CHAPTER 1

Commutative and Noncommutative Algebraic Geometry

3. Quantum Monoids and their Actions on Quantum Spaces

We use the orthogonal product introduced in the previous section as "product" to define the notion of a monoid (some may call it an algebra w.r.t. the orthogonal product). Observe that on the geometric level the orthogonal product consists only of commuting points. So whenever we define a morphism on the geometric side with domain an orthogonal product of quantum spaces $f : \mathcal{X} \perp \mathcal{Y} \to \mathcal{Z}$ then we only have to define what happens to *commuting* pairs of points. That makes it much easier to define such morphisms for noncommutative coordinate algebras.

We are going to define monoids in this sense and study their actions on quantum spaces.

Let E be the functor represented by $\mathbb{K}$. It maps each algebra H to the one-element set $\{\iota : \mathbb{K} \to H\}$.

Definition 1.3.1. Let $\mathcal{M}$ be a noncommutative space and let

$$m : \mathcal{M} \perp \mathcal{M} \to \mathcal{M} \text{ and } e : E \to \mathcal{M}$$

be morphisms in **QS** such that the diagrams

$$\begin{tikzcd}
\mathcal{M} \perp \mathcal{M} \perp \mathcal{M} \arrow[r, "m \perp 1"] \arrow[d, "1 \perp m"'] & \mathcal{M} \perp \mathcal{M} \arrow[d, "m"] \\
\mathcal{M} \perp \mathcal{M} \arrow[r, "m"'] & \mathcal{M}
\end{tikzcd}$$

and

$$\begin{tikzcd}
E \perp \mathcal{M} \cong \mathcal{M} \cong \mathcal{M} \perp E \arrow[r, "\mathrm{id} \perp \eta"] \arrow[d, "\eta \perp \mathrm{id}"'] \arrow[dr, "1_{\mathcal{M}}"] & \mathcal{M} \perp \mathcal{M} \arrow[d, "\nabla"] \\
\mathcal{M} \perp \mathcal{M} \arrow[r, "\nabla"'] & \mathcal{M}
\end{tikzcd}$$

commute. Then $(\mathcal{M}, m, e)$ is called a *quantum monoid.*

Proposition 1.3.2. *Let $\mathcal{M}$ be a noncommutative space with function algebra H. Then H is a bialgebra if and only if $\mathcal{M}$ is a quantum monoid.*

PROOF. Since the functors $\mathcal{M} \perp \mathcal{M}$, $\mathcal{M} \perp E$ and $E \perp \mathcal{M}$ are represented by $H \otimes H$ resp. $H \otimes \mathbb{K} \cong H$ resp. $\mathbb{K} \otimes H \cong H$ the Yoneda Lemma defines a bijection between the morphisms $m : \mathcal{M} \perp \mathcal{M} \to \mathcal{M}$ and the algebra homomorphisms $\Delta : H \to H \otimes H$ and similarly a bijection between the morphisms $e : E \to \mathcal{M}$ and the algebra homomorphisms $\varepsilon : H \to \mathbb{K}$. Again by the Yoneda Lemma the bialgebra diagrams in $\mathbb{K}$-**Alg** commute if and only if the corresponding diagrams for a quantum monoid commute. $\square$

Observe that a similar result cannot be formulated for Hopf algebras H since neither the antipode S nor the multiplication $\nabla : H \otimes H \to H$ are algebra homomorphisms. In contrast to affine algebraic groups (2.3.2) Hopf algebras in the category $\mathbb{K}$-$\mathbf{Alg}^{op} \cong QR$ are not groups. Nevertheless, one defines

Definition 1.3.3. A functor defined on the category of $\mathbb{K}$-algebras and represented by a Hopf algebra H is called a *quantum group.*

Definition 1.3.4. Let $\mathcal{X}$ be a noncommutative space and let $\mathcal{M}$ be a quantum monoid. A morphism (a natural transformation) of quantum spaces $\rho : \mathcal{M} \perp \mathcal{X} \to \mathcal{X}$ is called an *operation* of $\mathcal{M}$ on $\mathcal{X}$ if the diagrams

$$\begin{tikzcd}
\mathcal{M} \perp \mathcal{M} \perp \mathcal{X} \arrow[r, "m \perp 1"] \arrow[d, "1 \perp \rho"'] & \mathcal{M} \perp \mathcal{X} \arrow[d, "\rho"] \\
\mathcal{M} \perp \mathcal{X} \arrow[r, "\rho"'] & \mathcal{X}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{X} \cong E \perp \mathcal{X} \arrow[r, "\eta \perp \mathrm{id}"] \arrow[dr, "\mathrm{id}_{\mathcal{X}}"] & \mathcal{M} \perp \mathcal{X} \arrow[d, "\rho"] \\
 & \mathcal{X}
\end{tikzcd}$$

commute. We call $\mathcal{X}$ a noncommutative $\mathcal{M}$*-space.*

Proposition 1.3.5. *Let $\mathcal{X}$ be a noncommutative space with function algebra $A = \mathcal{O}(\mathcal{X})$. Let $\mathcal{M}$ be a quantum monoid with function algebra $B = \mathcal{O}(\mathcal{M})$. Let $\rho : \mathcal{M} \perp \mathcal{X} \to \mathcal{X}$ be a morphism in* $\mathbf{QS}$ *and let $f : A \to B \otimes A$ be the associated homomorphism of algebras. Then the following are equivalent*

1. $(\mathcal{X}, \mathcal{M}, \rho)$ is an operation of the quantum monoid $\mathcal{M}$ on the noncommutative space $\mathcal{X}$;

2. (A, H, f) define an H-comodule algebra.

Proof. The homomorphisms of algebras $\Delta \otimes 1_A$, $1_B \otimes f$, $\epsilon \otimes 1_A$ etc. represent the morphisms of quantum spaces $m \perp \mathrm{id}$, $\mathrm{id} \perp \rho$, $\eta \perp \mathrm{id}$ etc. Hence the required diagrams are transferred by the Yoneda Lemma. □

Example 1.3.6. 1. The quantum monoid of "quantum matrices":
We consider the algebra

$$M_q(2) := \mathbb{K}\langle a, b, c, d\rangle / I = \mathbb{K}\left\langle \begin{matrix} a & b \\ c & d \end{matrix} \right\rangle / I$$

where the two-sided ideal I is generated by the elements

$$ab - q^{-1}ba, ac - q^{-1}ca, bd - q^{-1}db, cd - q^{-1}dc, ad - da - (q^{-1} - q)bc, bc - cb.$$

The quantum space $\mathcal{M}_q(2)$ associated with the algebra $M_q(2)$ is given by

$$\begin{aligned}\mathcal{M}_q(2)(A) &= \mathbb{K}\text{-}\mathbf{Alg}(M_q(2), A)\\ &= \left\{ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} | a', b', c', d' \in A; a'b' = q^{-1}b'a', \ldots, b'c' = c'b' \right\}\end{aligned}$$

where each homomorphism of algebras $f : M_q(2) \longrightarrow A$ is described by the quadruple (a', b', c', d') of images of the algebra generators a, b, c, d. The images must satisfy the same relations that generate the two-sided ideal I hence

$$\begin{gathered} a'b' = q^{-1}b'a', a'c' = q^{-1}c'a', b'd' = q^{-1}d'b', c'd' = q^{-1}d'c', \\ b'c' = c'b', a'd' - q^{-1}b'c' = d'a' - qc'b'. \end{gathered}$$

We write these quadruples as 2×2-matrices and call them *quantum matrices*. The unusual commutation relations are chosen so that the following examples work.

The quantum space of quantum matrices turns out to be a quantum monoid. We give both the algebraic (with function algebras) and the geometric (with quantum spaces) approach to define the multiplication.

a) *The algebraic approach:*

The algebra $M_q(2)$ is a bialgebra with the diagonal

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

i.e. by $\Delta(a) = a \otimes a + b \otimes c$, $\Delta(b) = a \otimes b + b \otimes d$, $\Delta(c) = c \otimes a + d \otimes c$ and $\Delta(d) = c \otimes b + d \otimes d$, and with the counit

$$\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e. $\varepsilon(a) = 1$, $\varepsilon(b) = 0$, $\varepsilon(c) = 0$, and $\varepsilon(d) = 1$. We have to prove that Δ and ε are homomorphisms of algebras and that the coalgebra laws are satisfied. To obtain a homomorphism of algebras $\Delta : M_q(2) \longrightarrow M_q(2) \otimes M_q(2)$ we define $\Delta : \mathbb{K}\langle a, b, c, d\rangle \longrightarrow M_q(2) \otimes M_q(2)$ on the free algebra (the polynomial ring in noncommuting variables) $\mathbb{K}\langle a, b, c, d\rangle$ generated by the set $\{a, b, c, d\}$ and show that it vanishes on the ideal I or more simply on the generators of the ideal. Then it factors through a unique homomorphism of algebras $\Delta : M_q(2) \longrightarrow M_q(2) \otimes M_q(2)$. We check this only for one generator of the ideal I:

$$\begin{aligned} &\Delta(ab - q^{-1}ba) = \Delta(a)\Delta(b) - q^{-1}\Delta(b)\Delta(a) = \\ &= (a \otimes a + b \otimes c)(a \otimes b + b \otimes d) - q^{-1}(a \otimes b + b \otimes d)(a \otimes a + b \otimes c) \\ &= aa \otimes ab + ab \otimes ad + ba \otimes cb + bb \otimes cd - q^{-1}(aa \otimes ba + ab \otimes bc + ba \otimes da + bb \otimes dc) \\ &= aa \otimes (ab - q^{-1}ba) + ab \otimes (ad - q^{-1}bc) + ba \otimes (cb - q^{-1}da) + bb \otimes (cd - q^{-1}dc) \\ &= ba \otimes (q^{-1}ad - q^{-2}bc + cb - q^{-1}da) \equiv 0 \mod (I). \end{aligned}$$

The reader should check the other identities.

The coassociativity follows from

$$(\Delta \otimes 1)\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} =$$
$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (1 \otimes \Delta)\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The reader should check the properties of the counit.

b) *The geometric approach:*

$M_q(2)$ has a rather remarkable (and actually well known) comultiplication that is better understood by using the induced multiplication of commuting points. Given two commuting quantum matrices $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ in $\mathcal{M}_q(2)(A)$. Then their matrix product

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}$$

is again a quantum matrix. To prove this we only check one of the relations

$$\begin{aligned}(a_1a_2 + b_1c_2)(a_1b_2 + b_1d_2) &= a_1a_2a_1b_2 + a_1a_2b_1d_2 + b_1c_2a_1b_2 + b_1c_2b_1d_2 \\ &= a_1a_1a_2b_2 + a_1b_1a_2d_2 + b_1a_1c_2b_2 + b_1b_1c_2d_2 \\ &= q^{-1}a_1a_1b_2a_2 + q^{-1}b_1a_1(d_2a_2 + (q^{-1} - q)b_2c_2) + b_1a_1b_2c_2 + q^{-1}b_1b_1d_2c_2 \\ &= q^{-1}(a_1a_1b_2a_2 + a_1b_1b_2c_2 + b_1a_1d_2a_2 + b_1b_1d_2c_2) \\ &= q^{-1}(a_1b_2a_1a_2 + a_1b_2b_1c_2 + b_1d_2a_1a_2 + b_1d_2b_1c_2) \\ &= q^{-1}(a_1b_2 + b_1d_2)(a_1a_2 + b_1c_2)\end{aligned}$$

We have used that the two points are commuting points. This multiplication obviously is a natural transformation $\mathcal{M}_q(2) \perp \mathcal{M}_q(2)(A) \longrightarrow \mathcal{M}_q(2)(A)$ (natural in A). It is associative and has unit $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. For the associativity observe that by 1.2.14

$$((\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}), \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix})$$

is a pair of commuting points if and only if

$$(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, (\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}))$$

is a pair of commuting points.

Since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for all quantum matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_q(2)(B)$ we see that $\mathcal{M}_q(2)$ is a quantum monoid.

It remains to show that the multiplication of $\mathcal{M}_q(2)$ and the comultiplication of $M_q(2)$ correspond to each other by the Yoneda Lemma. The identity morphism of

$M_q(2) \otimes M_q(2)$ is given by the pair of commuting points
$(\iota_1, \iota_2) \in \mathcal{M}_q(2) \perp \mathcal{M}_q(2)(M_q(2) \otimes M_q(2)) = \mathbb{K}\text{-}\mathbf{Alg}(M_q(2) \otimes M_q(2), M_q(2) \otimes M_q(2))$.
Since $\iota_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes 1 = \begin{pmatrix} a \otimes 1 & b \otimes 1 \\ c \otimes 1 & d \otimes 1 \end{pmatrix}$ and $\iota_2 = 1 \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 \otimes a & 1 \otimes b \\ 1 \otimes c & 1 \otimes d \end{pmatrix}$ we have $\mathrm{id} = (\iota_1, \iota_2) = (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes 1, 1 \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix})$. The Yoneda Lemma defines the diagonal as the image of the identity under $\mathbb{K}\text{-}\mathbf{Alg}(M_q(2) \otimes M_q(2), M_q(2) \otimes M_q(2)) \longrightarrow \mathbb{K}\text{-}\mathbf{Alg}(M_q(2), M_q(2) \otimes M_q(2))$ by the multiplication. So $\Delta(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \Delta = \iota_1 * \iota_2 = (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes 1) * (1 \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Thus $M_q(2)$ defines a quantum monoid $\mathcal{M}_q(2)$ with

$$\mathcal{M}_q(2)(B) = \left\{ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} | a', b', c', d' \in B; a'b' = q^{-1}b'a', \dots, b'c' = c'b' \right\}.$$

This is the deformed version of $\mathcal{M}_2^{\times}$ the multiplicative monoid of the 2×2-matrices of commutative algebras.

2. Let $A_q^{2|0} = \mathbb{K}\langle x, y\rangle/(xy - q^{-1}yx)$ be the function algebra of the quantum plane $\mathbb{A}_q^{2|0}$. By the definition 1.2.5 we have

$$\mathbb{A}_q^{2|0}(A') = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | x, y \in A'; xy = q^{-1}yx \right\}.$$

The set

$$\mathcal{M}_q(2)(A') = \left\{ \begin{pmatrix} u & x \\ y & z \end{pmatrix} | u, x, y, z \in A'; ux = q^{-1}xu, \dots, xy = yx \right\}$$

operates on this quantum plane by matrix multiplication

$$\mathcal{M}_q(2)(A') \perp \mathbb{A}_q^{2|0}(A') \ni (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}_q^{2|0}(A').$$

Again one should check that the required equations are preserved. Since we have a matrix multiplication we get an operation as in the preceding proposition. In particular $A_q^{2|0}$ is a $M_q(2)$-comodule algebra.

As in example 1. we get the comultiplication as $\delta(\begin{pmatrix} x \\ y \end{pmatrix}) = \delta = (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes 1) * (1 \otimes \begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$.

3. Let $A_q^{0|2} = \mathbb{K}\langle \xi, \eta\rangle/(\xi^2, \eta^2, \xi\eta + q\eta\xi)$ be the function algebra of the dual quantum plane $\mathbb{A}_q^{0|2}$. By the definition 1.2.5 we have

$$\mathbb{A}_q^{0|2}(A') = \left\{ \begin{pmatrix} a' & b' \end{pmatrix} \middle| a', b' \in A'; a'^2 = 0, b'^2 = 0, a'b' = -qb'a' \right\}.$$

The quantum monoid $M_q(2)$ also operates on the dual quantum plane by matrix multiplication

$$\mathbb{A}_q^{0|2}(A') \perp \mathcal{M}_q(2)(A') \ni (\begin{pmatrix}\xi & \eta\end{pmatrix}, \begin{pmatrix} a & b \\ c & d\end{pmatrix}) \mapsto \begin{pmatrix}\xi & \eta\end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathbb{A}_q^{0|2}(A').$$

This gives another example of a $M_q(2)$-comodule algebra $A_q^{0|2} \longrightarrow A_q^{0|2} \otimes M_q(2)$ with $\delta(\begin{pmatrix}\xi & \eta\end{pmatrix}) = \delta = (\begin{pmatrix}\xi & \eta\end{pmatrix} \otimes 1) * (1 \otimes \begin{pmatrix} a & b \\ c & d\end{pmatrix}) = \begin{pmatrix}\xi & \eta\end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d\end{pmatrix}$.

What is now the reason for the remarkable relations on $\mathcal{M}_q(2)$? It is based on a fact that we will show later namely that $M_q(2)$ is the universal quantum monoid acting on the quantum plane $\mathbb{A}_q^{2|0}$ from the left and on the dual quantum plane $\mathbb{A}_q^{0|2}$ from the right. This however happens in the category of quantum planes represented by quadratic algebras. Here we will show a simpler theorem for finite dimensional algebras.

Problem 1.3.1. Determine the $\mathbb{H}$-points of the quantum plane $\mathbb{A}_q^{2|0}$ where $\mathbb{H}$ is the $\mathbb{R}$-algebra of the quaternions.

Definition 1.3.7. 1. Let $\mathcal{X}$ be a quantum space. A quantum space $\mathcal{M}(\mathcal{X})$ together with a morphism of quantum spaces $\mu : \mathcal{M}(\mathcal{X}) \perp \mathcal{X} \longrightarrow \mathcal{X}$ is called a *quantum space acting universally* on $\mathcal{X}$ (or simply a *universal quantum space for* $\mathcal{X}$) if for every quantum space $\mathcal{Y}$ and every morphism of quantum spaces $f : \mathcal{Y} \perp \mathcal{X} \longrightarrow \mathcal{X}$ there is a unique morphism of quantum spaces $g : \mathcal{Y} \longrightarrow \mathcal{M}(\mathcal{X})$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{Y} \perp \mathcal{X} \arrow[d, "g \perp 1_{\mathcal{X}}"'] \arrow[dr, "f"] & \\
\mathcal{M}(\mathcal{X}) \perp \mathcal{X} \arrow[r, "\mu"'] & \mathcal{X}.
\end{tikzcd}$$

2. Let A be a $\mathbb{K}$-algebra. A $\mathbb{K}$-algebra $M(A)$ together with a homomorphism of algebras $\delta : A \longrightarrow M(A) \otimes A$ is called an *algebra coacting universally on* A (or simply a *universal algebra for* A) if for every $\mathbb{K}$-algebra B and every homomorphism of $\mathbb{K}$-algebras $f : A \longrightarrow B \otimes A$ there exists a unique homomorphism of algebras $g : M(A) \longrightarrow B$ such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[dr, "f"'] & M(A) \otimes A \arrow[d, "g \otimes 1_A"] \\
& B \otimes A
\end{tikzcd}$$

By the universal properties the universal algebra $M(A)$ for A and the universal quantum space $\mathcal{M}(\mathcal{X})$ for $\mathcal{X}$ are unique up to isomorphism.

Proposition 1.3.8. 1. *Let A be a $\mathbb{K}$-algebra with universal algebra $M(A)$ and $\delta : A \longrightarrow M(A) \otimes A$. Then $M(A)$ is a bialgebra and A is an $M(A)$-comodule algebra by δ.*

2. *If B is a bialgebra and if $f : A \longrightarrow B \otimes A$ defines the structure of a B-comodule algebra on A then there is a unique homomorphism $g : M(A) \longrightarrow B$ of bialgebras such that the following diagram commutes*

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "f"'] & M(A) \otimes A \arrow[d, "g \otimes 1_A"] \\ & B \otimes A \end{tikzcd}$$

The corresponding statement for quantum spaces and quantum monoids is the following.

Proposition 1.3.9. 1. *Let $\mathcal{X}$ be a quantum space with universal quantum space $\mathcal{M}(\mathcal{X})$ and $\mu : \mathcal{M}(\mathcal{X}) \perp A \longrightarrow A$. Then $\mathcal{M}(\mathcal{X})$ is a quantum monoid and $\mathcal{X}$ is an $\mathcal{M}(\mathcal{X})$-space by μ.*

2. *If $\mathcal{Y}$ is another quantum monoid and if $f : \mathcal{Y} \perp \mathcal{X} \longrightarrow \mathcal{X}$ defines the structure of a $\mathcal{Y}$-space on $\mathcal{X}$ then there is a unique morphism of quantum monoids $g : \mathcal{Y} \longrightarrow \mathcal{M}(\mathcal{X})$ such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{Y} \perp \mathcal{X} \arrow[d, "g \perp 1_{\mathcal{X}}"'] \arrow[dr, "f"] & \\ \mathcal{M}(\mathcal{X}) \perp \mathcal{X} \arrow[r, "\mu"'] & \mathcal{X}. \end{tikzcd}$$

Proof. We give the proof for the algebra version of the proposition. Consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "\delta"'] & M(A) \otimes A \arrow[d, "\Delta \otimes 1_A"] \\ M(A) \otimes A \arrow[r, "1_{M(A)} \otimes \delta"'] & M(A) \otimes M(A) \otimes A \end{tikzcd}$$

where the morphism of algebras Δ is defined by the universal property of $M(A)$ with respect to the algebra morphism $(1_{M(A)} \otimes \delta)\delta$. Furthermore there is a unique morphism of algebras $\epsilon : M(A) \longrightarrow \mathbb{K}$ such that

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "1_A"'] & M(A) \otimes A \arrow[d, "\epsilon \otimes 1_A"] \\ & A \cong \mathbb{K} \otimes A \end{tikzcd}$$

commutes.

The coalgebra axioms arise from the following commutative diagrams

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[d, "\delta"] & M(A) \otimes A \arrow[d, "\Delta \otimes 1_A"] \\
M(A) \otimes A \arrow[r, "1_{M(A)} \otimes \delta"] \arrow[d, shift right, "\Delta \otimes 1_A"'] \arrow[d, shift left, "1_{M(A)} \otimes \delta"] & M(A) \otimes M(A) \otimes A \arrow[d, shift right, "\Delta \otimes 1_{M(A)} \otimes 1_A"'] \arrow[d, shift left, "1_{M(A)} \otimes \Delta \otimes 1_A"] \\
M(A) \otimes M(A) \otimes A \arrow[r, "1_{M(A)} \otimes 1_{M(A)} \otimes \delta"] & M(A) \otimes M(A) \otimes M(A) \otimes A
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[rr, "\delta"] \arrow[d, "\delta"] & & M(A) \otimes A \arrow[d, "\Delta \otimes 1_A"'] \arrow[dd, "1_{M(A)} \otimes 1_A"] \\
M(A) \otimes A \arrow[r, "1_{M(A)} \otimes \delta"] \arrow[drr, "1_{M(A)} \otimes 1_A"'] & M(A) \otimes M(A) \otimes A \arrow[dr, "1_{M(A)} \otimes \epsilon \otimes 1_A"] & \\
 & & M(A) \otimes A \cong M(A) \otimes \mathbb{K} \otimes A
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[d, "\delta"] \arrow[dd, bend right, "1_A"'] & M(A) \otimes A \arrow[d, "\Delta \otimes 1_A"] \\
M(A) \otimes A \arrow[r, "1_{M(A)} \otimes \delta"] \arrow[d, "\epsilon \otimes 1_A"] & M(A) \otimes M(A) \otimes A \arrow[d, "\epsilon \otimes 1_{M(A)} \otimes 1_A"] \\
A \arrow[r, "\delta"] & M(A) \otimes A \cong \mathbb{K} \otimes M(A) \otimes A.
\end{tikzcd}$$

In fact these diagrams imply by the uniqueness of the induced homomorphisms of algebras $(\Delta \otimes 1_{M(A)})\Delta = (1_{M(A)} \otimes \Delta)\Delta$, $(1_{M(A)} \otimes \epsilon)\Delta = 1_{M(A)}$ and $\epsilon \otimes (1_{M(A)})\Delta = 1_{M(A)}$. Finally A is an $M(A)$-comodule algebra by the definition of Δ and ϵ.

Now assume that a structure of a B-comodule algebra on A is given by a bialgebra B and $f : A \to B \otimes A$. Then there is a unique homomorphism of algebras $g : M(A) \to B$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[dr, "f"'] & M(A) \otimes A \arrow[d, "g \otimes 1_A"] \\
 & B \otimes A
\end{tikzcd}$$

commutes. Then the following diagram

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[dr, "f"'] & M(A) \otimes A \arrow[r, shift left, "\Delta \otimes 1_A"] \arrow[r, shift right, "1_{M(A)} \otimes \delta"'] \arrow[d, "g \otimes 1_A"] & M(A) \otimes M(A) \otimes A \arrow[d, "g \otimes g \otimes 1_A"] \\
& B \otimes A \arrow[r, shift left, "\Delta_B \otimes 1_A"] \arrow[r, shift right, "1_B \otimes f"'] & B \otimes B \otimes A
\end{tikzcd}$$

implies $((g \otimes g)\Delta \otimes 1_A)\delta = (g \otimes g \otimes 1_A)(\Delta \otimes 1_A)\delta = (g \otimes g \otimes 1_A)(1_{M(A)} \otimes \delta)\delta = (g \otimes (g \otimes 1_A)\delta)\delta = (1_B \otimes (g \otimes 1_A)\delta)(g \otimes 1_A)\delta = (1_B \otimes f)f = (\Delta_B \otimes 1_A)f = (\Delta_B \otimes 1_A)(g \otimes 1_A)\delta = (\Delta_B g \otimes 1_A)\delta$ hence $(g \otimes g)\Delta = \Delta_B g$. Furthermore the diagram

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[dr, "f"] \arrow[ddr, "1_A"'] & M(A) \otimes A \arrow[d, "g \otimes 1_A"'] \arrow[dd, shift left=4, "\epsilon \otimes 1_A"] \\
& B \otimes A \arrow[d, "\epsilon_B \otimes 1_A"'] \\
& A \cong \mathbb{K} \otimes A
\end{tikzcd}$$

implies $\epsilon_B g = \epsilon$. Thus g is a homomorphism of bialgebras. $\square$

Since universal algebras for algebras A tend to become very big they do not exist in general. But a theorem of Tambara's says that they exist for finite dimensional algebras (over a field $\mathbb{K}$).

Definition 1.3.10. If $\mathcal{X}$ is a quantum space with finite dimensional function algebra then we call $\mathcal{X}$ a *finite quantum space.*

The following theorem is the quantum space version and equivalent to a theorem of Tambara.

Theorem 1.3.11. *Let $\mathcal{X}$ be a finite quantum space. Then there exists a (universal) quantum space $\mathcal{M}(\mathcal{X})$ with morphism of quantum spaces $\mu : \mathcal{M}(\mathcal{X}) \perp \mathcal{X} \to \mathcal{X}$.*

The algebra version of this theorem is

Theorem 1.3.12. (Tambara) *Let A be a finite dimensional $\mathbb{K}$-algebra. Then there exists a (universal) $\mathbb{K}$-algebra $M(A)$ with homomorphism of algebras $\delta : A \to M(A) \otimes A$.*

Proof. We are going to construct the $\mathbb{K}$-algebra $M(A)$ quite explicitly. First we observe that $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ is a coalgebra (cf. problem A.6.8) with the structural morphism $\Delta : A^* \to (A \otimes A)^* \cong A^* \otimes A^*$. Denote the dual basis by $\sum_{i=1}^n a_i \otimes \bar{a}^i \in A \otimes A^*$. Now let $T(A \otimes A^*)$ be the tensor algebra of the vector space

$A \otimes A^*$. Consider elements of the tensor algebra

$$\begin{aligned}&xy \otimes \zeta \in A \otimes A^*,\\ &x \otimes y \otimes \Delta(\zeta) \in A \otimes A \otimes A^* \otimes A^* \cong A \otimes A^* \otimes A \otimes A^*,\\ &1 \otimes \zeta \in A \otimes A^*,\\ &\zeta(1) \in \mathbb{K}.\end{aligned}$$

The following elements

$$xy \otimes \zeta - x \otimes y \otimes \Delta(\zeta) \tag{1}$$

and

$$1 \otimes \zeta - \zeta(1) \tag{2}$$

generate a two-sided ideal $I \subseteq T(A \otimes A^*)$. Now we define

$$M(A) := T(A \otimes A^*)/I$$

and the cooperation $\delta : A \ni a \to \sum_{i=1}^n (a \otimes \bar{a}^i) \otimes a_i \in T(A \otimes A^*)/I \otimes A$. This is a well-defined linear map.

To show that this map is a homomorphism of algebras we first describe the multiplication of A by $a_i a_j = \sum_k \alpha_{ij}^k a_k$. Then the comultiplication of A^* is given by $\Delta(\bar{a}^k) = \sum_{ij} \alpha_{ij}^k \bar{a}^i \otimes \bar{a}^j$ since $(\Delta(\bar{a}^k), a_l \otimes a_m) = (\bar{a}^k, a_l a_m) = \sum_r \alpha_{lm}^r (\bar{a}^k, a_r) = \alpha_{lm}^k = \sum_{ij} \alpha_{ij}^k (\bar{a}^i, a_l)(\bar{a}^j, a_m) = (\sum_{ij} \alpha_{ij}^k \bar{a}^i \otimes \bar{a}^j, a_l \otimes a_m)$. Now write $1 = \sum \beta^k a_k$. Then we get $\epsilon(\bar{a}^i) = \beta^i$ since $\epsilon(\bar{a}^i) = (\bar{a}^i, 1) = \sum_j \beta^j (\bar{a}^i, a_j) = \beta^i$. So we have $\delta(a)\delta(b) = (\sum_{i=1}^n (a \otimes \bar{a}^i) \otimes a_i) \cdot (\sum_{j=1}^n (b \otimes \bar{a}^j) \otimes a_j) = \sum_{ij} (a \otimes b \otimes \bar{a}^i \otimes \bar{a}^j) \otimes a_i a_j = \sum_{ijk} \alpha_{ij}^k (a \otimes b \otimes \bar{a}^i \otimes \bar{a}^j) \otimes a_k = \sum_k (a \otimes b \otimes \Delta(\bar{a}^k)) \otimes a_k = \sum_k (ab \otimes \bar{a}^k) \otimes a_k = \delta(ab)$. Furthermore we have $\delta(1) = \sum_i (1 \otimes \bar{a}^i) \otimes a_i = \sum_i \bar{a}^i(1) \otimes a_i = 1 \otimes \sum_i \bar{a}^i(1) a_i = 1 \otimes 1$. Hence δ is a homomorphism of algebras.

Now we have to show that there is a unique g for each f. First of all $f : A \to B \otimes A$ induces uniquely determined linear maps $f_i : A \to B$ with $f(a) = \sum_i f_i(a) \otimes a_i$ since the a_i form a basis. Since f is a homomorphism of algebras we get from $\sum_k f_k(a) \otimes a_k = f(ab) = f(a)f(b) = \sum_{ij} (f_i(a) \otimes a_i)(f_j(b) \otimes a_j) = \sum_{ij} f_i(a) f_j(b) \otimes a_i a_j = \sum_{ijk} \alpha_{ij}^k f_i(a) f_j(b) \otimes a_k$ by comparison of coefficients

$$f_k(ab) = \sum_{ij} \alpha_{ij}^k f_i(a) f_j(b).$$

Furthermore we define $g(a \otimes \bar{a}) := (1 \otimes \bar{a}) f(a) \in B$. Then we get in particular $g(a \otimes \bar{a}^i) = (1 \otimes \bar{a}^i)(\sum_j f_j(a) \otimes a_j) = f_i(a)$. We can extend the map g to a homomorphism of algebras $g : T(A \otimes A^*) \to B$. Applied to the generators of the ideal we get $g(ab \otimes \bar{a}^k - a \otimes b \otimes \Delta(\bar{a}^k)) = (1 \otimes \bar{a}^k) \sum_l f_l(ab) \otimes a_l - \sum_{rsij} \alpha_{ij}^k (1 \otimes \bar{a}^i)(f_r(a) \otimes a_r) \cdot (1 \otimes \bar{a}^j)(f_s(b) \otimes a_s) = f_k(ab) - \sum_{ij} \alpha_{ij}^k f_i(a) f_j(b) = 0$. We have furthermore $g(1 \otimes \zeta - \zeta(1)) = (1 \otimes \zeta) f(1) - \zeta(1) = (1 \otimes \zeta)(1 \otimes 1) - \zeta(1) = 1\zeta(1) - \zeta(1) = 0$. Thus the homomorphism of algebras g vanishes on the ideal I so it may be factored through $M(A) = T(A)/I$. Denote this factorization also by g. Then the diagram commutes since $(g \otimes 1_A)\delta(a) = (g \otimes 1_A)(\sum_i (a \otimes \bar{a}^i) \otimes a_i) = \sum_i (1 \otimes \bar{a}^i) f(a) \otimes a_i = \sum_{ij} f_j(a)(\bar{a}^i, a_j) \otimes a_i = \sum_i f_i(a) \otimes a_i = f(a)$.

We still have to show that g is uniquely determined. Assume that we also have $(h \otimes 1_A)\delta = f$ then $\sum_i h(a \otimes \bar{a}^i) \otimes a_i = (h \otimes 1_A)\delta(a) = f(a) = \sum_i f_i(a) \otimes a_i$ hence $h(a \otimes \bar{a}^i) = f_i(a) = g(a \otimes \bar{a}^i)$, i.e. $g = h$. □

Definition 1.3.13. Let A be a $\mathbb{K}$-algebra. The universal algebra $M(A)$ for A that is a bialgebra is also called the *coendomorphism bialgebra* of A.

Problem 1.3.2.

1. Determine explicitly the dual coalgebra A^* of the algebra $A := \mathbb{K}\langle x \rangle/(x^2)$. (Hint: Find a basis for A.)
2. Determine and describe the coendomorphism bialgebra of A from problem 1.1. (Hint: Determine first a set of algebra generators of $M(A)$. Then describe the relations.)
3. Determine explicitly the dual coalgebra A^* of $A := \mathbb{K}\langle x \rangle/(x^3)$.
4. Determine and describe the coendomorphism bialgebra of A from problem 1.3.
5. (*) Determine explicitly the dual coalgebra A^* of $A := \mathbb{K}\langle x, y \rangle/I$ where the ideal I is generated as a two-sided ideal by the polynomials
$$xy - q^{-1}yx, x^2, y^2.$$
6. (*) Determine the coendomorphism bialgebra of A from problem 1.5.
7. Let A be a finite dimensional $\mathbb{K}$-algebra with universal bialgebra $A \longrightarrow B \otimes A$. Show
   i) that $A^{op} \longrightarrow B^{op} \otimes A^{op}$ is universal (where A^{op} has the multiplication $\nabla\tau : A \otimes A \longrightarrow A \otimes A \longrightarrow A$);
   ii) that $A \cong A^{op}$ implies $B \cong B^{op}$ (as bialgebras);
   iii) that for commutative algebras A the algebra B satisfies $B \cong B^{op}$ but that B need not be commutative.
   iv) Find an isomorphism $B \cong B^{op}$ for the bialgebra $B = \mathbb{K}\langle a, b \rangle/(a^2, ab+ba)$. (compare problem 1.2 2).