Problem set for
Quantum Groups and Noncommutative Geometry

(41) Let (M^*, ev) be a left dual for M. Show that there is a *unique* morphism $\mathrm{db} : I \longrightarrow M \otimes M^*$ satisfying the conditions

$$(M \xrightarrow{\mathrm{db}\otimes 1} M \otimes M^* \otimes M \xrightarrow{1\otimes \mathrm{ev}} M) = 1_M$$
$$(M^* \xrightarrow{1\otimes \mathrm{db}} M^* \otimes M \otimes M^* \xrightarrow{\mathrm{ev}\otimes 1} M^*) = 1_{M^*}.$$

(Uniqueness of the dual basis.)

(42) Let B be the bialgebra $\mathbb{K}\langle x, y\rangle/I$, where I is generated by $x^2, xy+yx$, with the diagonal $\Delta(y) = y\otimes y$, $\Delta(x) = x\otimes 1+y\otimes x$, and the counit $\varepsilon(y) = 1, \varepsilon(x) = 0$.

A *chain complex* has the form

$$M = (\ldots \xrightarrow{\partial_3} M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0)$$

with $\partial_{n-1}\partial_n = 0$.

Show that the category $\mathbb{K}$-$\mathcal{C}omp$ of chain complexes is equivalent to the category B-$\mathcal{C}omod$ of B-comodules.

Use the following construction. If M is a chain complex then define a B-comodule on $M = \oplus_{i\in\mathbb{N}} M_i$ with the structure map $\delta : M \longrightarrow B \otimes M$, $\delta(m) := y^i \otimes m + xy^{i-1} \otimes \partial_i(m)$ for all $m \in M_i$ and for all $i \in \mathbb{N}$ resp. $\delta(m) := 1 \otimes m$ for $m \in M_0$. Conversely if $M, \delta : M \longrightarrow B \otimes M$ is a B-comodule then we define $\mathbb{K}$-modules $M_i := \{m \in M | \exists m' \in M[\delta(m) = y^i \otimes m + xy^{i-1} \otimes m']\}$ and $\mathbb{K}$-linear maps $\partial_i : M_i \longrightarrow M_{i-1}$ by $\partial_i(m) := m'$ for $\delta(m) = y^i \otimes m+xy^{i-1} \otimes m'$. Check that this defines an equivalence of categories.

(Hint: Let $m \in M \in B$-$\mathcal{C}omod$. Since y^i, xy^i form a basis of B we have $\delta(m) = \sum_i y^i \otimes m_i + \sum_i xy^i \otimes m'_i$. We apply to this the equation $(1 \otimes \delta)\delta = (\Delta \otimes 1)\delta$ and compare coefficients to get

$$\delta(m_i) = y^i \otimes m_i + xy^{i-1} \otimes m'_{i-1}, \quad \delta(m'_i) = y^i \otimes m'_i$$

for all $i \in \mathbb{N}_0$ (with $m'_{-1} = 0$). Consequently for each $m_i \in M_i$ there is exactly one $\partial(m_i) = m'_{i-1} \in M$ such that

$$\delta(m_i) = y^i \otimes m_i + xy^{i-1} \otimes \partial(m_i).$$

Since $\delta(m'_{i-1}) = y^{i-1} \otimes m'_{i-1}$ for all $i \in \mathbb{N}$ we see that $\partial(m_i) \in M_{i-1}$. So we have defined $\partial : M_i \longrightarrow M_{i-1}$. Furthermore we see from this equation that $\partial^2(m_i) = 0$ for all $i \in \mathbb{N}$. Hence we have obtained a chain complex from (M, δ).

If we apply $(\epsilon \otimes 1)\delta(m) = m$ then we get $m = \sum m_i$ with $m_i \in M_i$ hence $M = \bigoplus_{i \in \mathbb{N}} M_i$. This together with the inverse construction leads to the required equivalence.)

(43) Find an example of an object M in a monoidal category $\mathcal{C}$ that has a left dual but no right dual.

(44) (a) In the category of $\mathbb{N}$-graded vector spaces determine all objects M that have a left dual.
  (b) In the category of chain complexes $\mathbb{K}$-$\mathcal{C}omp$ determine all objects M that have a left dual.
  (c) In the category of cochain complexes $\mathbb{K}$-$\mathcal{C}ocomp$ determine all objects M that have a left dual.

Due date: Tuesday, 09.07.2002, 16:15 in Lecture Hall E41