CHAPTER 1

Commutative and Noncommutative Algebraic Geometry

2. Quantum Spaces and Noncommutative Geometry

Now we come to noncommutative geometric spaces and their function algebras. Many of the basic principles of commutative algebraic geometry as introduced in 1.1 carry over to noncommutative geometry. Our main aim, however, is to study the symmetries (automorphisms) of noncommutative spaces which lead to the notion of a quantum group.

Since the construction of noncommutative geometric spaces has deep applications in theoretical physics we will also call these spaces quantum spaces.

Definition 1.2.1. Let A be a (not necessarily commutative) $\mathbb{K}$-algebra. Then the functor $\mathcal{X} := \mathbb{K}\text{-}\mathbf{Alg}(A, \text{-}) : \mathbb{K}\text{-}\mathbf{Alg} \to \mathbf{Set}$ represented by A is called *(affine) noncommutative (geometric) space* or *quantum space*. The elements of $\mathbb{K}\text{-}\mathbf{Alg}(A, B)$ are called *B-points* of $\mathcal{X}$. A *morphism of noncommutative spaces* $f : \mathcal{X} \to \mathcal{Y}$ is a natural transformation.

This definition implies immediately

Corollary 1.2.2. *The noncommutative spaces form a category* **QS** *that is dual to the category of* $\mathbb{K}$*-algebras.*

Remark 1.2.3. Thus one often calls the dual category $\mathbb{K}\text{-}\mathbf{Alg}^{op}$ category of noncommutative spaces.

If A is a finitely generated algebra then it may be considered as a residue class algebra $A \cong \mathbb{K}\langle x_1, \ldots, x_n\rangle / I$ of a polynomial algebra in noncommuting variables (cf. A.6). If $I = (p_1(x_1, \ldots, x_n), \ldots, p_m(x_1, \ldots, x_n))$ is the two-sided ideal generated by the polynomials $p_1, \ldots, p_m$ then the sets $\mathbb{K}\text{-}\mathbf{Alg}(A, B)$ can be considered as sets of zeros of these polynomials in B^n. In fact, we have $\mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}\langle x_1, \ldots, x_n\rangle, B) \cong \mathrm{Map}(\{x_1, \ldots, x_n\}, B) = B^n$. Thus $\mathbb{K}\text{-}\mathbf{Alg}(A, B)$ can be considered as the set of those homomorphisms of algebras from $\mathbb{K}\langle x_1, \ldots, x_n\rangle$ to B that vanish on the ideal I or as the set of zeros of these polynomials in B^n.

Similar to Theorem 1.1.13 one shows also in the noncommutative case that morphisms between noncommutative spaces are described by polynomials.

The Theorem 1.1.11 on the operation of the affine algebra $A = \mathcal{O}(\mathcal{X})$ on $\mathcal{X}$ as function algebra can be carried over to the noncommutative case as well: the natural transformation $\psi(B) : A \times \mathcal{X}(B) \to B$ (natural in B) is given by $\psi(B)(a, p) := p(a)$ and comes from the isomorphism $A \cong \mathrm{Nat}(\mathcal{X}, \mathbb{A})$.

Now we come to a claim on the function algebra A that we did not prove in the commutative case, but that holds in the commutative as well as in the noncommutative situation.

Lemma 1.2.4. *Let D be a set and $\phi : D \times \mathcal{X}(\text{-}) \longrightarrow \mathbb{A}(\text{-})$ be a natural transformation. Then there exists a unique map $f : D \longrightarrow A$ such that the diagram*

$$\begin{tikzcd} D \times \mathcal{X}(B) \arrow[d, "f\times 1"'] \arrow[dr, "\phi(B)"] & \\ A \times \mathcal{X}(B) \arrow[r, "\psi(B)"'] & B \end{tikzcd}$$

commutes.

Proof. Let $\phi : D \times \mathcal{X} \longrightarrow \mathbb{A}$ be given. We first define a map $f' : D \longrightarrow \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ by $f'(d)(B)(p) := \phi(B)(d,p)$.

We claim that $f'(d) : \mathcal{X} \longrightarrow \mathbb{A}$ is a natural transformation. Observe that the diagram

$$\begin{tikzcd} D \times \mathcal{X}(B) \arrow[r, "\phi(B)"] \arrow[d, "D \times \mathcal{X}(g)"'] & \mathbb{A}(B) = B \arrow[d, "g"] \\ D \times \mathcal{X}(B') \arrow[r, "\phi(B')"'] & \mathbb{A}(B') = B' \end{tikzcd}$$

commutes for any $g : B \longrightarrow B'$, since ϕ is a natural transformation. Thus the diagram

$$\begin{tikzcd} \mathcal{X}(B) \arrow[r, "f'(d)(B)"] \arrow[d, "\mathcal{X}(g)"'] & \mathbb{A}(B) = B \arrow[d, "g"] \\ \mathcal{X}(B') \arrow[r, "f'(d)(B')"'] & \mathbb{A}(B') = B' \end{tikzcd}$$

commutes since

$$\begin{aligned} (g \circ f'(d)(B))(p) &= (g \circ \phi(B))(d,p) \\ &= \phi(B') \circ (1 \times \mathcal{X}(g))(d,p) \\ &= \phi(B')(d, \mathcal{X}(g)(p)) \\ &= f'(d)(B')(\mathcal{X}(g)(p)). \end{aligned}$$

Hence $f'(d) \in \mathrm{Nat}(X, \mathbb{A})$ and $f' : D \longrightarrow \mathrm{Nat}(\mathcal{X}, \mathbb{A})$.

Now we define $f : D \longrightarrow A$ as $D \xrightarrow{f'} \mathrm{Nat}(\mathcal{X}, \mathbb{A}) \cong A$. By using the isomorphism from 1.1.11 we get $f(d) = f'(d)(A)(1)$. (Actually we get $f(d) = f'(d)(A)(1)(x)$ but we identify $\mathbb{A}(B)$ and B by $\mathbb{A}(B) \ni p \mapsto p(x) \in B$.)

Then we get

$$\begin{aligned}\psi(B)(f \times 1)(d,p) &= \psi(B)(f(d),p)\\ &= \psi(B)(f'(d)(A)(1)(x),p) \text{ (by definition of } f)\\ &= p \circ f'(d)(A)(1) \text{ (since we may omit } x)\\ &= p \circ \phi(A)(d,1) \text{ (by definition of } f')\\ &= \phi(B)(D \times \mathcal{X}(p))(d,1) \text{ (since } \phi \text{ is a natural transformation)}\\ &= \phi(B)(d,p).\end{aligned}$$

Hence the diagram in the Lemma commutes.

To show the uniqueness of f let $g : D \longrightarrow A$ be a homomorphism such that $\psi(B)(g \times 1) = \phi(B)$. Then we have

$$f(d) = f'(d)(A)(1) = \phi(A)(d,1) = \psi(A)(g\times 1)(d,1) = \psi(A)(f(d),1) = 1\circ g(d) = g(d)$$

hence $f = g$. $\square$

Problem 1.2.1. ***Definition:*** Let D be an algebra. A natural transformation $\phi : D \times \mathcal{X} \longrightarrow \mathbb{A}$ is called an *algebra action* if $\phi(B)(\text{-},p) : D \longrightarrow \mathbb{A}(B) = B$ is an algebra homomorphism for all B and all $p \in \mathcal{X}(B)$.

Lemma: The natural transformation $\psi : A \times \mathcal{X} \longrightarrow \mathbb{A}$ is an algebra action.

Theorem: Let D be an algebra and $\phi : D \times \mathcal{X}(\text{-}) \longrightarrow \mathbb{A}(\text{-})$ be an algebra action. Then there exists a unique algebra homomorphism $f : D \longrightarrow A$ such that the diagram

$$\begin{tikzcd} D \times \mathcal{X}(B) \arrow[d, "f\times 1"'] \arrow[dr, "\phi(B)"] & \\ A \times \mathcal{X}(B) \arrow[r, "\psi(B)"'] & B \end{tikzcd}$$

commutes.

Definition 1.2.5. The noncommutative space $\mathbb{A}_q^{2|0}$ with the function algebra

$$\mathcal{O}(\mathbb{A}_q^{2|0}) := \mathbb{K}\langle x,y\rangle/(xy - q^{-1}yx)$$

with $q \in \mathbb{K}\setminus\{0\}$ is called the *(deformed) quantum plane.* The noncommutative space $\mathbb{A}_q^{0|2}$ with the function algebra

$$\mathcal{O}(\mathbb{A}_q^{0|2}) := \mathbb{K}\langle \xi,\eta\rangle/(\xi^2,\eta^2,\xi\eta + q\eta\xi)$$

is called the *dual (deformed) quantum plane.* We have

$$\mathbb{A}_q^{2|0}(A) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x,y \in A; xy = q^{-1}yx \right\}$$

and

$$\mathbb{A}_q^{0|2}(A) = \left\{ \begin{pmatrix} \xi, & \eta \end{pmatrix} \middle| \xi,\eta \in A; \xi^2 = 0, \eta^2 = 0, \xi\eta = -q\eta\xi \right\}.$$

Definition 1.2.6. Let $\mathcal{X}$ be a noncommutative space with function algebra A and let $\mathcal{X}_c$ be the restriction of the functor $\mathcal{X} : \mathbb{K}\text{-}\mathbf{Alg} \to \mathbf{Set}$ to the category of commutative algebras: $\mathcal{X}_c : \mathbb{K}\text{-}\mathbf{cAlg} \to \mathbf{Set}$. Then we call $\mathcal{X}_c$ the *commutative part* of the noncommutative space $\mathcal{X}$.

Lemma 1.2.7. *The commutative part $\mathcal{X}_c$ of a noncommutative space $\mathcal{X}$ is an affine variety.*

PROOF. The underlying functor $\mathbb{A} : \mathbb{K}\text{-}\mathbf{cAlg} \to \mathbb{K}\text{-}\mathbf{Alg}$ has a left adjoint functor $\mathbb{K}\text{-}\mathbf{Alg} \ni A \mapsto A/[A,A] \in \mathbb{K}\text{-}\mathbf{cAlg}$ where $[A,A]$ denotes the two-sided ideal of A generated by the elements $ab - ba$. In fact for each homomorphism of algebras $f : A \to B$ with a commutative algebra B there is a factorization through $A/[A,A]$ since f vanishes on the elements $ab - ba$.

Hence if $A = \mathcal{O}(\mathcal{X})$ is the function algebra of $\mathcal{X}$ then $A/[A,A]$ is the representing algebra for $\mathcal{X}_c$. $\square$

Remark 1.2.8. For any commutative algebra (of coefficients) B the spaces $\mathcal{X}$ and $\mathcal{X}_c$ have the same B-points: $\mathcal{X}(B) = \mathcal{X}_c(B)$. The two spaces differ only for noncommutative algebras of coefficients. In particular for commutative fields B as algebras of coefficients the quantum plane $\mathbb{A}_q^{2|0}$ has only B-points on the two axes since the function algebra $\mathbb{K}\langle x, y\rangle/(xy - q^{-1}yx, xy - yx) \cong K[x,y]/(xy)$ defines only B-points (b_1, b_2) where at least one of the coefficients is zero.

Problem 1.2.2. Let S_3 be the symmetric group and $A := \mathbb{K}[S_3]$ be the group algebra on S_3. Describe the points of $\mathcal{X}(B) = \mathbb{K}\text{-}\mathbf{Alg}(A, B)$ as a subspace of $\mathbb{A}^2(B)$. What is $\mathcal{X}_c(B)$ and what is the affine algebra of $\mathcal{X}_c$?

To understand how Hopf algebras fit into the context of noncommutative spaces we have to better understand the tensor product in $\mathbb{K}\text{-}\mathbf{Alg}$.

Definition 1.2.9. Let $A = \mathcal{O}(\mathcal{X})$ and $A' = \mathcal{O}(\mathcal{Y})$ be the function algebras of the noncommutative spaces $\mathcal{X}$ resp. $\mathcal{Y}$. Two B-points $p : A \to B$ in $\mathcal{X}(B)$ and $p' : A' \to B$ in $\mathcal{Y}(B)$ are called *commuting points* if we have for all $a \in A$ and all $a' \in A'$

$$p(a)p'(a') = p'(a')p(a),$$

i.e. if the images of the two homomorphisms p and p' commute.

Remark 1.2.10. To show that the points p and p' commute, it is sufficient to check that the images of the algebra generators $p(x_1), \ldots, p(x_m)$ commute with the images of the algebra generators $p'(y_1), \ldots, p'(y_n)$ under the multiplication. This means that we have

$$b_i b'_j = b'_j b_i$$

for the B-points $(b_1, \ldots, b_m) \in \mathcal{X}(B)$ and $(b'_1, \ldots, b'_n) \in \mathcal{Y}(B)$.

Definition 1.2.11. The functor

$$(\mathcal{X} \perp \mathcal{Y})(B) := \{(p, p') \in \mathcal{X}(B) \times \mathcal{Y}(B) | p, p' \text{ commute}\}$$

is called the *orthogonal product* of the noncommutative spaces $\mathcal{X}$ and $\mathcal{Y}$.

Remark 1.2.12. Together with $\mathcal{X}$ and $\mathcal{Y}$ the orthogonal product $\mathcal{X} \perp \mathcal{Y}$ is again a functor, since homomorphisms $f : B \longrightarrow B'$ are compatible with the multiplication and thus preserve commuting points. Hence $\mathcal{X} \perp \mathcal{Y}$ is a subfunctor of $\mathcal{X} \times \mathcal{Y}$.

Lemma 1.2.13. *If $\mathcal{X}$ and $\mathcal{Y}$ are noncommutative spaces, then $\mathcal{X} \perp \mathcal{Y}$ is a noncommutative space with function algebra $\mathcal{O}(\mathcal{X} \perp \mathcal{Y}) = \mathcal{O}(\mathcal{X}) \otimes \mathcal{O}(\mathcal{Y})$.*

If $\mathcal{X}$ and $\mathcal{Y}$ have finitely generated function algebras then the function algebra of $\mathcal{X} \perp \mathcal{Y}$ is also finitely generated.

PROOF. Let $A := \mathcal{O}(\mathcal{X})$ and $A' := \mathcal{O}(\mathcal{Y})$. Let $(p, p') \in (\mathcal{X} \perp \mathcal{Y})(B)$ be a pair of commuting points. Then there is a unique homomorphism of algebras $h : A \otimes A' \longrightarrow B$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[dr, "p"'] & A \otimes A' \arrow[d, "h"] & A' \arrow[l, "\iota'"'] \arrow[dl, "p'"] \\ & B. & \end{tikzcd}$$

Define $h(a \otimes a') := p(a)p'(a')$ and check the necessary properties. Observe that for an arbitrary homomorphism of algebras $h : A \otimes A' \longrightarrow B$ the images of elements of the form $a \otimes 1$ and $1 \otimes a'$ commute since these elements already commute in $A \otimes A'$. Thus we have

$$(\mathcal{X} \perp \mathcal{Y})(B) \cong \mathbb{K}\text{-}\mathbf{Alg}(A \otimes A', B).$$

If the algebra A is generated by the elements $a_1, \ldots, a_m$ and the algebra A' is generated by the elements $a'_1, \ldots, a'_n$ then the algebra $A \otimes A'$ is generated by the elements $a_i \otimes 1$ and $1 \otimes a'_j$. $\square$

Proposition 1.2.14. *The orthogonal product of noncommutative spaces is associative, i.e. for noncommutative spaces $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ we have*

$$(\mathcal{X} \perp \mathcal{Y}) \perp \mathcal{Z} \cong \mathcal{X} \perp (\mathcal{Y} \perp \mathcal{Z}).$$

PROOF. Let B be a coefficient algebra and let $p_x \in \mathcal{X}(B)$, $p_y \in \mathcal{Y}(B)$, and $p_z \in \mathcal{Z}(B)$ be points such that $((p_x, p_y), p_z)$ is a pair of commuting points in $((\mathcal{X} \perp \mathcal{Y}) \perp \mathcal{Z})(B)$. In particular (p_x, p_y) is also a pair of commuting points. Thus we have for all $a \in A := \mathcal{O}(\mathcal{X})$, $a' \in A' := \mathcal{O}(\mathcal{Y})$, and $a'' \in A'' := \mathcal{O}(\mathcal{Z})$

$$p_x(a)p_y(a')p_z(a'') = (p_x, p_y)(a \otimes a')p_z(a'') = p_z(a'')(p_x, p_y)(a \otimes a') = p_z(a'')p_x(a)p_y(a')$$

and

$$p_x(a)p_y(a') = p_y(a')p_x(a).$$

If we choose $a = 1$ then we get $p_y(a')p_z(a'') = p_z(a'')p_y(a')$. For arbitrary a, a', a'' we then get

$$p_x(a)p_y(a')p_z(a'') = p_z(a'')p_x(a)p_y(a') = p_z(a'')p_y(a')p_x(a) = p_y(a')p_z(a'')p_x(a)$$

hence (p_y, p_z) and $(p_x, (p_y, p_z))$ are also pairs of commuting points. $\square$

Problem 1.2.3. Show that the orthogonal product of quantum spaces $\mathcal{X} \perp \mathcal{Y}$ is a tensor product for the category **QS** (in the sense of monoidal categories – if you know already what that is).