(2)$, hence S is a left antipode by 2.1.3. Symmetrically S is a right antipode. Thus the bialgebra $SL_q(2)$ is a Hopf algebra or a quantum group.

Example 2.5.6. The affine algebraic group $\mathbb{GL}(n) : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbf{Gr}$ defined by $\mathbb{GL}(n)(A)$, the group of invertible $n \times n$-matrices with coefficients in the commutative algebra A, is represented by the algebra $\mathcal{O}(\mathbb{GL}(n)) = GL(n) = \mathbb{K}[x_{ij}, t]/(\det(x_{ij})t - 1)$ i.e.

$$\mathbb{GL}(n)(A) \cong \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x_{ij}, t]/(\det(x_{ij})t - 1), A)).$$

Since $\mathbb{GL}(n)(A)$ has a group structure by the multiplication of matrices, the representing commutative algebra has a Hopf algebra structure with the diagonal $\Delta = \iota_1 * \iota_2$ hence

$$\Delta(x_{ik}) = \sum x_{ij} \otimes x_{jk},$$

the counit $\varepsilon(x_{ij}) = \delta_{ij}$ and the antipode $S(x_{ij}) = t \cdot adj(X)_{ij}$ where $adj(X)$ is the adjoint matrix of $X = (x_{ij})$. We leave the verification of these facts from linear algebra to the reader. The diagonal applied to t gives

$$\Delta(t) = t \otimes t.$$

Hence $t(= \det(X)^{-1})$ is a grouplike element in $GL(n)$. This reflects the rule $\det(AB) = \det(A)\det(B)$ hence $\det(AB)^{-1} = \det(A)^{-1}\det(B)^{-1}$.

Example 2.5.7. Let $M_q(2)$ be as in the example 2.5.5. We define

$$GL_q(2) := M_q(2)[t]/J$$

with J generated by the elements $t \cdot (ad - q^{-1}bc) - 1$. The algebra $GL_q(2)$ represents the functor

$$\mathbb{GL}_q(2)(A) = \{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}_q(2)(A) | \det_q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \text{ invertible in } A\}.$$

In fact there is a canonical homomorphism of algebras $M_q(2) \longrightarrow GL_q(2)$. A homomorphism of algebras $\varphi : M_q(2) \longrightarrow A$ has a unique continuation to $GL_q(2)$ iff $\det_q(\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix})$ is invertible:

$$\begin{tikzcd} M_q(2) \arrow[r] \arrow[dr] & M_q(2)[t] \arrow[r] \arrow[d] & G_q(2) \arrow[dl] \\ & A & \end{tikzcd}$$

with $t \mapsto \det_q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^{-1}$. Thus $\mathbb{GL}_q(2)(A)$ is a subset of $\mathcal{M}_q(2)(A)$. Observe that $M_q(2) \longrightarrow GL_q(2)$ is *not* surjective.

Since the quantum determinant preserves products and the product of invertible elements is again invertible we get $\mathbb{GL}_q(2)$ is a quantum submonoid of $\mathcal{M}_q(2)$, hence $\Delta : GL_q(2) \longrightarrow GL_q(2) \otimes GL_q(2)$ with $\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\Delta(t) = t \otimes t$.

We construct the antipode for $GL_q(2)$. We define $T : M_q(2)[t] \longrightarrow M_q(2)[t]^{op}$ by

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} := t \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} \qquad \text{and} \qquad T(t) := \det_q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - q^{-1}bc.$$

As in 2.5.5 T defines a homomorphism of algebras. We obtain an induced homomorphism of algebras $S : GL_q(2) \longrightarrow GL_q(2)^{op}$ or a $GL_q(2)^{op}$-point in $GL_q(2)$ since $S(t(ad - q^{-1}bc) - 1) = (S(d)S(a) - q^{-1}S(c)S(b))S(t) - S(1) = (t^2ad - q^{-1}t^2cb)(ad - q^{-1}bc) - 1 = t^2(ad - q^{-1}bc)^2 - 1 = 0$.

Since S satisfies $\sum S(x_{(1)})x_{(2)} = \varepsilon(x)$ for all given generators, S is a left antipode by 2.1.3. Symmetrically S is a right antipode. Thus the bialgebra $GL_q(2)$ is a Hopf algebra or a quantum group.

Example 2.5.8. Let $sl(2)$ be the 3-dimensional vector space generated by the matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $sl(2)$ is a subspace of the algebra $M(2)$ of 2×2-matrices over $\mathbb{K}$. We easily verify $[X, Y] = XY - YX = H$, $[H, X] = HX - XH = 2X$, and $[H, Y] = HY - YH = -2Y$, so that $sl(2)$ becomes a Lie subalgebra of $M(2)^L$, which is the Lie algebra of matrices of trace zero. The universal enveloping algebra $U(sl(2))$ is a Hopf algebra generated as an algebra by the elements X, Y, H with the relations

$$[X,Y] = H, \quad [H,X] = 2X, \quad [H,Y] = -2Y.$$

As a consequence of the Poincaré-Birkhoff-Witt Theorem (that we don't prove) the Hopf algebra $U(sl(2))$ has the basis $\{X^iY^jH^k | i,j,k \in \mathbb{N}\}$. Furthermore one can prove that all finite dimensional $U(sl(2))$-modules are semisimple.

Example 2.5.9. We define the so-called q-deformed version $U_q(sl(2))$ of $U(sl(2))$ for any $q \in \mathbb{K}$, $q \neq 1, -1$ and q invertible. Let $U_q(sl(2))$ be the algebra generated by the elements E, F, K, K' with the relations

$$\begin{gathered} KK' = K'K = 1, \\ KEK' = q^2E, \qquad KFK' = q^{-2}F, \\ EF - FE = \frac{K - K'}{q - q^{-1}}. \end{gathered}$$

Since K' is the inverse of K in $U_q(sl(2))$ we write $K^{-1} = K'$. The representation theory of this algebra is fundamentally different depending on whether q is a root of unity or not.

We show that $U_q(sl(2))$ is a Hopf algebra or quantum group. We define

$$\begin{gathered} \Delta(E) = 1 \otimes E + E \otimes K, \qquad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \\ \Delta(K) = K \otimes K, \\ \varepsilon(E) = \varepsilon(F) = 0, \qquad \varepsilon(K) = 1, \\ S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}. \end{gathered}$$

.

First we show that Δ can be expanded in a unique way to an algebra homomorphism $\Delta : U_q(sl(2)) \longrightarrow U_q(sl(2)) \otimes U_q(sl(2))$. Write $U_q(sl(2))$ as the residue class algebra $\mathbb{K}\langle E, F, K, K^{-1}\rangle / I$ where I is generated by

$$\begin{gathered} KK^{-1} - 1, \qquad K^{-1}K - 1, \\ KEK^{-1} - q^2E, \qquad KFK^{-1} - q^{-2}F, \\ EF - FE - \frac{K - K^{-1}}{q - q^{-1}}. \end{gathered}$$

Since K^{-1} must be mapped to the inverse of $\Delta(K)$ we must have $\Delta(K^{-1}) = K^{-1} \otimes K^{-1}$. Now Δ can be expanded in a unique way to the free algebra $\Delta : K\langle E, F, K, K^{-1}\rangle \to U_q(sl(2)) \otimes U_q(sl(2))$. We have $\Delta(KK^{-1}) = \Delta(K)\Delta(K^{-1}) = 1$ and similarly $\Delta(K^{-1}K) = 1$. Furthermore we have $\Delta(KEK^{-1}) = \Delta(K)\Delta(E)\Delta(K^{-1}) = (K \otimes K)$ $(1 \otimes E + E \otimes K)(K^{-1} \otimes K^{-1}) = KK^{-1} \otimes KEK^{-1} + KEK^{-1} \otimes K^2K^{-1} = q^2(1 \otimes E +$

$E \otimes K) = q^2\Delta(E) = \Delta(q^2E)$ and similarly $\Delta(KFK^{-1}) = \Delta(q^{-2}F)$. Finally we have

$$\begin{aligned}
\Delta(EF - FE) &= (1 \otimes E + E \otimes K)(K' \otimes F + F \otimes 1) \\
&\quad -(K' \otimes F + F \otimes 1)(1 \otimes E + E \otimes K) \\
&= K' \otimes EF + F \otimes E + EK' \otimes KF + EF \otimes K \\
&\quad -K' \otimes FE - K'E \otimes FK - F \otimes E - FE \otimes K \\
&= K' \otimes (EF - FE) + (EF - FE) \otimes K \\
&= \frac{K' \otimes (K - K') + (K - K') \otimes K}{q - q^{-1}} \\
&= \Delta\left(\frac{K - K'}{q - q^{-1}}\right)
\end{aligned}$$

hence Δ vanishes on I and can be factorized through a unique algebra homomorphism

$$\Delta : U_q(sl(2)) \longrightarrow U_q(sl(2)) \otimes U_q(sl(2)).$$

In a similar way, actually much simpler, one gets an algebra homomorphism

$$\varepsilon : U_q(sl(2)) \longrightarrow \mathbb{K}.$$

To check that Δ is coassociative it suffices to check this for the generators of the algebra. We have $(\Delta \otimes 1)\Delta(E) = (\Delta \otimes 1)(1 \otimes E + E \otimes K) = 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K = (1 \otimes \Delta)(1 \otimes E + E \otimes K) = (1 \otimes \Delta)\Delta(E)$. Similarly we get $(\Delta \otimes 1)\Delta(F) = (1 \otimes \Delta)\Delta(F)$. For K the claim is obvious. The counit axiom is easily checked on the generators.

Now we show that S is an antipode for $U_q(sl(2))$. First define $S : \mathbb{K}\langle E, F, K, K^{-1}\rangle \to U_q(sl(2))^{op}$ by the definition of S on the generators. We have

$$\begin{gathered}
S(KK^{-1}) = 1 = S(K^{-1}K), \\
S(KEK^{-1}) = -KEK^{-1}K^{-1} = -q^2EK^{-1} = S(q^2E), \\
S(KFK^{-1}) = -KKFK^{-1} = -q^{-2}KF = S(q^{-2}F), \\
S(EF - FE) = KFEK^{-1} - EK^{-1}KF = KFK^{-1}KEK - EF \\
= \frac{K^{-1} - K}{q - q^{-1}} = S\left(\frac{K - K^{-1}}{q - q^{-1}}\right).
\end{gathered}$$

So S defines a homomorphism of algebras $S : U_q(sl(2)) \longrightarrow U_q(sl(2))$. Since S satisfies $\sum S(x_{(1)})x_{(2)} = \varepsilon(x)$ for all given generators, S is a left antipode by 2.1.3. Symmetrically S is a right antipode. Thus the bialgebra $U_q(sl(2))$ is a Hopf algebra or a quantum group.

This quantum group is of central interest in theoretical physics. Its representation theory is well understood. If q is not a root of unity then the finite dimensional $U_q(sl(2))$-modules are semisimple. Many more properties can be found in [Kassel: Quantum Groups].

6. Quantum Automorphism Groups

Lemma 2.6.1. *The category* $\mathbb{K}$-**Alg** *of* $\mathbb{K}$-*algebras has arbitrary coproducts.*

Proof. This is a well known fact from universal algebra. In fact all equationally defined algebraic categories are complete and cocomplete. We indicate the construction of the coproduct of a family $(A_i|i \in I)$ of $\mathbb{K}$-algebras.

Define $\coprod_{i\in I} A_i := T(\bigoplus_{i\in I} A_i)/L$ where T denotes the tensor algebra and where L is the two sided ideal in $T(\bigoplus_{i\in I} A_i)$ generated by the set

$$J := \{\iota j_k(x_k y_k) - \iota(j_k(x_k))\iota(j_k(y_k)), 1_{T(\bigoplus A_i)} - \iota j_k(1_{A_k})|x_k, y_k \in A_k, k \in I\}.$$

Then one checks easily for a family of algebra homomorphisms $(f_k : A_k \to B|k \in I)$ that the following diagram gives the required universal property

$$\begin{tikzcd}
A_k \arrow[r, "j_k"] \arrow[drrr, "f_k"'] & \bigoplus A_i \arrow[r, "\iota"] \arrow[drr, "f"'] & T(\bigoplus A_i) \arrow[r, "\nu"] \arrow[dr, "f'"'] & T(\bigoplus A_i)/L \arrow[d, "\bar{f}"] \\
 & & & B
\end{tikzcd}$$

$\square$

Corollary 2.6.2. *The category of bialgebras has finite coproducts.*

Proof. The coproduct $\coprod B_i$ of bialgebras $(B_i|i \in I)$ in $\mathbb{K}$-**Alg** is an algebra. For the diagonal and the counit we obtain the following commutative diagrams

$$\begin{tikzcd}
B_k \arrow[r, "j_k"] \arrow[d, "\Delta_k"'] & \coprod B_i \arrow[d, "\exists_1 \Delta"] \\
B_k \otimes B_k \arrow[r, "j_k \otimes j_k"'] & \coprod B_i \otimes \coprod B_i
\end{tikzcd}$$

$$\begin{tikzcd}
B_k \arrow[r, "j_k"] \arrow[dr, "\varepsilon_k"'] & \coprod B_i \arrow[d, "\exists_1 \varepsilon"] \\
 & \mathbb{K}
\end{tikzcd}$$

since in both cases $\coprod B_i$ is a coproduct in $\mathbb{K}$-**Alg**. Then it is easy to show that these homomorphisms define a bialgebra structure on $\coprod B_i$ and that $\coprod B_i$ satisfies the universal property for bialgebras. $\square$

Theorem 2.6.3. *Let B be a bialgebra. Then there exists a Hopf algebra $H(B)$ and a homomorphism of bialgebras $\iota : B \to H(B)$ such that for every Hopf algebra H and for every homomorphism of bialgebras $f : B \to H$ there is a unique* *homomorphism of Hopf algebras $g : H(B) \to H$ such that the diagram*

$$\begin{tikzcd} B \arrow[r, "\iota"] \arrow[dr, "f"'] & H(B) \arrow[d, "g"] \\ & H \end{tikzcd}$$

commutes.

Proof. Define a sequence of bialgebras $(B_i|i \in \mathbb{N})$ by

$$\begin{aligned} &B_0 := B, \\ &B_{i+1} := B_i^{opcop}, i \in \mathbb{N}. \end{aligned}$$

Let B' be the coproduct of the family $(B_i|i \in \mathbb{N})$ with injections $\iota_i : B_i \to B'$. Because B' is a coproduct of bialgebras there is a unique homomorphism of bialgebras $S' : B' \to B'^{opcop}$ such that the diagrams

$$\begin{tikzcd} B_i \arrow[r, "\iota_i"] \arrow[d, "\mathrm{id}"'] & B' \arrow[d, "S'"] \\ B_{i+1}^{opcop} \arrow[r, "\iota_{i+1}"'] & B'^{opcop} \end{tikzcd}$$

commute.

Now let I be the two sided ideal in B' generated by

$$\{(S' * 1 - u\varepsilon)(x_i), (1 * S' - u\varepsilon)(x_i)|x_i \in \iota_i(B_i), i \in \mathbb{N}\}.$$

I is a coideal, i.e. $\varepsilon_{B'}(I) = 0$ and $\Delta_{B'}(I) \subseteq I \otimes B' + B' \otimes I$.

Since $\varepsilon_{B'}$ and $\Delta_{B'}$ are homomorphisms of algebras it suffices to check this for the generating elements of I. Let $x \in B_i$ be given. Then we have $\varepsilon((1 * S')\iota_i(x)) = \varepsilon(\nabla(1 \otimes S')\Delta\iota_i(x)) = \nabla_{\mathbb{K}}(\varepsilon \otimes \varepsilon S')(\iota_i \otimes \iota_i)\Delta_i(x) = (\varepsilon\iota_i \otimes \varepsilon\iota_i)\Delta_i(x) = \varepsilon_i(x) = \varepsilon(u\varepsilon\iota_i(x))$. Symmetrically we have $\varepsilon((S' * 1)\iota_i(x)) = \varepsilon(u\varepsilon\iota_i(x))$. Furthermore we have

$$\begin{aligned} &\Delta((1 * S')\iota_i(x)) \\ &= \Delta\nabla(1 \otimes S')\Delta\iota_i(x) \\ &= (\nabla \otimes \nabla)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta)(1 \otimes S')(\iota_i \otimes \iota_i)\Delta_i(x) \\ &= (\nabla \otimes \nabla)(1 \otimes \tau \otimes 1)(\Delta \otimes \tau(S' \otimes S')\Delta)(\iota_i \otimes \iota_i)\Delta_i(x) \\ &= \sum(\nabla \otimes \nabla)(1 \otimes \tau \otimes 1)(\iota_i(x_{(1)}) \otimes \iota_i(x_{(2)}) \otimes S'\iota_i(x_{(4)}) \otimes S'\iota_i(x_{(3)})) \\ &= \sum \iota_i(x_{(1)})S'\iota_i(x_{(4)}) \otimes \iota_i(x_{(2)})S'\iota_i(x_{(3)}) \\ &= \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes (1 * S')\iota_i(x_{(2)}). \end{aligned}$$

Hence we have

$$\begin{aligned}
\Delta((1 * S'-u\varepsilon)\iota_i(x))& \\
&= \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes (1 * S')\iota_i(x_{(2)}) - \Delta u\varepsilon\iota_i(x)\\
&= \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes ((1 * S') - u\varepsilon)\iota_i(x_{(2)})\\
&\qquad + \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes u\varepsilon\iota_i(x_{(2)}) - \Delta u\varepsilon\iota_i(x)\\
&= \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes (1 * S' - u\varepsilon)\iota_i(x_{(2)})\\
&\qquad + \sum \iota_i(x_{(1)})S'\iota_i(x_{(2)}) \otimes 1_{B'} - u\varepsilon\iota_i(x) \otimes 1_{B'}\\
&= \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes (1 * S' - u\varepsilon)\iota_i(x_{(2)})\\
&\qquad +(1 * S' - u\varepsilon)\iota_i(x) \otimes 1_{B'}\\
&\in B' \otimes I + I \otimes B'.
\end{aligned}$$

Thus I is a coideal and a biideal of B'.

Now let $H(B) := B'/I$ and let $\nu : B' \to H(B)$ be the residue class homomorphism. We show that $H(B)$ is a bialgebra and ν is a homomorphism of bialgebras. $H(B)$ is an algebra and ν is a homomorphism of algebras since I is a two sided ideal. Since $I \subseteq \mathrm{Ker}(\varepsilon)$ there is a unique factorization

$$\begin{tikzcd}
B' \arrow[r, "\nu"] \arrow[dr, "\varepsilon'"'] & B'/I \arrow[d, "\varepsilon"] \\
& \mathbb{K}
\end{tikzcd}$$

where $\varepsilon : B'/I \to \mathbb{K}$ is a homomorphism of algebras. Since $\Delta(I) \subseteq B' \otimes I + I \otimes B' \subseteq \mathrm{Ker}(\nu \otimes \nu : B' \otimes B' \to B'/I \otimes B'/I)$ and thus $I \subseteq \mathrm{Ker}(\Delta(\nu \otimes \nu))$ we have a unique factorization

$$\begin{tikzcd}
B' \arrow[r, "\nu"] \arrow[d, "\Delta_{B'}"'] & B'/I \arrow[d, "\Delta"] \\
B' \otimes B' \arrow[r, "\nu\otimes\nu"'] & B'/I \otimes B'/I
\end{tikzcd}$$

by an algebra homomorphism $\Delta : B'/I \to B'/I \otimes B'/I$. Now it is easy to verify that B'/I becomes a bialgebra and ν a bialgebra homomorphism.

We show that the map $\nu S' : B' \to B'/I$ can be factorized through B'/I in the commutative diagram

$$\begin{tikzcd}
B' \arrow[r, "\nu"] \arrow[dr, "\nu S'"'] & B'/I \arrow[d, "S"] \\
& B'/I
\end{tikzcd}$$

This holds if $I \subseteq \mathrm{Ker}(\nu S')$. Since $\mathrm{Ker}(\nu) = I$ it suffices to show $S'(I) \subseteq I$. We have

$$\begin{aligned} & S'((S' * 1)\iota_i(x)) = \\ & = \nabla\tau(S'^2\iota_i \otimes S'\iota_i)\Delta_i(x) \\ & = \nabla\tau(S' \otimes 1)(\iota_{i+1} \otimes \iota_{i+1})\Delta_i(x) \\ & = \nabla(1 \otimes S')(\iota_{i+1} \otimes \iota_{i+1})\tau\Delta_i(x) \\ & = \nabla(1 \otimes S')(\iota_{i+1} \otimes \iota_{i+1})\Delta_{i+1}(x) \\ & = (1 * S')\iota_{i+1}(x) \end{aligned}$$

and

$$S'(u\varepsilon\iota_i(x)) = S'(1)\varepsilon_i(x) = S'(1)\varepsilon_{i+1}(x) = S'(u\varepsilon\iota_{i+1}(x))$$

hence we get

$$S'((S' * 1 - u\varepsilon)\iota_i(x)) = (1 * S' - u\varepsilon)\iota_{i+1}(x) \in I.$$

This shows $S'(I) \subseteq I$. So there is a unique homomorphism of bialgebras $S : H(B) \to H(B)^{opcop}$ such that the diagram

$$\begin{tikzcd} B' \arrow[r, "\nu"] \arrow[d, "S'"'] & H(B) \arrow[d, "S"] \\ B'^{opcop} \arrow[r, "\nu"'] & H(B)^{opcop} \end{tikzcd}$$

commutes.

Now we show that $H(B)$ is a Hopf algebra with antipode S. By Proposition 2.1.3 it suffices to test on generators of $H(B)$ hence on images $\nu\iota_i(x)$ of elements $x \in B_i$. We have

$$\begin{aligned} (1 * S)\nu\iota_i(x) = \nabla(\nu \otimes S\nu)\Delta\iota_i(x) = \nabla(\nu \otimes \nu)(1 \otimes S')\Delta\iota_i(x) = \\ = \nu(1 * S')\iota_i(x) = \nu u\varepsilon\iota_i(x) = u\varepsilon\nu\iota_i(x). \end{aligned}$$

By Proposition 2.1.3 S is an antipode for $H(B)$.

We prove now that $H(B)$ together with $\iota := \nu\iota_0 : B \to H(B)$ is a free Hopf algebra over B. Let H be a Hopf algebra and let $f : B \to H$ be a homomorphism of bialgebras. We will show that there is a unique homomorphism $\bar{f} : H(B) \to H$ such that

$$\begin{tikzcd} B \arrow[r, "\iota"] \arrow[dr, "f"'] & H(B) \arrow[d, "\bar{f}"] \\ & H \end{tikzcd}$$

commutes.

We define a family of homomorphisms of bialgebras $f_i : B_i \to H$ by

$$\begin{aligned} & f_0 := f, \\ & f_{i+1} := S_H f_i, i \in \mathbb{N}. \end{aligned}$$

We have in particular $f_i = S_H^i f$ for all $i \in \mathbb{N}$. Thus there is a unique homomorphism of bialgebras $f' : B' = \coprod B_i \to H$ such that $f'\iota_i = f_i$ for all $i \in \mathbb{N}$.

We show that $f'(I) = 0$. Let $x \in B_i$. Then

$$\begin{aligned} f'((1 * S')\iota_i(x)) &= f'(\nabla(1 \otimes S')(\iota_i \otimes \iota_i)\Delta_i(x)) \\ &= \sum f'\iota_i(x_{(1)}) f'S'\iota_i(x_{(2)}) \\ &= \sum f'\iota_i(x_{(1)}) f'\iota_{i+1}(x_{(2)}) \\ &= \sum f_i(x_{(1)}) f_{i+1}(x_{(2)}) \\ &= \sum f_i(x_{(1)}) S f_i(x_{(2)}) \\ &= (1 * S) f_i(x) = u\varepsilon f_i(x) = u\varepsilon_i(x) \\ &= f'(u\varepsilon\iota_i(x)). \end{aligned}$$

This together with the symmetric statement gives $f'(I) = 0$. Hence there is a unique factorization through a homomorphism of algebras $\bar{f} : H(B) \to H$ such that $f' = \bar{f}\nu$.

The homomorphism $\bar{f} : H(B) \to H$ is a homomorphism of bialgebras since the diagram

$$\begin{tikzcd} B' \arrow[rr, bend left, "f'"] \arrow[r, "\nu"'] \arrow[d, "\Delta"'] & B'/I \arrow[r, "\bar{f}"'] \arrow[d, "\Delta'"] & H \arrow[d, "\Delta_H"] \\ B' \otimes B' \arrow[r, "\nu\otimes\nu"] \arrow[rr, bend right, "f'\otimes f'"'] & B'/I \otimes B'/I \arrow[r, "\bar{f}\otimes\bar{f}"] & H \otimes H \end{tikzcd}$$

commutes with the possible exception of the right hand square $\Delta\bar{f}$ and $(\bar{f}\otimes\bar{f})\Delta'$. But ν is surjective so also the last square commutes. Similarly we get $\varepsilon_H\bar{f} = \varepsilon_{H(B)}$. Thus $\bar{f}$ is a homomorphism of bialgebras and hence a homomorphism of Hopf algebras. $\square$

Remark 2.6.4. In chapter 1 we have constructed universal bialgebras $M(A)$ with coaction $\delta : A \to M(A) \otimes A$ for certain algebras A (see 1.3.12). This induces a homomorphism of algebras

$$\delta' : A \to H(M(A)) \otimes A$$

such that A is a comodule-algebra over the Hopf algebra $H(M(A))$. If H is a Hopf algebra and A is an H-comodule algebra by $\partial : A \to H \otimes A$ then there is a unique homomorphism of bialgebras $f : M(A) \to H$ such that

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "\partial"'] & M(A) \otimes A \arrow[d, "f\otimes 1"] \\ & H \otimes A \end{tikzcd}$$

commutes. Since the $f : M(A) \longrightarrow H$ factorizes uniquely through $\bar{f} : H(M(A)) \longrightarrow H$ we get a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\delta'"] \arrow[dr, "\partial"'] & H(M(A)) \otimes A \arrow[d, "\bar{f}\otimes 1"] \\ & H \otimes A \end{tikzcd}$$

with a unique homomorphism of Hopf algebras $\bar{f} : H(M(A)) \longrightarrow H$.

This proof depends only on the existence of a universal algebra $M(A)$ for the algebra A. Hence we have

Corollary 2.6.5. *Let $\mathcal{X}$ be a quantum space with universal quantum space (and quantum monoid) $\mathcal{M}(\mathcal{X})$. Then there is a unique (up to isomorphism) quantum group $\mathcal{H}(\mathcal{M}(\mathcal{X}))$ acting universally on $\mathcal{X}$.*

This quantum group $\mathcal{H}(\mathcal{M}(\mathcal{X}))$ can be considered as the "quantum subgroup of invertible elements" of $\mathcal{M}(\mathcal{X})$ or the quantum group of "quantum automorphisms" of $\mathcal{X}$.

7. Duality of Hopf Algebras

In 2.4.8 we have seen that the dual Hopf algebra H^* of a finite dimensional Hopf algebra H satisfies certain relations w.r.t. the evaluation map. The multiplication of H^* is derived from the comultiplication of H and the comultiplication of H^* is derived from the multiplication of H.

This kind of duality is restricted to the finite-dimensional situation. Nevertheless one wants to have a process that is close to the finite-dimensional situation. This short section is devoted to several approaches of duality for Hopf algebras.

First we use the relations of the finite-dimensional situation to give a general definition.

Definition 2.7.1. Let H and L be Hopf algebras. Let

$$\mathrm{ev} : L \otimes H \ni a \otimes h \mapsto \langle a, h\rangle \in \mathbb{K}$$

be a bilinear form satisfying

$$\langle a \otimes b, \sum h_{(1)} \otimes h_{(2)}\rangle = \langle ab, h\rangle, \quad \langle 1, h\rangle = \varepsilon(h) \tag{3}$$

$$\langle \sum a_{(1)} \otimes a_{(2)}, h \otimes j\rangle = \langle a, hj\rangle, \quad \langle a, 1\rangle = \varepsilon(a) \tag{4}$$

$$\langle a, S(h)\rangle = \langle S(a), h\rangle \tag{5}$$

Such a map is called a *weak duality of Hopf algebras.* The bilinear form is called *left (right) nondegenerate* if $\langle a, H\rangle = 0$ implies $a = 0$ ($\langle L, h\rangle = 0$ implies $h = 0$). A *duality of Hopf algebras* is a weak duality that is left and right nondegenerate.

Remark 2.7.2. If H is a finite dimensional Hopf algebra then the usual evaluation $\mathrm{ev} : H^* \otimes H \longrightarrow \mathbb{K}$ defines a duality of Hopf algebras.

Remark 2.7.3. Assume that $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ defines a weak duality. By A.4.15 we have isomorphisms $\mathrm{Hom}(L \otimes H, \mathbb{K}) \cong \mathrm{Hom}(L, \mathrm{Hom}(H, \mathbb{K}))$ and $\mathrm{Hom}(L \otimes H, \mathbb{K}) \cong \mathrm{Hom}(H, \mathrm{Hom}(L, \mathbb{K}))$. Denote the homomorphisms associated with $\mathrm{ev} : L \otimes K \longrightarrow \mathbb{K}$ by $\varphi : L \longrightarrow \mathrm{Hom}(H, \mathbb{K})$ resp. $\psi : H \longrightarrow \mathrm{Hom}(L, \mathbb{K})$. They satisfy $\varphi(a)(h) = \mathrm{ev}(a \otimes h) = \psi(h)(a)$.

$\mathrm{ev} : L \otimes K \longrightarrow \mathbb{K}$ is left nondegenerate iff $\varphi : L \longrightarrow \mathrm{Hom}(H, \mathbb{K})$ is injective. $\mathrm{ev} : L \otimes K \longrightarrow \mathbb{K}$ is right nondegenerate iff $\psi : H \longrightarrow \mathrm{Hom}(L, \mathbb{K})$ is injective.

Lemma 2.7.4. 1. *The bilinear form* $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ *satisfies (3) if and only if* $\varphi : L \longrightarrow \mathrm{Hom}(H, \mathbb{K})$ *is a homomorphism of algebras.*

2. *The bilinear form* $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ *satisfies (4) if and only if* $\psi : H \longrightarrow \mathrm{Hom}(L, \mathbb{K})$ *is a homomorphism of algebras.*

Proof. $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ satisfies the right equation of (3) iff $\varphi(ab)(h) = \langle ab, h\rangle = \langle a \otimes b, \sum h_{(1)} \otimes h_{(2)}\rangle = \sum \langle a, h_{(1)}\rangle\langle b, h_{(2)}\rangle = \sum \varphi(a)(h_{(1)})\varphi(b)(h_{(2)}) = (\varphi(a) * \varphi(b))(h)$ by the definition of the algebra structure on $\mathrm{Hom}(H, \mathbb{K})$.

$\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ satisfies the left equation of (3) iff $\varphi(1)(h) = \langle 1, h\rangle = \varepsilon(h)$.

The second part of the Lemma follows by symmetry. □

Example 2.7.5. There is a weak duality between the quantum groups $\mathbb{SL}_q(2)$ and $U_q(sl(2))$. (Kassel: Chapter VII.4).

Proposition 2.7.6. *Let* $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ *be a weak duality of Hopf algebras. Let* $I := \mathrm{Ker}(\varphi : L \longrightarrow \mathrm{Hom}(H, \mathbb{K}))$ *and* $J := \mathrm{Ker}(\psi : H \longrightarrow \mathrm{Hom}(L, \mathbb{K}))$. *Let* $\overline{L} := L/I$ *and* $\overline{H} := H/J$. *Then* $\overline{L}$ *and* $\overline{H}$ *are Hopf algebras and the induced bilinear form* $\overline{\mathrm{ev}} : \overline{L} \otimes \overline{H} \longrightarrow \mathbb{K}$ *is a duality.*

Proof. First observe that I and J are two sided ideals hence $\overline{L}$ and $\overline{H}$ are algebras. Then $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ can be factored through $\overline{\mathrm{ev}} : \overline{L} \otimes \overline{H} \longrightarrow \mathbb{K}$ and the equations (3) and (4) are still satisfied for the residue classes.

The ideals I and J are biideals. In fact, let $x \in I$ then $\langle \Delta(x), a \otimes b\rangle = \langle x, ab\rangle = 0$ hence $\Delta(x) \in \mathrm{Ker}(\varphi \otimes \varphi : L \otimes L \longrightarrow \mathrm{Hom}(H \otimes H, \mathbb{K}) = I \otimes L + L \otimes I$ (the last equality is an easy exercise in linear algebra) and $\varepsilon(x) = \langle x, 1\rangle = 0$. Hence as in the proof of Theorem 2.6.3 we get that $\overline{L} = L/I$ and $\overline{H} = H/J$ are bialgebras. Since $\langle S(x), a\rangle = \langle x, S(a)\rangle = 0$ we have an induced homomorphism $\overline{S} : \overline{L} \longrightarrow \overline{L}$. The identities satisfied in L hold also for the residue classes in $\overline{L}$ so that L and similarly $\overline{H}$ become Hopf algebras. Finally we have by definition of I that $\langle \overline{x}, \overline{a}\rangle = \langle x, a\rangle = 0$ for all $a \in H$ iff $a \in I$ or $\overline{a} = 0$. Thus the bilinear form $\overline{\mathrm{ev}} : \overline{L} \otimes \overline{H} \longrightarrow K$ defines a duality. □

Problem 2.7.8. (in Linear Algebra)

1. For $U \subseteq V$ define $U^\perp := \{f \in V^* | f(U) = 0\}$. For $Z \subseteq V^*$ define $Z^\perp := \{v \in V | Z(v) = 0\}$. Show that the following hold:
   (a) $U \subseteq V \Longrightarrow U = U^{\perp\perp}$;
   (b) $Z \subseteq V^*$ and $\dim Z < \infty \Longrightarrow Z = Z^{\perp\perp}$;
   (c) $\{U \subseteq V | \dim V/U < \infty\} \cong \{Z \subseteq V^* | \dim Z < \infty\}$ under the maps $U \mapsto U^\perp$ and $Z \mapsto Z^\perp$.
2. Let $V = \bigoplus_{i=1}^\infty \mathbb{K}x_i$ be an infinite-dimensional vector space. Find an element $g \in (V \otimes V)^*$ that is not in $V^* \otimes V^* \quad (\subseteq (V \otimes V)^*)$.

Definition 2.7.7. Let A be an algebra. We define $A^o := \{f \in A^* | \exists \text{ ideal } {}_AI_A \subseteq A : \dim(A/I) < \infty \text{ and } f(I) = 0\}$.

Lemma 2.7.8. *Let* A *be an algebra and* $f \in A^*$. *The following are equivalent:*

1. $f \in A^o$;
2. *there exists* $I_A \subseteq A$ *such that* $\dim A/I < \infty$ *and* $f(I) = 0$;
3. $A \cdot f \subseteq {}_A\mathrm{Hom}_{\mathbb{K}}(.A_A, .\mathbb{K})$ *is finite dimensional;*
4. $A \cdot f \cdot A$ *is finite dimensional;*
5. $\nabla^*(f) \in A^* \otimes A^*$.

Proof. 1. $\Longrightarrow$ 2. and 4. $\Longrightarrow$ 3. are trivial.

2. $\Longrightarrow$ 3. Let $I_A \subseteq A$ with $f(I) = 0$ and $\dim A/I < \infty$. Write $A^* \otimes A \to \mathbb{K}$ as $\langle g, a\rangle$. Then $\langle af, i\rangle = \langle f, ia\rangle = 0$ hence $Af \subset I^\perp$ and $\dim Af < \infty$.

3. $\Longrightarrow$ 2. Let $\dim Af < \infty$. Then $I_A := (Af)^\perp$ is an ideal of finite codimension in A and $f(I) = 0$ holds.

2. $\Longrightarrow$ 1. Let $I_A \subset A$ with $\dim A/I_A < \infty$ and $f(I) = 0$ be given. Then right multiplication induces $\varphi : A \to \mathrm{Hom}_{\mathbb{K}}(A/I., A/I.)$ and $\dim \mathrm{End}_{\mathbb{K}}(A/I) < \infty$. Thus $J = \mathrm{Ker}(\varphi) \subseteq A$ is a two sided ideal of finite codimension and $J \subset I$ (since $\varphi(j)(\bar{1}) = 0 = \bar{1} \cdot j = \bar{j}$ implies $j \in I$). Furthermore we have $f(J) \subseteq f(I) = 0$.

1. $\Longrightarrow$ 4. $\langle afb, i\rangle = \langle f, bia\rangle = 0$ implies $A \cdot f \cdot A \subseteq {}_A I_A^\perp$ hence $\dim AfA < \infty$.

3. $\Longrightarrow$ 5. We observe that $\nabla^*(f) = f\nabla \in (A \otimes A)^*$. We want to show that $\nabla^*(f) \in A^* \otimes A^*$. Let $g_1, \ldots, g_n$ be a basis of Af. Then there exist $h_1, \ldots, h_n \in A^*$ such that $bf = \sum h_i(b)g_i$. Let $a, b \in A$. Then $\langle \nabla^*(f), a \otimes b\rangle = \langle f, ab\rangle = \langle bf, a\rangle = \sum h_i(b)g_i(a) = \langle \sum g_i \otimes h_i, a \otimes b\rangle$ so that $\nabla^*(f) = \sum g_i \otimes h_i \in A^* \otimes A^*$.

5. $\Longrightarrow$ 3. Let $\nabla^*(f) = \sum g_i \otimes h_i \in A^* \otimes A^*$. Then $bf = \sum h_i(b)g_i$ for all $b \in A$ as before. Thus Af is generated by the $g_1, \ldots, g_n$. □

Proposition 2.7.9. *Let (A, m, u) be an algebra. Then we have $m^*(A^o) \subseteq A^o \otimes A^o$. Furthermore $(A^o, \Delta, \varepsilon)$ is a coalgebra with $\Delta = m^*$ and $\varepsilon = u^*$.*

Proof. Let $f \in A^o$ and let $g_1, \ldots, g_n$ be a basis for Af. Then we have $m^*(f) = \sum g_i \otimes h_i$ for suitable $h_i \in A^*$ as in the proof of the previous proposition. Since $g_i \in Af$ we get $Ag_i \subseteq Af$ and $\dim(Ag_i) < \infty$ and hence $g_i \in A^o$. Choose $a_1, \ldots, a_n \in A$ such that $g_i(a_j) = \delta_{ij}$. Then $(fa_j)(a) = f(a_j a) = \langle m^*(f), a_j \otimes a\rangle = \sum g_i(a_j)h_i(a) = h_j(a)$ implies $fa_j = h_j \in fA$. Observe that $\dim(fA) < \infty$ hence $\dim(h_j A) < \infty$, so that $h_j \in A^o$. This proves $m^*(f) \in A^o \otimes A^o$.

One checks easily that counit law and coassociativity hold. □

Theorem 2.7.10. (The Sweedler dual:) *Let $(B, m, u, \Delta, \varepsilon)$ be a bialgebra. Then $(B^o, \Delta^*, \varepsilon^*, m^*, u^*)$ again is a bialgebra. If $B = H$ is a Hopf algebra with antipode S, then S^* is an antipode for $B^o = H^o$.*

Proof. We know that $(B^*, \Delta^*, \varepsilon^*)$ is an algebra and that (B^o, m^*, u^*) is a coalgebra. We show now that $B^o \subseteq B^*$ is a subalgebra. Let $f, g \in B^o$ with $\dim(Bf) < \infty$ and $\dim(Bg) < \infty$. Let $a \in B$. Then we have $(a(fg))(b) = (fg)(ba) = \sum f(b_{(1)}a_{(1)})g(b_{(2)}a_{(2)}) = \sum(a_{(1)}f)(b_{(1)})(a_{(2)}g)(b_{(2)}) = \sum((a_{(1)}f)(a_{(2)}g))(b)$ hence $a(fg) = \sum(a_{(1)}f)(a_{(2)}g) \in (Bf)(Bg)$. Since $\dim(Bf)(Bg) < \infty$ we have $\dim(B(fg)) < \infty$ so that $fg \in B^o$. Furthermore we have $\varepsilon \in B^o$, since $\mathrm{Ker}(\varepsilon)$ has codimension 1. Thus $B^o \subseteq B^*$ is a subalgebra. It is now easy to see that B^o is a bialgebra.

Now let S be the antipode of H. We show $S^*(H^o) \subseteq H^o$. Let $a \in H$, $f \in H^o$. Then $\langle aS^*(f), b\rangle = \langle S^*(f), ba\rangle = \langle f, S(ba)\rangle = \langle f, S(a)S(b)\rangle = \langle fS(a), S(b)\rangle = \langle S^*(fS(a)), b\rangle$. This implies $aS^*(f) = S^*(fS(a))$ and $HS^*(f) = S^*(fS(H)) \subseteq S^*(fH)$. Since $f \in H^o$ we get $\dim(fH) < \infty$ so that $\dim(S^*(fH)) < \infty$ and $\dim(HS^*(f)) < \infty$. This shows $S^*(f) \in H^o$. The rest of the proof is now trivial. □

Definition 2.7.11. Let $G = \mathbb{K}\text{-}\mathbf{cAlg}(H, \text{-})$ be an affine group and $R \in \mathbb{K}\text{-}\mathbf{cAlg}$. We define $G \otimes_{\mathbb{K}} R := G|_{R\text{-}\mathbf{cAlg}}$ to be the restriction to commutative R-algebras. The functor $G \otimes_{\mathbb{K}} R$ is represented by $H \otimes R \in R\text{-}\mathbf{cAlg}$:

$$G|_{R\text{-}\mathbf{cAlg}}(A) = \mathbb{K}\text{-}\mathbf{cAlg}(H, A) \cong R\text{-}\mathbf{cAlg}(H \otimes R, A).$$

Theorem 2.7.12. (The Cartier dual:) *Let H be a finite dimensional commutative cocommutative Hopf algebra. Let $G = \mathbb{K}\text{-}\mathbf{cAlg}(H, \text{-})$ be the associated affine group and let $D(G) := \mathbb{K}\text{-}\mathbf{cAlg}(H^*, \text{-})$ be the dual group. Then we have*

$$D(G) = \mathcal{G}r(G, G_m)$$

where $\mathcal{G}r(G, G_m)(R) = \mathrm{Gr}(G \otimes_{\mathbb{K}} R, G_m \otimes_{\mathbb{K}} R)$ is the set of group (-functor) homomorphisms and G_m is the multiplicative group.

Proof. We have $\mathcal{G}r(G, G_m)(R) = \mathrm{Gr}(G \otimes_{\mathbb{K}} R, G_M \otimes_{\mathbb{K}} R) \cong R\text{-Hopf-Alg}(\mathbb{K}[t, t^{-1}] \otimes R, H \otimes R) \cong R\text{-Hopf-Alg}(R[t, t^{-1}], H \otimes R) \cong \{x \in U(H \otimes R) | \Delta(x) = x \otimes x, \varepsilon(x) = 1\}$, since $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$ imply $xS(x) = \varepsilon(x) = 1$.

Consider $x \in \mathrm{Hom}_R((H \otimes R)^*, R) = \mathrm{Hom}_R(H^* \otimes R, R)$. Then $\Delta(x) = x \otimes x$ iff $x(v^*w^*) = \langle x, v^*w^* \rangle = \langle \Delta(x), v^* \otimes w^* \rangle = x(v^*)x(w^*)$ and $\varepsilon(x) = 1$ iff $\langle x, \varepsilon \rangle = 1$. Hence $x \in R\text{-}\mathbf{cAlg}((H \otimes R)^*, R) \cong \mathbb{K}\text{-}\mathbf{cAlg}(H^*, R) = D(G)(R)$. $\square$