Lectures on

QUANTUM GROUPS AND NONCOMMUTATIVE GEOMETRY

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CHAPTER 1

Commutative and Noncommutative Algebraic Geometry

Introduction

Throughout we will fix a base field $\mathbb{K}$. The reader may consider it as real numbers or complex numbers or any other of his most favorite fields.

A fundamental and powerful tool for geometry is to associate with each space X the algebra of functions $\mathcal{O}(X)$ from X to the base field (of coefficients). The dream of geometry is that this construction is bijective, i.e. that two different spaces are mapped to two different function algebras and that each algebra is the function algebra of some space.

Actually the spaces and the algebras will form a category. There are admissible maps. For algebras it is quite clear what these maps will be. For spaces this is less obvious, partly due to the fact that we did not say clearly what spaces exactly are. Then the *dream of geometry* would be that these two categories, the category of (certain) spaces and the category of (certain) algebras, are dual to each other.

Algebraic geometry, noncommutative geometry, and theoretical physics have as a basis this fundamental idea, the duality of two categories, the category of spaces (state spaces in physics) and the category of function algebras (algebras of observables) in physics. We will present this duality in the 1. chapter. Certainly the type of spaces as well as the type of algebras will have to be specified.

Theoretical physics uses the categories of locally compact Hausdorff spaces and of commutative C^*-algebras. A famous theorem of Gelfand-Naimark says that these categories are duals of each other.

(Affine) algebraic geometry uses a duality between the categories of affine algebraic schemes and of (reduced) finitely generated commutative algebras.

To get the whole framework of algebraic geometry one needs to go to more general spaces by patching affine spaces together. On the algebra side this amounts to considering sheaves of commutative algebras. We shall not pursue this more general approach to algebraic geometry, since generalizations to noncommutative geometry are still in the state of development and incomplete.

Noncommutative geometry uses either (imaginary) noncommutative spaces and not necessarily commutative algebras or (imaginary) noncommutative spaces and not necessarily commutative C^*-algebras.

We will take an approach to the duality between geometry and algebra that heavily uses functorial tools, especially representable functors. The affine (algebraic) spaces we use will be given in the form of sets of common zeros of certain polynomials, where the zeros can be taken in arbitrary (commutative) $\mathbb{K}$-algebras B. So an affine space will consist of many different sets of zeros, depending on the choice of the coefficient algebra B.

We first give a short introduction to commutative algebraic geometry in this setup and develop a duality between the category of affine (algebraic) spaces and the category of (finitely generated) commutative algebras.

Then we will transfer it to the noncommutative situation. The functorial approach to algebraic geometry is not too often used but it lends itself particularly well to the study of the noncommutative situation. Even in that situation one obtains space-like objects.

The chapter will close with a first step to construct automorphism "groups" of noncommutative spaces. Since the construction of inverses presents special problems we will only construct endomorphism "monoids" in this chapter and postpone the study of invertible endomorphisms or automorphisms to the next chapter.

At the end of the chapter you should

- know how to construct an affine scheme from a commutative algebra,
- know how to construct the function algebra of an affine scheme,
- know what a noncommutative space is and know examples of such,
- understand and be able to construct endomorphism quantum monoids of certain noncommutative spaces,
- understand, why endomorphism quantum monoids are not made out of endomorphisms of a noncommutative space.

1. The Principles of Commutative Algebraic Geometry

We will begin with simplest form of (commutative) geometric spaces and see a duality between these very simple "spaces" and certain commutative algebras. This example will show how the concept of a function algebra can be used to fulfill the dream of geometry in this situation. It will also show the functorial methods that will be applied throughout this text. It is a particularly simple example of a duality as mentioned in the introduction. This example will not be used later on, so we will only sketch the proofs for some of the statements.

Example 1.1.1. Consider a set of points without any additional geometric structure. So the geometric space is just a set. We introduce the notion of its algebra of functions.

Let X be a set. Then $\mathbb{K}^X := \mathrm{Map}(X, \mathbb{K})$ is a $\mathbb{K}$-algebra with componentwise addition and multiplication: $(f+g)(x) := f(x) + g(x)$ and $(fg)(x) := f(x)g(x)$. We study this fact in more detail.

The set $\mathbb{K}^X$ considered as a vector space with the addition $(f+g)(x) := f(x)+g(x)$ and the scalar multiplication $(\alpha f)(x) := \alpha f(x)$ defines a representable contravariant functor

$$\mathbb{K}^{-} : \mathcal{S}et \longrightarrow \mathcal{V}ec\,.$$

This functor is a representable functor represented by $\mathbb{K}$. In fact $\mathbb{K}^h : \mathbb{K}^Y \longrightarrow \mathbb{K}^X$ is a linear map for every map $h : X \longrightarrow Y$ since $\mathbb{K}^h(\alpha f + \beta g)(x) = (\alpha f + \beta g)(h(x)) = \alpha f(h(x)) + \beta g(h(x)) = (\alpha f h + \beta g h)(x) = (\alpha \mathbb{K}^h(f) + \beta \mathbb{K}^h(g))(x)$ hence $\mathbb{K}^h(\alpha f + \beta g) = \alpha \mathbb{K}^h(f) + \beta \mathbb{K}^h(g)$.

Consider the homomorphism $\tau : \mathbb{K}^X \otimes \mathbb{K}^Y \longrightarrow \mathbb{K}^{X \times Y}$, defined by $\tau(f \otimes g)(x,y) := f(x)g(y)$. In order to obtain a unique homomorphism τ defined on the tensor product we have to show that $\tau' : \mathbb{K}^X \times \mathbb{K}^Y \longrightarrow \mathbb{K}^{X \times Y}$ is a bilinear map : $\tau'(f + f', g)(x,y) = (f + f')(x)g(y) = (f(x) + f'(x))g(y) = f(x)g(y) + f'(x)g(y) = (\tau'(f,g) + \tau'(f',g))(x,y)$ gives the additivity in the left hand argument. The additivity in the right hand argument and the bilinearity is checked similarly. One can check that τ is always injective. If X or Y are finite then τ is bijective.

As a special example we obtain a multiplication $\nabla : \mathbb{K}^X \otimes \mathbb{K}^X \xrightarrow{\tau} \mathbb{K}^{X \times X} \xrightarrow{\mathbb{K}^{\Delta}} \mathbb{K}^X$ where $\Delta : X \longrightarrow X \times X$ in $\mathcal{S}et$ is the diagonal map $\Delta(x) := (x,x)$. Furthermore we get a unit $\eta : \mathbb{K}^{\{*\}} \xrightarrow{\mathbb{K}^{\epsilon}} \mathbb{K}^X$ where $\epsilon : X \longrightarrow \{*\}$ is the unique map into the one element set. One verifies easily that $(\mathbb{K}^X, \eta, \nabla)$ is a $\mathbb{K}$-algebra. Two properties are essential here, the associativity and the unit of $\mathbb{K}$ and the fact that (X, Δ, ϵ) is a "comonoid" in the category $\mathcal{S}et$:

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X \times X \arrow[d, "1 \times \Delta"] \\
X \times X \arrow[r, "\Delta \times 1"'] & X \times X \times X
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] \arrow[drr, "1_X"] & & X \times X \arrow[d, "1 \times \epsilon"] \\
X \times X \arrow[r, "\epsilon \times 1"'] & \{*\} \times X \cong & X \cong X \times \{*\}.
\end{tikzcd}$$

Since $\mathbb{K}^{-}$ is a functor these two diagrams carry over to the category $\mathcal{V}ec$ and produce the required diagrams for a $\mathbb{K}$-algebra.

For a map $f : X \longrightarrow Y$ we obtain a homomorphism of algebras $\mathbb{K}^f : \mathbb{K}^Y \longrightarrow \mathbb{K}^X$ because the diagrams

$$\begin{tikzcd}
\mathbb{K}^Y \otimes \mathbb{K}^Y \arrow[r, "\tau"] \arrow[d, "\mathbb{K}^f \otimes \mathbb{K}^f"] & \mathbb{K}^{Y \times Y} \arrow[r, "\mathbb{K}^{\Delta}"] \arrow[d, "\mathbb{K}^{f \times f}"] & \mathbb{K}^Y \arrow[d, "\mathbb{K}^f"] \\
\mathbb{K}^X \otimes \mathbb{K}^X \arrow[r, "\tau"] & \mathbb{K}^{X \times X} \arrow[r, "\mathbb{K}^{\Delta}"] & \mathbb{K}^X
\end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small] & \mathbb{K}^{\{*\}} \cong \mathbb{K} \arrow[dl, "\eta"'] \arrow[dr, "\eta"] & \\ \mathbb{K}^{Y} \arrow[rr, "\mathbb{K}^{f}"] & & \mathbb{K}^{X} \end{tikzcd}$$

commute.

Thus

$$\mathbb{K}^{-} : \mathcal{S}et \longrightarrow \mathbb{K}\text{-}c\mathcal{A}lg$$

is a contravariant functor.

By the definition of the set-theoretic (cartesian) product we know that $\mathbb{K}^X = \prod_X \mathbb{K}$. This identity does not only hold on the set level, it holds also for the algebra structures on $\mathbb{K}^X$ resp. $\prod_X \mathbb{K}$.

We now construct an inverse functor

$$\operatorname{Spec} : \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{S}et.$$

For each point $x \in X$ there is a maximal ideal m_x of $\prod_X \mathbb{K}$ defined by $m_x := \{f \in \operatorname{Map}(X, \mathbb{K}) | f(x) = 0\}$. If X is a finite set then these are exactly all maximal ideals of $\prod_X \mathbb{K}$. To show this we observe the following. The surjective homomorphism $p_x : \prod_X \mathbb{K} \longrightarrow \mathbb{K}$ has kernel m_x hence m_x is a maximal ideal. If $m \subseteq \prod_X \mathbb{K}$ is a maximal ideal and $a = (\alpha_1, \ldots, \alpha_n) \in m$ then for any $\alpha_i \neq 0$ we get $(0, \ldots, 0, 1_i, 0, \ldots, 0) = (0, \ldots, 0, \alpha_i^{-1}, 0, \ldots, 0)(\alpha_1, \ldots, \alpha_n) \in m$ hence the i-th factor $0 \times \ldots \times \mathbb{K} \times \ldots \times 0$ of $\prod_X \mathbb{K}$ is in m. So the elements $a \in m$ must have at least one common component $\alpha_j = 0$ since $m \neq \mathbb{K}$. But more than one such a component is impossible since we would get zero divisors in the residue class algebra. Thus $m = m_x$ where $x \in X$ is the j-th elements of the set.

One can easily show more namely that the ideals m_x are precisely all prime ideals of $\operatorname{Map}(X, \mathbb{K})$.

With each commutative algebra A we can associate the set $\operatorname{Spec}(A)$ of all prime ideals of A. That defines a functor Spec: $\mathbb{K}\text{-}\mathcal{A}lg \longrightarrow \mathcal{S}et$. Applied to algebras of the form $\mathbb{K}^X = \prod_X \mathbb{K}$ with a finite set X this functor recovers X as $X \cong \operatorname{Spec}(\mathbb{K}^X)$. Thus the dream of geometry is satisfied in this particular example.

The above example shows that we may hope to gain some information on the space (set) X by knowing its algebra of functions $\mathbb{K}^X$ and applying the functor Spec to it. For finite sets and certain algebras the functors $\mathbb{K}^{-}$ and Spec actually define a category duality. We are going to expand this duality to larger categories.

We shall carry some geometric structure into the sets X and will study the connection between these geometric spaces and their algebras of functions. For this purpose we will describe sets of points by their coordinates. Examples are the circle or the parabola. More generally the geometric spaces we are going to consider are so called affine schemes described by polynomial equations. We will see that such geometric spaces are completely described by their algebras of functions. Here the Yoneda Lemma will play a central rôle.

We will, however, take a different approach to functions algebras and geometric spaces, than one does in algebraic geometry. We use the functorial approach, which lends itself to an easier access to the principles of noncommutative geometry. We will define geometric spaces as certain functors from the category of commutative algebras to the category of sets. These sets will have a strong geometrical meaning. The functors will associate with each algebra A the set of points of a "geometric variety", where the points have coordinates in the algebra A.

Definition 1.1.2. The functor $\mathbb{A} = \mathbb{A}^1 : \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{S}et$ (the underlying functor) that associates with each commutative $\mathbb{K}$-algebra A its space (set) of points (elements) A is called the *affine line*.

Lemma 1.1.3. *The functor "affine line" is a representable functor.*

Proof. By Lemma 2.3.5 the representing object is $\mathbb{K}[x]$. Observe that it is unique up to isomorphism. $\square$

Definition 1.1.4. The functor $\mathbb{A}^2 : \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{S}et$ that associates with each commutative algebra A the space (set) of points (elements) of the plane A^2 is called the *affine plane*.

Lemma 1.1.5. *The functor "affine plane" is a representable functor.*

Proof. Similar to Lemma 2.3.9 the representing object is $\mathbb{K}[x_1, x_2]$. This algebra is unique up to isomorphism. $\square$

Let $p_1(x_1, \ldots, x_n), \ldots, p_m(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n]$ be a family of polynomials. We want to consider the (geometric) variety of zeros of these polynomials. Observe that $\mathbb{K}$ may not contain sufficiently many zeros for these polynomials. Thus we are going to admit zeros in extension fields of $\mathbb{K}$ or more generally in arbitrary commutative $\mathbb{K}$-algebras.

In the following rather simple buildup of commutative algebraic geometry, the reader should carefully verify in which statements and proofs the commutativity is really needed. Most of the following will be verbally generalized to not necessarily commutative algebras.

Definition 1.1.6. Given a set of polynomials $\{p_1, \ldots, p_m\} \subseteq \mathbb{K}[x_1, \ldots, x_n]$. The functor $\mathcal{X}$ that associates with each commutative algebra A the set $\mathcal{X}(A)$ of zeros of the polynomials (p_i) in A^n is called an *affine algebraic variety* or an *affine scheme* (in $\mathbb{A}^n$) with defining polynomials $p_1, \ldots, p_m$. The elements in $\mathcal{X}(A)$ are called the *A-points* of $\mathcal{X}$.

Theorem 1.1.7. *The affine scheme $\mathcal{X}$ in $\mathbb{A}^n$ with defining polynomials $p_1, \ldots, p_m$ is a representable functor with representing algebra*

$$\mathcal{O}(\mathcal{X}) := \mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m),$$

called the affine algebra *of the functor* $\mathcal{X}$.

PROOF. First we show that the affine scheme $\mathcal{X} : \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{S}et$ with the defining polynomials $p_1, \ldots, p_m$ is a functor. Let $f : A \longrightarrow B$ be a homomorphism of commutative algebras. The induced map $f^n : A^n \longrightarrow B^n$ defined by application of f on the components restricts to $\mathcal{X}(A) \subseteq A^n$ as $\mathcal{X}(f) : \mathcal{X}(A) \longrightarrow \mathcal{X}(B)$. This map is well-defined for let $(a_1, \ldots, a_n) \in \mathcal{X}(A)$ be a zero for all polynomials $p_1, \ldots, p_m$ then $p_i(f(a_1), \ldots, f(a_n)) = f(p_i(a_1, \ldots, a_n)) = f(0) = 0$ for all i hence $f^n(a_1, \ldots, a_n) = (f(a_1), \ldots, f(a_n)) \in B^n$ is a zero for all polynomials. Thus $\mathcal{X}(f) : \mathcal{X}(A) \longrightarrow \mathcal{X}(B)$ is well-defined. Functoriality of $\mathcal{X}$ is clear now.

Now we show that $\mathcal{X}$ is representable by $\mathcal{O}(\mathcal{X}) = \mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m)$. Observe that $(p_1, \ldots, p_m)$ denotes the (two-sided) ideal in $\mathbb{K}[x_1, \ldots, x_n]$ generated by the polynomials $p_1, \ldots, p_m$. We know that each n-tupel $(a_1, \ldots, a_n) \in A^n$ uniquely determines an algebra homomorphism $f : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow A$ by $f(x_1) = a_1, \ldots, f(x_n) = a_n$. (The polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ in $\mathbb{K}\text{-}c\mathcal{A}lg$ is free over the set $\{x_1, \ldots, x_n\}$, or $\mathbb{K}[x_1, \ldots, x_n]$ together with the embedding $\iota : \{x_1, \ldots, x_n\} \longrightarrow \mathbb{K}[x_1, \ldots, x_n]$ is a couniversal solution of the problem given by the underlying functor $\mathbb{A} : \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{S}et$ and the set $\{x_1, \ldots, x_n\} \in \mathcal{S}et$.) This homomorphism of algebras maps polynomials $p(x_1, \ldots, x_n)$ into $f(p) = p(a_1, \ldots, a_n)$. Hence $(a_1, \ldots, a_n)$ is a common zero of the polynomials $p_1, \ldots, p_m$ if and only if $f(p_i) = p_i(a_1, \ldots, a_n) = 0$, i.e. $p_1, \ldots, p_m$ are in the kernel of f. This happens if and only if f vanishes on the ideal $(p_1, \ldots, p_m)$ or in other word can be factorized through the residue class map

$$\nu : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow \mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m)$$

This induces a bijection

$$\mathrm{Mor}_{\mathbb{K}\text{-}c\mathcal{A}lg}(\mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m), A) \ni f \mapsto (f(x_1), \ldots, f(x_n)) \in \mathcal{X}(A).$$

Now it is easy to see that this bijection is a natural isomorphism (in A). □

If no polynomials are given for the above construction, then the functor under this construction is the affine space $\mathbb{A}^n$ of dimension n. By giving polynomials the functor $\mathcal{X}$ becomes a subfunctor of $\mathbb{A}^n$, because it defines subsets $\mathcal{X}(A) \subseteq \mathbb{A}^n(A) = A^n$. Both functors are representable functors. The embedding is induced by the homomorphism of algebras $\nu : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow \mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m)$.

Problem 1.1.1. (1) Determine the affine algebra of the functor "unit circle" S^1 in $\mathbb{A}^2$.

(2) Determine the affine algebra of the functor "unit sphere" S^{n-1} in $\mathbb{A}^n$.

(3) Determine the affine algebra of the functor "torus" $\mathcal{T}$ and find an "embedding" of $\mathcal{T}$ into $\mathbb{A}^3$.

Actually $\mathcal{T}$ can be considered as product $\S^1 \times \S^1(B) = \S^1(B) \times \S^1(B)$. Take the first copy of $\S^1(B)$ as the circle with radius 2, then we have $\S^1_1(B) \times S^1_2(B) = \{(u, v, x, y) | u^2 + v^2 = 4; x^2 + y^2 = 1\}$. The embedding is

$$(u, v, x, y) \mapsto (u, v, 0) + 1/2x(u, v, 0) + (0, 0, y)$$

hence the algebra map is

$$\mathbb{K}[x,y,z] \longrightarrow \mathbb{K}[u,v]/(u^2+v^2-4) \otimes \mathbb{K}[x,y]/(x^2+y^2-1)$$

with

$$\begin{aligned} x &\mapsto u + 1/2xu, \\ y &\mapsto v + 1/2xv, \\ z &\mapsto y. \end{aligned}$$

(4) Let $\mathcal{U}$ denote the plane curve $xy = 1$. Then $\mathcal{U}$ is not isomorphic to the affine line. (Hint: An isomorphism $\mathbb{K}[x,x^{-1}] \longrightarrow \mathbb{K}[y]$ sends x to a polynomial $p(y)$ which must be invertible. Consider the highest coefficient of $p(y)$ and show that $p(y) \in \mathbb{K}$. But that means that the map cannot be bijective.)
$\mathcal{U}$ is also called the *unit functor*. Can you explain, why?

(5) Let $\mathcal{X}$ denote the plane curve $y = x^2$. Then $\mathcal{X}$ is isomorphic to the affine line.

(6) Let $\mathbb{K} = \mathbb{C}$ be the field of complex numbers. Show that the unit functor $\mathcal{U}: \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{S}et$ in Problem (3) is naturally isomorphic to the unit circle S^1. (Hint: There is an algebra isomorphism between the representing algebras $\mathbb{K}[e,e^{-1}]$ and $\mathbb{K}[c,s]/(c^2+s^2-1)$.)

(7) * Let $\mathbb{K}$ be an algebraically closed field. Let p be an irreducible square polynomial in $\mathbb{K}[x,y]$. Let $\mathcal{Z}$ be the conic section defined by p with the affine algebra $\mathbb{K}[x,y]/(p)$. Show that $\mathcal{Z}$ is naturally isomorphic either to $\mathcal{X}$ or to $\mathcal{U}$ from parts (4) resp. (5).

Remark 1.1.8. Affine algebras of affine schemes are finitely generated commutative algebras and any such algebra is an affine algebra of some affine scheme, since $A \cong \mathbb{K}[x_1,\ldots,x_n]/(p_1,\ldots,p_m)$ (Hilbert basis theorem).

The polynomials $p_1,\ldots,p_m$ are not uniquely determined by the affine algebra of an affine scheme. Not even the ideal generated by the polynomials in the polynomial ring $\mathbb{K}[x_1,\ldots,x_n]$ is uniquely determined. Also the number of variables $x_1,\ldots,x_n$ is not uniquely determined.

The $\mathbb{K}$-points $(\alpha_1,\ldots,\alpha_n) \in \mathcal{X}(\mathbb{K})$ of an affine scheme $\mathcal{X}$ (with coefficients in the base field $\mathbb{K}$) are called *rational points*. They do not suffice to completely describe the affine scheme.

Let for example $\mathbb{K} = \mathbb{R}$ the set of rational numbers. If $\mathcal{X}$ and $\mathcal{Y}$ are affine schemes with affine algebras $\mathcal{O}(\mathcal{X}) := \mathbb{K}[x,y]/(x^2+y^2+1)$ and $\mathcal{O}(\mathcal{Y}) := \mathbb{K}[x]/(x^2+1)$ then both schemes have no rational points. The scheme $\mathcal{Y}$, however, has exactly two complex points (with coefficients in the field $\mathbb{C}$ of complex numbers) and the scheme $\mathcal{X}$ has infinitely many complex points, hence $\mathcal{X}(\mathbb{C}) \not\cong \mathcal{Y}(\mathbb{C})$. This does not result from the embeddings into different spaces $\mathbb{A}^2$ resp. $\mathbb{A}^1$. In fact we also have $\mathcal{O}(\mathcal{Y}) = \mathbb{K}[x]/(x^2+1) \cong \mathbb{K}[x,y]/(x^2+1,y)$, so $\mathcal{Y}$ can be considered as an affine scheme in $\mathbb{A}^2$.

Since each affine scheme $\mathcal{X}$ is isomorphic to the functor $\mathrm{Mor}_{\mathbb{K}\text{-}cAlg}(\mathcal{O}(\mathcal{X}), \text{-})$ we will henceforth identify these two functors, thus removing annoying isomorphisms.

Definition 1.1.9. Let $\mathbb{K}$-$\mathcal{A}\!f\!f$ denote the category of all commutative finitely generated (or affine cf. 1.1.8) $\mathbb{K}$-algebras. An *affine algebraic variety* is a representable functor $\mathbb{K}\text{-}\mathcal{A}\!f\!f(A,\text{-}) : \mathbb{K}\text{-}\mathcal{A}\!f\!f \to \mathcal{S}et$. The affine algebraic varieties together with the natural transformations form the *category of affine algebraic varieties* $\mathbb{K}$-$\mathcal{V}ar$ over $\mathbb{K}$. The functor that associates with each affine algebra A its affine algebraic variety represented by A is denoted by $\operatorname{Spec} : \mathbb{K}\text{-}\mathcal{A}\!f\!f \to \mathbb{K}\text{-}\mathcal{V}ar$, $\operatorname{Spec}(A) = \mathbb{K}\text{-}\mathcal{A}\!f\!f(A,\text{-})$.

By the Yoneda Lemma the functor

$$\operatorname{Spec} : \mathbb{K}\text{-}\mathcal{A}\!f\!f \to \mathbb{K}\text{-}\mathcal{V}ar$$

is an antiequivalence (or duality) of categories with inverse functor

$$\mathcal{O} : \mathbb{K}\text{-}\mathcal{V}ar \to \mathbb{K}\text{-}\mathcal{A}\!f\!f.$$

An affine algebraic variety is completely described by its affine algebra $\mathcal{O}(\mathcal{X})$. Thus the dream of geometry is realized.

Arbitrary (not necessarily finitely generated) commutative algebras also define representable functors (defined on the category of all commutative algebras). Thus we also have "infinite dimensional" varieties which we will call *geometric spaces* or *affine varieties*. We denote their category by $\mathcal{G}eom(\mathbb{K})$ and get a commutative diagram

$$\begin{tikzcd}
\mathbb{K}\text{-}\mathcal{A}\!f\!f \arrow[r, "\operatorname{Spec}"] \arrow[d] & \mathbb{K}\text{-}\mathcal{V}ar \arrow[d] \\
\mathbb{K}\text{-}c\mathcal{A}lg \arrow[r, "\cong^o"'] & \mathcal{G}eom(\mathbb{K})
\end{tikzcd}$$

We call the representable functors $\mathcal{X} : \mathbb{K}\text{-}c\mathcal{A}lg \to \mathcal{S}et$ *geometric spaces* or *affine varieties*, and the representable functors $\mathcal{X} : \mathbb{K}\text{-}\mathcal{A}\!f\!f \to \mathcal{S}et$ *affine schemes* or *affine algebraic varieties*. This is another realization of the dream of geometry.

The geometric spaces can be viewed as sets of zeros in arbitrary commutative $\mathbb{K}$-algebras B of arbitrarily many polynomials with arbitrarily many variables. The function algebra of $\mathcal{X}$ will be called the *affine algebra* of $\mathcal{X}$ in both cases.

Example 1.1.10. A somewhat less trivial example is the state space of a circular pendulum (of length 1). The location is in $L = \{(a,b) \in A^2 | a^2 + b^2 = 1\}$, the momentum is in $M = \{p \in A\}$ which is a straight line. So the whole geometric space for the pendulum is $(L \times M)(A) = \{(a,b,p) | a,b,p \in A; a^2 + b^2 = 1\}$. This geometric space is represented by $\mathbb{K}[x,y,z]/(x^2 + y^2 - 1)$ since

$$(L \times M)(A) = \{(a,b,p) | a,b,p \in A; a^2 + b^2 = 1\} \cong \mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}[x,y,z]/(x^2 + y^2 - 1), A).$$

The two antiequivalences of categories above give rise to the question for the function algebra. If a representable functor $\mathcal{X} = \mathbb{K}\text{-}c\mathcal{A}lg(A,\text{-})$ is viewed as geometric sets of zeros of certain polynomials, i.e. as spaces with coordinates in arbitrary commutative algebras B, (plus functorial behavior), then it is not clear why the representing algebra A should be anything like an algebra of functions on these geometric sets. It is not even clear where these functions should assume their values. Only if we can show that A can be viewed as a reasonable algebra of functions, we should talk about a realization of the dream of geometry. But this will be done in the following theorem. We will consider functions as maps (coordinate functions) from the geometric set $\mathcal{X}(B)$ to the set of coordinates B, maps that are natural in B. Such coordinate functions are just natural transformations from $\mathcal{X}$ to the underlying functor $\mathbb{A}$.

Theorem 1.1.11. *Let $\mathcal{X}$ be a geometric space with the affine algebra $A = \mathcal{O}(\mathcal{X})$. Then $A \cong \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ as $\mathbb{K}$-algebras, where $\mathbb{A} : \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{S}et$ is the underlying functor or affine line. The isomorphism $A \cong \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ induces a natural transformation $A \times \mathcal{X}(B) \longrightarrow B$ (natural in B).*

Proof. First we define an isomorphism between the sets A and $\mathrm{Nat}(\mathcal{X}, \mathbb{A})$. Because of $\mathcal{X} = \mathrm{Mor}_{\mathbb{K}\text{-}cAlg}(A, \text{-}) =: \mathbb{K}\text{-}c\mathcal{A}lg(A, \text{-})$ and $\mathbb{A} = \mathrm{Mor}_{\mathbb{K}\text{-}cAlg}(\mathbb{K}[x], \text{-}) =: \mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}[x], \text{-})$ the Yoneda Lemma gives us

$$\mathrm{Nat}(\mathcal{X}, \mathbb{A}) = \mathrm{Nat}(\mathbb{K}\text{-}c\mathcal{A}lg(A, \text{-}), \mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}[x], \text{-})) \cong \mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}[x], A) = \mathbb{A}(A) \cong A$$

on the set level. Let $\phi : A \longrightarrow \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ denote the given isomorphism. ϕ is defined by $\phi(a)(B)(p)(x) := p(a)$. By the Yoneda Lemma its inverse is given by $\phi^{-1}(\alpha := \alpha((A)(1)(x)$.

$\mathrm{Nat}(\mathcal{X}, \mathbb{A})$ carries an algebra structure given by the algebra structure of the coefficients. For a coefficient algebra B, a B-point $p : A \longrightarrow B$ in $\mathcal{X}(B) = \mathbb{K}\text{-}\mathcal{A}lg(A, B)$, and $\alpha, \beta \in \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ we have $\alpha(B)(p) \in \mathbb{A}(B) = B$. Hence $(\alpha + \beta)(B)(p) := (\alpha(B) + \beta(B))(p) = \alpha(B)(p) + \beta(B)(p)$ and $(\alpha \cdot \beta)(B)(p) := (\alpha(B) \cdot \beta(B))(p) = \alpha(B)(p) \cdot \beta(B)(p)$ make $\mathrm{Nat}(\mathcal{X}, \mathbb{A})$ an algebra.

Let a be an arbitrary element in A. By the isomorphism given above this element induces an algebra homomorphism $g_a : \mathbb{K}[x] \longrightarrow A$ mapping x onto a. This algebra homomorphism induces the natural transformation $\phi(a) : \mathcal{X} \longrightarrow \mathbb{A}$. On the B-level it is just the composition with g_a, i.e. $\phi(a)(B)(p) = (\mathbb{K}[x] \xrightarrow{g_a} A \xrightarrow{p} B)$. Since such a homomorphism is completely described by the image of x we get $\phi(a)(B)(p)(x) = p(a)$. To compare the algebra structures of A and $\mathrm{Nat}(\mathcal{X}, \mathbb{A})$ let $a, a' \in A$. We have $\phi(a)(B)(p)(x) = p(a)$ and $\phi(a')(B)(p)(x) = p(a')$, hence $\phi(a + a')(B)(p)(x) = p(a + a') = p(a) + p(a') = \phi(a)(B)(p)(x) + \phi(a')(B)(p)(x) = (\phi(a)(B)(p)+\phi(a')(B)(p))(x) = (\phi(a)(B)+\phi(a')(B))(p)(x) = (\phi(a)+\phi(a'))(B)(p)(x)$. Analogously we get $\phi(aa')(B)(p)(x) = p(aa') = p(a)p(a') = (\phi(a) \cdot \phi(a'))(B)(p)(x)$, and thus $\phi(a + a') = \phi(a) + \phi(a')$ and $\phi(aa') = \phi(a) \cdot \phi(a')$. Hence addition and multiplication in $\mathrm{Nat}(\mathcal{X}, \mathbb{A})$ are defined by the addition and the multiplication of the values $p(a) + p(a')$ resp. $p(a)p(a')$.

We describe the action $\psi(B) : A \times \mathcal{X}(B) \longrightarrow B$ of A on $\mathcal{X}(B)$. Let $p : A \longrightarrow B$ be a B-point in $\mathbb{K}\text{-}c\mathcal{A}lg(A, B) = \mathcal{X}(B)$. For each $a \in A$ the image $\phi(a) : \mathcal{X} \longrightarrow \mathbb{A}$ is a natural transformation hence we have maps $\psi(B) : A \times \mathcal{X}(B) \longrightarrow B$ such that $\psi(B)(a, p) = p(a)$. Finally each homomorphism of algebras $f : B \longrightarrow B'$ induces a commutative diagram

$$\begin{tikzcd} A \times \mathcal{X}(B) \arrow[r, "\psi(B)"] \arrow[d, "A \times \mathcal{X}(f)"'] & B \arrow[d, "f"] \\ A \times \mathcal{X}(B') \arrow[r, "\psi(B')"'] & B' \end{tikzcd}$$

Thus $\psi(B) : A \times \mathcal{X}(B) \longrightarrow B$ is a natural transformation. $\square$

Remark 1.1.12. Observe that the isomorphism $A \cong \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ induces a natural transformation $A \times \mathcal{X}(B) \longrightarrow B$ (natural in B). In particular the affine algebra A can be viewed as the set of functions from the set of B-points $\mathcal{X}(B)$ into the "base" ring B (functions which are natural in B). In this sense the algebra A may be considered as function algebra of the geometric space $\mathcal{X}$. Thus we will call A the *function algebra* of $\mathcal{X}$.

One can show that the algebra A is universal with respect to the property, that for each commutative algebra D and each natural transformation $\rho : D \times \mathcal{X}(\text{-}) \longrightarrow \text{-}$ there is a unique homomorphism of algebras $f : D \longrightarrow A$, such that the triangle

$$\begin{tikzcd} D \times \mathcal{X}(B) \arrow[d, "f \times 1_{\mathcal{X}(B)}"'] \arrow[dr, "\rho(B)"] & \\ A \times \mathcal{X}(B) \arrow[r, "\psi(B)"'] & B \end{tikzcd}$$

commutes. We will show this result later on for noncommutative algebras. The universal property implies that the function algebra A of an geometric space $\mathcal{X}$ is unique up to isomorphism.

Let $\mathcal{X}$ be an geometric space with function algebra $A = \mathcal{O}(\mathcal{X})$. If $p : A \longrightarrow \mathbb{K}$ is a rational point of $\mathcal{X}$, i.e. a homomorphism of algebras, then $\mathrm{Im}(p) = \mathbb{K}$ hence $\mathrm{Ker}(p)$ is a maximal ideal of A of codimension 1. Conversely let $\mathfrak{m}$ be a maximal ideal of A of codimension 1 then this defines a rational point $p : A \longrightarrow A/\mathfrak{m} \cong \mathbb{K}$. If $\mathbb{K}$ is algebraicly closed and $\mathfrak{m}$ an arbitrary maximal ideal of A, then $A/\mathfrak{m}$ is a finitely generated $\mathbb{K}$-algebra and a field extension of $\mathbb{K}$, hence it coincides with $\mathbb{K}$. Thus the codimension of $\mathfrak{m}$ is 1. The set of maximal ideals of A is called the *maximal spectrum* $\mathrm{Spec}_m(A)$. This is the approach of algebraic geometry to recover the geometric space of (rational) points from the function algebra A. We will not follow this approach since it does not easily extend to noncommutative geometry.

Problem 1.1.2. 1. Let $\mathcal{X}$ be an geometric space with affine algebra A. Show that the algebra A is universal with respect to the property, that for each commutative algebra D and each natural transformation $\rho : D \times \mathcal{X}(\text{-}) \longrightarrow \text{-}$ there is a unique homomorphism of algebras $f : D \longrightarrow A$, such that the triangle

$$\begin{tikzcd} D \times \mathcal{X}(B) \arrow[d, "f\times 1_{\mathcal{X}(B)}"'] \arrow[dr, "\rho(B)"] & \\ A \times \mathcal{X}(B) \arrow[r, "\psi(B)"'] & B. \end{tikzcd}$$

2. Let $\mathcal{X}$ be an affine scheme with affine algebra

$$A = \mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m).$$

Define "coordinate functions" $q_i : \mathcal{X}(B) \longrightarrow B$ which describe the coordinates of B-points and identify these coordinate functions with elements of A.

Now we will study morphisms between geometric spaces.

Theorem 1.1.13. *Let $\mathcal{X} \subseteq \mathbb{A}^r$ and $\mathcal{Y} \subseteq \mathbb{A}^s$ be affine algebraic varieties and let $\phi : \mathcal{X} \longrightarrow \mathcal{Y}$ be a natural transformation. Then there are polynomials*

$$p_1(x_1, \ldots, x_r), \ldots, p_s(x_1, \ldots, x_r) \in \mathbb{K}[x_1, \ldots, x_r],$$

such that

$$\phi(A)(a_1, \ldots, a_r) = (p_1(a_1, \ldots, a_r), \ldots, p_s(a_1, \ldots, a_r)),$$

for all $A \in \mathbb{K}$-$\mathcal{A}\!f\!f$ and all $(a_1, \ldots, a_r) \in \mathcal{X}(A)$, i.e. the morphisms between affine algebraic varieties are of polynomial type.

Proof. Let $\mathcal{O}(\mathcal{X}) = \mathbb{K}[x_1, \ldots, x_r]/I$ and $\mathcal{O}(\mathcal{Y}) = \mathbb{K}[y_1, \ldots, y_s]/J$. For $A \in \mathbb{K}$-$\mathcal{A}lg$ and $(a_1, \ldots, a_r) \in \mathcal{X}(A)$ let $f : \mathbb{K}[x_1, \ldots, x_r]/I \longrightarrow A$ with $f(x_i) = a_i$ be the homomorphism obtained from $\mathcal{X}(A) \cong \mathbb{K}$-$\mathcal{A}lg(\mathbb{K}[x_1, \ldots, x_r]/I, A)$. The natural transformation ϕ is given by composition with a homomorphism $g : \mathbb{K}[y_1, \ldots, y_s]/J \longrightarrow \mathbb{K}[x_1, \ldots, x_r]/I$ hence we get

$$\phi(A) : \mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}[x_1, \ldots, x_r]/I, A) \ni f \mapsto fg \in \mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}[y_1, \ldots, y_s]/J, A).$$

Since g is described by $g(y_i) = p_i(x_1, \ldots, x_r) \in \mathbb{K}[x_1, \ldots, x_r]$ we get

$$\begin{aligned} \phi(A)(a_1, \ldots, a_s) &= (fg(y_1), \ldots, fg(y_s)) \\ &= (f(p_1(x_1, \ldots, x_r)), \ldots, f(p_s(x_1, \ldots, x_r))) \\ &= (p_1(a_1, \ldots, a_r), \ldots, p_s(a_1, \ldots, a_r)). \end{aligned}$$

□

An analogous statement holds for geometric spaces.

Example 1.1.14. The isomorphism between the affine line (1.1.2) and the parabola is given by the isomorphism $f : \mathbb{K}[x, y]/(y - x^2) \longrightarrow \mathbb{K}[z]$, $f(x) = z$, $f(y) = z^2$ that has the inverse function $f^{-1}(z) = x$. On the affine schemes $\mathbb{A}$, the affine line, and $\mathbb{P}$, the parabola, the induced map is $f : \mathbb{A}(A) \ni a \mapsto (a, a^2) \in \mathbb{P}(A)$ resp. $f^{-1} : \mathbb{P}(A) \ni (a, b) \mapsto a \in \mathbb{A}(A)$.

2. Quantum Spaces and Noncommutative Geometry

Now we come to noncommutative geometric spaces and their function algebras. Many of the basic principles of commutative algebraic geometry as introduced in 1.1 carry over to noncommutative geometry. Our main aim, however, is to study the symmetries (automorphisms) of noncommutative spaces which lead to the notion of a quantum group.

Since the construction of noncommutative geometric spaces has deep applications in theoretical physics we will also call these spaces quantum spaces.

Definition 1.2.1. Let A be a (not necessarily commutative) $\mathbb{K}$-algebra. Then the functor $\mathcal{X} := \mathbb{K}\text{-}\mathcal{A}lg(A,\text{-}) : \mathbb{K}\text{-}\mathcal{A}lg \longrightarrow \mathcal{S}et$ represented by A is called *(affine) noncommutative (geometric) space* or *quantum space*. The elements of $\mathbb{K}\text{-}\mathcal{A}lg(A,B)$ are called *B-points* of $\mathcal{X}$. A *morphism of noncommutative spaces* $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a natural transformation.

This definition implies immediately

Corollary 1.2.2. *The noncommutative spaces form a category $\mathcal{QS}$ that is dual to the category of $\mathbb{K}$-algebras.*

Remark 1.2.3. Thus one often calls the dual category $\mathbb{K}\text{-}\mathcal{A}lg^{op}$ category of noncommutative spaces.

If A is a finitely generated algebra then it may be considered as a residue class algebra $A \cong \mathbb{K}\langle x_1, \ldots, x_n\rangle/I$ of a polynomial algebra in noncommuting variables (cf. [Advanced Algebra] 2.2). If $I = (p_1(x_1, \ldots, x_n), \ldots, p_m(x_1, \ldots, x_n))$ is the two-sided ideal generated by the polynomials $p_1, \ldots, p_m$ then the sets $\mathbb{K}\text{-}\mathcal{A}lg(A,B)$ can be considered as sets of zeros of these polynomials in B^n. In fact, we have $\mathbb{K}\text{-}\mathcal{A}lg(\mathbb{K}\langle x_1, \ldots, x_n\rangle, B) \cong \mathrm{Map}(\{x_1, \ldots, x_n\}, B) = B^n$. Thus $\mathbb{K}\text{-}\mathcal{A}lg(A,B)$ can be considered as the set of those homomorphisms of algebras from $\mathbb{K}\langle x_1, \ldots, x_n\rangle$ to B that vanish on the ideal I or as the set of zeros of these polynomials in B^n.

Similar to Theorem 1.1.13 one shows also in the noncommutative case that morphisms between noncommutative spaces are described by polynomials.

The Theorem 1.1.11 on the operation of the affine algebra $A = \mathcal{O}(\mathcal{X})$ on $\mathcal{X}$ as function algebra can be carried over to the noncommutative case as well: the natural transformation $\psi(B) : A \times \mathcal{X}(B) \longrightarrow B$ (natural in B) is given by $\psi(B)(a,p) := p(a)$ and comes from the isomorphism $A \cong \mathrm{Nat}(\mathcal{X}, \mathbb{A})$.

Now we come to a claim on the function algebra A that we did not prove in the commutative case, but that holds in the commutative as well as in the noncommutative situation.

Lemma 1.2.4. *Let D be a set and $\phi : D \times \mathcal{X}(\text{-}) \longrightarrow \mathbb{A}(\text{-})$ be a natural transformation. Then there exists a unique map $f : D \longrightarrow A$ such that the diagram*

$$\begin{tikzcd} D \times \mathcal{X}(B) \arrow[d, "f\times 1"'] \arrow[dr, "\phi(B)"] & \\ A \times \mathcal{X}(B) \arrow[r, "\psi(B)"'] & B \end{tikzcd}$$

commutes.

Proof. Let $\phi : D \times \mathcal{X} \longrightarrow \mathbb{A}$ be given. We first define a map $f' : D \longrightarrow \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ by $f'(d)(B)(p) := \phi(B)(d, p)$.

We claim that $f'(d) : \mathcal{X} \longrightarrow \mathbb{A}$ is a natural transformation. Observe that the diagram

$$\begin{tikzcd} D \times \mathcal{X}(B) \arrow[r, "\phi(B)"] \arrow[d, "D \times \mathcal{X}(g)"'] & \mathbb{A}(B) = B \arrow[d, "g"] \\ D \times \mathcal{X}(B') \arrow[r, "\phi(B')"'] & \mathbb{A}(B') = B' \end{tikzcd}$$

commutes for any $g : B \longrightarrow B'$, since ϕ is a natural transformation. Thus the diagram

$$\begin{tikzcd} \mathcal{X}(B) \arrow[r, "f'(d)(B)"] \arrow[d, "\mathcal{X}(g)"'] & \mathbb{A}(B) = B \arrow[d, "g"] \\ \mathcal{X}(B') \arrow[r, "f'(d)(B')"'] & \mathbb{A}(B') = B' \end{tikzcd}$$

commutes since

$$\begin{aligned} (g \circ f'(d)(B))(p) &= (g \circ \phi(B))(d, p) \\ &= \phi(B') \circ (1 \times \mathcal{X}(g))(d, p) \\ &= \phi(B')(d, \mathcal{X}(g)(p)) \\ &= f'(d)(B')(\mathcal{X}(g)(p)). \end{aligned}$$

Hence $f'(d) \in \mathrm{Nat}(X, \mathbb{A})$ and $f' : D \longrightarrow \mathrm{Nat}(\mathcal{X}, \mathbb{A})$.

Now we define $f : D \longrightarrow A$ as $D \xrightarrow{f'} \mathrm{Nat}(\mathcal{X}, \mathbb{A}) \cong A$. By using the isomorphism from 1.1.11 we get $f(d) = f'(d)(A)(1)$. (Actually we get $f(d) = f'(d)(A)(1)(x)$ but we identify $\mathbb{A}(B)$ and B by $\mathbb{A}(B) \ni p \mapsto p(x) \in B$.)

Then we get

$$\begin{aligned}\psi(B)(f\times 1)(d,p) &= \psi(B)(f(d),p)\\ &= \psi(B)(f'(d)(A)(1)(x),p) \text{ (by definition of } f)\\ &= p\circ f'(d)(A)(1) \text{ (since we may omit } x)\\ &= p\circ \phi(A)(d,1) \text{ (by definition of } f')\\ &= \phi(B)(D\times \mathcal{X}(p))(d,1) \text{ (since } \phi \text{ is a natural transformation)}\\ &= \phi(B)(d,p).\end{aligned}$$

Hence the diagram in the Lemma commutes.

To show the uniqueness of f let $g : D \longrightarrow A$ be a homomorphism such that $\psi(B)(g\times 1) = \phi(B)$. Then we have

$$f(d) = f'(d)(A)(1) = \phi(A)(d,1) = \psi(A)(g\times 1)(d,1) = \psi(A)(f(d),1) = 1\circ g(d) = g(d)$$

hence $f = g$. □

Problem 1.2.3. Definition: Let $\mathcal{X}$ be an geometric space with affine algebra A. Let D be an algebra. A natural transformation $\rho : D\times\mathcal{X} \longrightarrow \mathbb{A}$ is called an *algebra action* if $\rho(B)(\text{-},p) : D \longrightarrow \mathbb{A}(B) = B$ is an algebra homomorphism for all B and all $p\in\mathcal{X}(B)$.

Give proofs for:

Lemma: The natural transformation $\psi : A\times\mathcal{X} \longrightarrow \mathbb{A}$ is an algebra action.

Theorem: Let D be an algebra and $\rho : D\times\mathcal{X}(\text{-}) \longrightarrow \mathbb{A}(\text{-})$ be an algebra action. Then there exists a unique algebra homomorphism $f : D \longrightarrow A$ such that the diagram

$$\begin{tikzcd} D\times\mathcal{X}(B) \arrow[d, "f\times 1"'] \arrow[dr, "\rho(B)"] & \\ A\times\mathcal{X}(B) \arrow[r, "\psi(B)"'] & B \end{tikzcd}$$

commutes.

Definition 1.2.5. The noncommutative space $\mathbb{A}_q^{2|0}$ with the function algebra

$$\mathcal{O}(\mathbb{A}_q^{2|0}) := \mathbb{K}\langle x,y\rangle/(xy - q^{-1}yx)$$

with $q\in\mathbb{K}\setminus\{0\}$ is called the *(deformed) quantum plane*. The noncommutative space $\mathbb{A}_q^{0|2}$ with the function algebra

$$\mathcal{O}(\mathbb{A}_q^{0|2}) := \mathbb{K}\langle \xi,\eta\rangle/(\xi^2,\eta^2,\xi\eta + q\eta\xi)$$

is called the *dual (deformed) quantum plane*. We have

$$\mathbb{A}_q^{2|0}(A) = \left\{ \begin{pmatrix} x\\ y\end{pmatrix} \middle| x,y\in A; xy = q^{-1}yx \right\}$$

and

$$\mathbb{A}_q^{0|2}(A) = \left\{ \begin{pmatrix}\xi, & \eta\end{pmatrix} \middle| \xi,\eta\in A; \xi^2 = 0, \eta^2 = 0, \xi\eta = -q\eta\xi \right\}.$$

Definition 1.2.6. Let $\mathcal{X}$ be a noncommutative space with function algebra A and let $\mathcal{X}_c$ be the restriction of the functor $\mathcal{X} : \mathbb{K}\text{-}\mathcal{A}lg \longrightarrow \mathcal{S}et$ to the category of commutative algebras: $\mathcal{X}_c : \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{S}et$. Then we call $\mathcal{X}_c$ the *commutative part* of the noncommutative space $\mathcal{X}$.

Lemma 1.2.7. *The commutative part $\mathcal{X}_c$ of a noncommutative space $\mathcal{X}$ is an affine variety.*

PROOF. The underlying functor $\mathbb{A} : \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathbb{K}\text{-}\mathcal{A}lg$ has a left adjoint functor $\mathbb{K}\text{-}\mathcal{A}lg \ni A \mapsto A/[A,A] \in \mathbb{K}\text{-}c\mathcal{A}lg$ where $[A,A]$ denotes the two-sided ideal of A generated by the elements $ab - ba$. In fact for each homomorphism of algebras $f : A \longrightarrow B$ with a commutative algebra B there is a factorization through $A/[A,A]$ since f vanishes on the elements $ab - ba$.

Hence if $A = \mathcal{O}(\mathcal{X})$ is the function algebra of $\mathcal{X}$ then $A/[A,A]$ is the representing algebra for $\mathcal{X}_c$. □

Remark 1.2.8. For any commutative algebra (of coefficients) B the spaces $\mathcal{X}$ and $\mathcal{X}_c$ have the same B-points: $\mathcal{X}(B) = \mathcal{X}_c(B)$. The two spaces differ only for noncommutative algebras of coefficients. In particular for commutative fields B as algebras of coefficients the quantum plane $\mathbb{A}_q^{2|0}$ has only B-points on the two axes since the function algebra $\mathbb{K}\langle x,y\rangle/(xy - q^{-1}yx, xy - yx) \cong K[x,y]/(xy)$ defines only B-points (b_1, b_2) where at least one of the coefficients is zero.

Problem 1.2.4. Let S_3 be the symmetric group and $A := \mathbb{K}[S_3]$ be the group algebra on S_3. Describe the points of $\mathcal{X}(B) = \mathbb{K}\text{-}\mathcal{A}lg(A,B)$ as a subspace of $\mathbb{A}^2(B)$. What is the commutative part $\mathcal{X}_c(B)$ of $\mathcal{X}$ and what is the affine algebra of $\mathcal{X}_c$?

To understand how Hopf algebras fit into the context of noncommutative spaces we have to better understand the tensor product in $\mathbb{K}\text{-}\mathcal{A}lg$.

Definition 1.2.9. Let $A = \mathcal{O}(\mathcal{X})$ and $A' = \mathcal{O}(\mathcal{Y})$ be the function algebras of the noncommutative spaces $\mathcal{X}$ resp. $\mathcal{Y}$. Two B-points $p : A \longrightarrow B$ in $\mathcal{X}(B)$ and $p' : A' \longrightarrow B$ in $\mathcal{Y}(B)$ are called *commuting points* if we have for all $a \in A$ and all $a' \in A'$

$$p(a)p'(a') = p'(a')p(a),$$

i.e. if the images of the two homomorphisms p and p' commute.

Remark 1.2.10. To show that the points p and p' commute, it is sufficient to check that the images of the algebra generators $p(x_1), \ldots, p(x_m)$ commute with the images of the algebra generators $p'(y_1), \ldots, p'(y_n)$ under the multiplication. This means that we have

$$b_i b'_j = b'_j b_i$$

for the B-points $(b_1, \ldots, b_m) \in \mathcal{X}(B)$ and $(b'_1, \ldots, b'_n) \in \mathcal{Y}(B)$.

Definition 1.2.11. The functor

$$(\mathcal{X} \perp \mathcal{Y})(B) := \{(p,p') \in \mathcal{X}(B) \times \mathcal{Y}(B) | p, p' \text{ commute}\}$$

is called the *orthogonal product* of the noncommutative spaces $\mathcal{X}$ and $\mathcal{Y}$.

Remark 1.2.12. Together with $\mathcal{X}$ and $\mathcal{Y}$ the orthogonal product $\mathcal{X} \perp \mathcal{Y}$ is again a functor, since homomorphisms $f : B \to B'$ are compatible with the multiplication and thus preserve commuting points. Hence $\mathcal{X} \perp \mathcal{Y}$ is a subfunctor of $\mathcal{X} \times \mathcal{Y}$.

Lemma 1.2.13. *If $\mathcal{X}$ and $\mathcal{Y}$ are noncommutative spaces, then $\mathcal{X} \perp \mathcal{Y}$ is a noncommutative space with function algebra $\mathcal{O}(\mathcal{X} \perp \mathcal{Y}) = \mathcal{O}(\mathcal{X}) \otimes \mathcal{O}(\mathcal{Y})$.*

If $\mathcal{X}$ and $\mathcal{Y}$ have finitely generated function algebras then the function algebra of $\mathcal{X} \perp \mathcal{Y}$ is also finitely generated.

Proof. Let $A := \mathcal{O}(\mathcal{X})$ and $A' := \mathcal{O}(\mathcal{Y})$. Let $(p, p') \in (\mathcal{X} \perp \mathcal{Y})(B)$ be a pair of commuting points. Then there is a unique homomorphism of algebras $h : A \otimes A' \to B$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[dr, "p"'] & A \otimes A' \arrow[d, "h"] & A' \arrow[l, "\iota'"'] \arrow[dl, "p'"] \\ & B. & \end{tikzcd}$$

Define $h(a \otimes a') := p(a)p'(a')$ and check the necessary properties. Observe that for an arbitrary homomorphism of algebras $h : A \otimes A' \to B$ the images of elements of the form $a \otimes 1$ and $1 \otimes a'$ commute since these elements already commute in $A \otimes A'$. Thus we have

$$(\mathcal{X} \perp \mathcal{Y})(B) \cong \mathbb{K}\text{-}\mathcal{A}lg(A \otimes A', B).$$

If the algebra A is generated by the elements $a_1, \ldots, a_m$ and the algebra A' is generated by the elements $a'_1, \ldots, a'_n$ then the algebra $A \otimes A'$ is generated by the elements $a_i \otimes 1$ and $1 \otimes a'_j$. $\square$

Proposition 1.2.14. *The orthogonal product of noncommutative spaces is associative, i.e. for noncommutative spaces $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ we have*

$$(\mathcal{X} \perp \mathcal{Y}) \perp \mathcal{Z} \cong \mathcal{X} \perp (\mathcal{Y} \perp \mathcal{Z}).$$

Proof. Let B be a coefficient algebra and let $p_x \in \mathcal{X}(B)$, $p_y \in \mathcal{Y}(B)$, and $p_z \in \mathcal{Z}(B)$ be points such that $((p_x, p_y), p_z)$ is a pair of commuting points in $((\mathcal{X} \perp \mathcal{Y}) \perp \mathcal{Z})(B)$. In particular (p_x, p_y) is also a pair of commuting points. Thus we have for all $a \in A := \mathcal{O}(\mathcal{X})$, $a' \in A' := \mathcal{O}(\mathcal{Y})$, and $a'' \in A'' := \mathcal{O}(\mathcal{Z})$

$$p_x(a)p_y(a')p_z(a'') = (p_x, p_y)(a \otimes a')p_z(a'') = p_z(a'')(p_x, p_y)(a \otimes a') = p_z(a'')p_x(a)p_y(a')$$

and

$$p_x(a)p_y(a') = p_y(a')p_x(a).$$

If we choose $a = 1$ then we get $p_y(a')p_z(a'') = p_z(a'')p_y(a')$. For arbitrary a, a', a'' we then get

$$p_x(a)p_y(a')p_z(a'') = p_z(a'')p_x(a)p_y(a') = p_z(a'')p_y(a')p_x(a) = p_y(a')p_z(a'')p_x(a)$$

hence (p_y, p_z) and $(p_x, (p_y, p_z))$ are also pairs of commuting points. $\square$

Problem 1.2.5. Show that the orthogonal product of quantum spaces $\mathcal{X} \perp \mathcal{Y}$ is a tensor product for the category $\mathcal{QS}$ (in the sense of monoidal categories – if you know already what that is).

3. Quantum Monoids and their Actions on Quantum Spaces

We use the orthogonal product introduced in the previous section as "product" to define the notion of a monoid (some may call it an algebra w.r.t. the orthogonal product). Observe that on the geometric level the orthogonal product consists only of commuting points. So whenever we define a morphism on the geometric side with domain an orthogonal product of quantum spaces $f : \mathcal{X} \perp \mathcal{Y} \longrightarrow \mathcal{Z}$ then we only have to define what happens to *commuting* pairs of points. That makes it much easier to define such morphisms for noncommutative coordinate algebras.

We are going to define monoids in this sense and study their actions on quantum spaces.

Let E be the functor represented by $\mathbb{K}$. It maps each algebra H to the one-element set $\{\iota : \mathbb{K} \to H\}$.

Definition 1.3.1. Let $\mathcal{M}$ be a noncommutative space and let

$$m : \mathcal{M} \perp \mathcal{M} \longrightarrow \mathcal{M} \text{ and } e : E \longrightarrow \mathcal{M}$$

be morphisms in $\mathcal{QS}$ such that the diagrams

$$\begin{tikzcd}
\mathcal{M} \perp \mathcal{M} \perp \mathcal{M} \arrow[r, "m \perp 1"] \arrow[d, "1 \perp m"'] & \mathcal{M} \perp \mathcal{M} \arrow[d, "m"] \\
\mathcal{M} \perp \mathcal{M} \arrow[r, "m"'] & \mathcal{M}
\end{tikzcd}$$

and

$$\begin{tikzcd}
E \perp \mathcal{M} \cong \mathcal{M} \cong \mathcal{M} \perp E \arrow[r, "\mathrm{id} \perp \eta"] \arrow[d, "\eta \perp \mathrm{id}"'] \arrow[dr, "1_{\mathcal{M}}"] & \mathcal{M} \perp \mathcal{M} \arrow[d, "\nabla"] \\
\mathcal{M} \perp \mathcal{M} \arrow[r, "\nabla"'] & \mathcal{M}
\end{tikzcd}$$

commute. Then $(\mathcal{M}, m, e)$ is called a *quantum monoid.*

Proposition 1.3.2. *Let $\mathcal{M}$ be a noncommutative space with function algebra H. Then H is a bialgebra if and only if $\mathcal{M}$ is a quantum monoid.*

Proof. Since the functors $\mathcal{M} \perp \mathcal{M}$, $\mathcal{M} \perp E$ and $E \perp \mathcal{M}$ are represented by $H \otimes H$ resp. $H \otimes \mathbb{K} \cong H$ resp. $\mathbb{K} \otimes H \cong H$ the Yoneda Lemma defines a bijection between the morphisms $m : \mathcal{M} \perp \mathcal{M} \longrightarrow \mathcal{M}$ and the algebra homomorphisms $\Delta : H \longrightarrow H \otimes H$ and similarly a bijection between the morphisms $e : E \longrightarrow \mathcal{M}$ and the algebra homomorphisms $\varepsilon : H \longrightarrow \mathbb{K}$. Again by the Yoneda Lemma the bialgebra diagrams in $\mathbb{K}$-$\mathcal{A}lg$ commute if and only if the corresponding diagrams for a quantum monoid commute. $\square$

Observe that a similar result cannot be formulated for Hopf algebras H since neither the antipode S nor the multiplication $\nabla : H \otimes H \longrightarrow H$ are algebra homomorphisms. In contrast to affine algebraic groups (2.3.2) Hopf algebras in the category $\mathbb{K}$-$\mathcal{A}lg^{op} \cong QR$ are not groups. Nevertheless, one defines

Definition 1.3.3. A functor defined on the category of $\mathbb{K}$-algebras and represented by a Hopf algebra H is called a *quantum group.*

Definition 1.3.4. Let $\mathcal{X}$ be a noncommutative space and let $\mathcal{M}$ be a quantum monoid. A morphism (a natural transformation) of quantum spaces $\rho : \mathcal{M} \perp \mathcal{X} \longrightarrow \mathcal{X}$ is called an *operation* of $\mathcal{M}$ on $\mathcal{X}$ if the diagrams

$$\begin{tikzcd} \mathcal{M} \perp \mathcal{M} \perp \mathcal{X} \arrow[r, "m \perp 1"] \arrow[d, "1 \perp \rho"'] & \mathcal{M} \perp \mathcal{X} \arrow[d, "\rho"] \\ \mathcal{M} \perp \mathcal{X} \arrow[r, "\rho"'] & \mathcal{X} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{X} \cong E \perp \mathcal{X} \arrow[r, "\eta \perp \mathrm{id}"] \arrow[dr, "\mathrm{id}_{\mathcal{X}}"'] & \mathcal{M} \perp \mathcal{X} \arrow[d, "\rho"] \\ & \mathcal{X} \end{tikzcd}$$

commute. We call $\mathcal{X}$ a noncommutative $\mathcal{M}$-*space.*

Proposition 1.3.5. *Let $\mathcal{X}$ be a noncommutative space with function algebra $A = \mathcal{O}(\mathcal{X})$. Let $\mathcal{M}$ be a quantum monoid with function algebra $B = \mathcal{O}(\mathcal{M})$. Let $\rho : \mathcal{M} \perp \mathcal{X} \longrightarrow \mathcal{X}$ be a morphism in $\mathcal{QS}$ and let $f : A \longrightarrow B \otimes A$ be the associated homomorphism of algebras. Then the following are equivalent*

1. $(\mathcal{X}, \mathcal{M}, \rho)$ is an operation of the quantum monoid $\mathcal{M}$ on the noncommutative space $\mathcal{X}$;

2. (A, H, f) define an H-comodule algebra.

PROOF. The homomorphisms of algebras $\Delta \otimes 1_A$, $1_B \otimes f$, $\epsilon \otimes 1_A$ etc. represent the morphisms of quantum spaces $m \perp \mathrm{id}$, $\mathrm{id} \perp \rho$, $\eta \perp \mathrm{id}$ etc. Hence the required diagrams are transferred by the Yoneda Lemma. $\square$

Example 1.3.6. 1. The quantum monoid of "quantum matrices":
We consider the algebra

$$M_q(2) := \mathbb{K}\langle a, b, c, d \rangle / I = \mathbb{K} \left\langle \begin{matrix} a & b \\ c & d \end{matrix} \right\rangle / I$$

where the two-sided ideal I is generated by the elements

$$ab - q^{-1}ba, ac - q^{-1}ca, bd - q^{-1}db, cd - q^{-1}dc, ad - da - (q^{-1} - q)bc, bc - cb.$$

The quantum space $\mathcal{M}_q(2)$ associated with the algebra $M_q(2)$ is given by

$$\begin{aligned}\mathcal{M}_q(2)(A) &= \mathbb{K}\text{-}\mathcal{A}lg\,(M_q(2), A)\\ &= \left\{ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} | a', b', c', d' \in A; a'b' = q^{-1}b'a', \ldots, b'c' = c'b' \right\}\end{aligned}$$

where each homomorphism of algebras $f : M_q(2) \longrightarrow A$ is described by the quadruple (a', b', c', d') of images of the algebra generators a, b, c, d. The images must satisfy the same relations that generate the two-sided ideal I hence

$$\begin{gathered} a'b' = q^{-1}b'a', a'c' = q^{-1}c'a', b'd' = q^{-1}d'b', c'd' = q^{-1}d'c',\\ b'c' = c'b', a'd' - q^{-1}b'c' = d'a' - qc'b'. \end{gathered}$$

We write these quadruples as 2×2-matrices and call them *quantum matrices*. The unusual commutation relations are chosen so that the following examples work.

The quantum space of quantum matrices turns out to be a quantum monoid. We give both the algebraic (with function algebras) and the geometric (with quantum spaces) approach to define the multiplication.

a) *The algebraic approach:*

The algebra $M_q(2)$ is a bialgebra with the diagonal

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

i.e. by $\Delta(a) = a \otimes a + b \otimes c$, $\Delta(b) = a \otimes b + b \otimes d$, $\Delta(c) = c \otimes a + d \otimes c$ and $\Delta(d) = c \otimes b + d \otimes d$, and with the counit

$$\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e. $\varepsilon(a) = 1$, $\varepsilon(b) = 0$, $\varepsilon(c) = 0$, and $\varepsilon(d) = 1$. We have to prove that Δ and ε are homomorphisms of algebras and that the coalgebra laws are satisfied. To obtain a homomorphism of algebras $\Delta : M_q(2) \longrightarrow M_q(2) \otimes M_q(2)$ we define $\Delta : \mathbb{K}\langle a, b, c, d\rangle \longrightarrow M_q(2) \otimes M_q(2)$ on the free algebra (the polynomial ring in noncommuting variables) $\mathbb{K}\langle a, b, c, d\rangle$ generated by the set $\{a, b, c, d\}$ and show that it vanishes on the ideal I or more simply on the generators of the ideal. Then it factors through a unique homomorphism of algebras $\Delta : M_q(2) \longrightarrow M_q(2) \otimes M_q(2)$. We check this only for one generator of the ideal I:

$$\begin{aligned}
&\Delta(ab - q^{-1}ba) = \Delta(a)\Delta(b) - q^{-1}\Delta(b)\Delta(a) =\\
&= (a \otimes a + b \otimes c)(a \otimes b + b \otimes d) - q^{-1}(a \otimes b + b \otimes d)(a \otimes a + b \otimes c)\\
&= aa \otimes ab + ab \otimes ad + ba \otimes cb + bb \otimes cd - q^{-1}(aa \otimes ba + ab \otimes bc + ba \otimes da + bb \otimes dc)\\
&= aa \otimes (ab - q^{-1}ba) + ab \otimes (ad - q^{-1}bc) + ba \otimes (cb - q^{-1}da) + bb \otimes (cd - q^{-1}dc)\\
&= ba \otimes (q^{-1}ad - q^{-2}bc + cb - q^{-1}da) \equiv 0 \mod (I).
\end{aligned}$$

The reader should check the other identities.

The coassociativity follows from

$$(\Delta\otimes 1)\Delta\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \Delta\begin{pmatrix} a & b\\ c & d\end{pmatrix}\otimes\begin{pmatrix} a & b\\ c & d\end{pmatrix} = (\begin{pmatrix} a & b\\ c & d\end{pmatrix}\otimes\begin{pmatrix} a & b\\ c & d\end{pmatrix})\otimes\begin{pmatrix} a & b\\ c & d\end{pmatrix} =$$
$$= \begin{pmatrix} a & b\\ c & d\end{pmatrix}\otimes(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\otimes\begin{pmatrix} a & b\\ c & d\end{pmatrix}) = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\otimes\Delta\begin{pmatrix} a & b\\ c & d\end{pmatrix} = (1\otimes\Delta)\Delta\begin{pmatrix} a & b\\ c & d\end{pmatrix}.$$

The reader should check the properties of the counit.

b) *The geometric approach:*

$M_q(2)$ has a rather remarkable (and actually well known) comultiplication that is better understood by using the induced multiplication of commuting points. Given two commuting quantum matrices $\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}$ and $\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}$ in $\mathcal{M}_q(2)(A)$. Then their matrix product

$$\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix} = \begin{pmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2\\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2\end{pmatrix}$$

is again a quantum matrix. To prove this we only check one of the relations

$$\begin{aligned}(a_1a_2+b_1c_2)(a_1b_2+b_1d_2) &= a_1a_2a_1b_2+a_1a_2b_1d_2+b_1c_2a_1b_2+b_1c_2b_1d_2\\
&= a_1a_1a_2b_2+a_1b_1a_2d_2+b_1a_1c_2b_2+b_1b_1c_2d_2\\
&= q^{-1}a_1a_1b_2a_2+q^{-1}b_1a_1(d_2a_2+(q^{-1}-q)b_2c_2)+b_1a_1b_2c_2+q^{-1}b_1b_1d_2c_2\\
&= q^{-1}(a_1a_1b_2a_2+a_1b_1b_2c_2+b_1a_1d_2a_2+b_1b_1d_2c_2)\\
&= q^{-1}(a_1b_2a_1a_2+a_1b_2b_1c_2+b_1d_2a_1a_2+b_1d_2b_1c_2)\\
&= q^{-1}(a_1b_2+b_1d_2)(a_1a_2+b_1c_2)\end{aligned}$$

We have used that the two points are commuting points. This multiplication obviously is a natural transformation $\mathcal{M}_q(2)\perp\mathcal{M}_q(2)(A)\longrightarrow\mathcal{M}_q(2)(A)$ (natural in A). It is associative and has unit $\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$. For the associativity observe that by 1.2.14

$$((\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix},\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}),\begin{pmatrix} a_3 & b_3\\ c_3 & d_3\end{pmatrix})$$

is a pair of commuting points if and only if

$$(\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix},(\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix},\begin{pmatrix} a_3 & b_3\\ c_3 & d_3\end{pmatrix}))$$

is a pair of commuting points.

Since $\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$ for all quantum matrices $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathcal{M}_q(2)(B)$ we see that $\mathcal{M}_q(2)$ is a quantum monoid.

It remains to show that the multiplication of $\mathcal{M}_q(2)$ and the comultiplication of $M_q(2)$ correspond to each other by the Yoneda Lemma. The identity morphism of $M_q(2) \otimes M_q(2)$ is given by the pair of commuting points

$(\iota_1, \iota_2) \in \mathcal{M}_q(2) \perp \mathcal{M}_q(2)(M_q(2) \otimes M_q(2)) = \mathbb{K}\text{-}\mathcal{A}lg\,(M_q(2) \otimes M_q(2), M_q(2) \otimes M_q(2))$.

Since $\iota_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes 1 = \begin{pmatrix} a\otimes 1 & b\otimes 1 \\ c\otimes 1 & d\otimes 1 \end{pmatrix}$ and $\iota_2 = 1 \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1\otimes a & 1\otimes b \\ 1\otimes c & 1\otimes d \end{pmatrix}$ we have $\mathrm{id} = (\iota_1, \iota_2) = (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes 1, 1 \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix})$. The Yoneda Lemma defines the diagonal as the image of the identity under $\mathbb{K}\text{-}\mathcal{A}lg\,(M_q(2) \otimes M_q(2), M_q(2) \otimes M_q(2)) \to \mathbb{K}\text{-}\mathcal{A}lg\,(M_q(2), M_q(2) \otimes M_q(2))$ by the multiplication. So $\Delta(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \Delta = \iota_1 * \iota_2 = (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes 1) * (1 \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Thus $M_q(2)$ defines a quantum monoid $\mathcal{M}_q(2)$ with

$$\mathcal{M}_q(2)(B) = \left\{ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} | a', b', c', d' \in B; a'b' = q^{-1}b'a', \ldots, b'c' = c'b' \right\}.$$

This is the deformed version of $\mathcal{M}_2^{\times}$ the multiplicative monoid of the 2×2-matrices of commutative algebras.

2. Let $A_q^{2|0} = \mathbb{K}\langle x, y\rangle/(xy - q^{-1}yx)$ be the function algebra of the quantum plane $\mathbb{A}_q^{2|0}$. By the definition 1.2.5 we have

$$\mathbb{A}_q^{2|0}(A') = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | x, y \in A'; xy = q^{-1}yx \right\}.$$

The set

$$\mathcal{M}_q(2)(A') = \left\{ \begin{pmatrix} u & x \\ y & z \end{pmatrix} | u, x, y, z \in A'; ux = q^{-1}xu, \ldots, xy = yx \right\}$$

operates on this quantum plane by matrix multiplication

$$\mathcal{M}_q(2)(A') \perp \mathbb{A}_q^{2|0}(A') \ni (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}_q^{2|0}(A').$$

Again one should check that the required equations are preserved. Since we have a matrix multiplication we get an operation as in the preceding proposition. In particular $A_q^{2|0}$ is a $M_q(2)$-comodule algebra.

As in example 1. we get the comultiplication as $\delta(\begin{pmatrix} x \\ y \end{pmatrix}) = \delta = (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes 1) * (1 \otimes \begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$.

3. Let $A_q^{0|2} = \mathbb{K}\langle \xi, \eta \rangle / (\xi^2, \eta^2, \xi\eta + q\eta\xi)$ be the function algebra of the dual quantum plane $\mathbb{A}_q^{0|2}$. By the definition 1.2.5 we have

$$\mathbb{A}_q^{0|2}(A') = \left\{ \begin{pmatrix} a' & b' \end{pmatrix} \middle| a', b' \in A'; a'^2 = 0, b'^2 = 0, a'b' = -qb'a' \right\}.$$

The quantum monoid $M_q(2)$ also operates on the dual quantum plane by matrix multiplication

$$\mathbb{A}_q^{0|2}(A') \perp \mathcal{M}_q(2)(A') \ni (\begin{pmatrix} \xi & \eta \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \mapsto \begin{pmatrix} \xi & \eta \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{A}_q^{0|2}(A').$$

This gives another example of a $M_q(2)$-comodule algebra $A_q^{0|2} \longrightarrow A_q^{0|2} \otimes M_q(2)$ with $\delta(\begin{pmatrix} \xi & \eta \end{pmatrix}) = \delta = (\begin{pmatrix} \xi & \eta \end{pmatrix} \otimes 1) * (1 \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} \xi & \eta \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

What is now the reason for the remarkable relations on $\mathcal{M}_q(2)$? It is based on a fact that we will show later namely that $M_q(2)$ is the universal quantum monoid acting on the quantum plane $\mathbb{A}_q^{2|0}$ from the left and on the dual quantum plane $\mathbb{A}_q^{0|2}$ from the right. This however happens in the category of quantum planes represented by quadratic algebras. Here we will show a simpler theorem for finite dimensional algebras.

Problem 1.3.6. (1) Consider the following subset $\mathcal{H}$ of the set of complex 2×2-matrices:

$$\mathcal{H} := \left\{ \begin{pmatrix} x & -y \\ \bar{y} & \bar{x} \end{pmatrix} \in M_{\mathbb{C}}(2 \times 2) | x, y \in \mathbb{C} \right\}$$

We call $\mathcal{H}$ the set of *Hamiltonian quaternions.* For

$$h = \begin{pmatrix} x & -y \\ \bar{y} & \bar{x} \end{pmatrix}$$

we define:

$$\bar{h} := \begin{pmatrix} \bar{x} & y \\ -\bar{y} & x \end{pmatrix}$$

Show:

(a) $h\bar{h} = (|x|^2 + |y|^2)E$ (E the unit matrix),

(b) $\mathcal{H}$ is a real subalgebra of the complex algebra of 2×2-matrices.

(c) $\mathcal{H}$ is a division algebra, i. e. each element different from zero has an inverse under the multiplication.

(d) Let

$$I := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad K := \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

Then E, I, J, K is an $\mathcal{R}$-basis of $\mathcal{H}$ an we have the following multiplication table:

$$I^2 = J^2 = K^2 = -1$$

$$IJ = -JI = K \quad JK = -KJ = I \quad KI = -IK = J.$$

(2) Compute the $\mathcal{H}$-points $A^{2|0}(\mathcal{H})$ of the quantum plane.

Definition 1.3.7. (1) Let $\mathcal{X}$ be a quantum space. A quantum space $\mathcal{M}(\mathcal{X})$ together with a morphism of quantum spaces $\mu : \mathcal{M}(\mathcal{X}) \perp \mathcal{X} \to \mathcal{X}$ is called a *quantum space acting universally* on $\mathcal{X}$ (or simply a *universal quantum space for* $\mathcal{X}$) if for every quantum space $\mathcal{Y}$ and every morphism of quantum spaces $f : \mathcal{Y} \perp \mathcal{X} \to \mathcal{X}$ there is a unique morphism of quantum spaces $g : \mathcal{Y} \to \mathcal{M}(\mathcal{X})$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{Y} \perp \mathcal{X} \arrow[d, "g\perp 1_{\mathcal{X}}"'] \arrow[dr, "f"] & \\
\mathcal{M}(\mathcal{X}) \perp \mathcal{X} \arrow[r, "\mu"'] & \mathcal{X}.
\end{tikzcd}$$

(2) Let A be a $\mathbb{K}$-algebra. A $\mathbb{K}$-algebra $M(A)$ together with a homomorphism of algebras $\delta : A \to M(A) \otimes A$ is called an *algebra coacting universally on* A (or simply a *universal algebra for* A) if for every $\mathbb{K}$-algebra B and every homomorphism of $\mathbb{K}$-algebras $f : A \to B \otimes A$ there exists a unique homomorphism of algebras $g : M(A) \to B$ such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[dr, "f"'] & M(A) \otimes A \arrow[d, "g\otimes 1_A"] \\
& B \otimes A
\end{tikzcd}$$

By the universal properties the universal algebra $M(A)$ for A and the universal quantum space $\mathcal{M}(\mathcal{X})$ for $\mathcal{X}$ are unique up to isomorphism.

Proposition 1.3.8. (1) *Let A be a $\mathbb{K}$-algebra with universal algebra $M(A)$ and $\delta : A \to M(A) \otimes A$. Then $M(A)$ is a bialgebra and A is an $M(A)$-comodule algebra by δ.*

(2) *If B is a bialgebra and if $f : A \to B \otimes A$ defines the structure of a B-comodule algebra on A then there is a unique homomorphism $g : M(A) \to B$ of bialgebras such that the following diagram commutes*

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[dr, "f"'] & M(A) \otimes A \arrow[d, "g\otimes 1_A"] \\
& B \otimes A
\end{tikzcd}$$

The corresponding statement for quantum spaces and quantum monoids is the following.

Proposition 1.3.9. (1) *Let $\mathcal{X}$ be a quantum space with universal quantum space $\mathcal{M}(\mathcal{X})$ and $\mu : \mathcal{M}(\mathcal{X}) \perp A \to A$. Then $\mathcal{M}(\mathcal{X})$ is a quantum monoid and $\mathcal{X}$ is an $\mathcal{M}(\mathcal{X})$-space by μ.*

(2) *If $\mathcal{Y}$ is another quantum monoid and if $f : \mathcal{Y} \perp \mathcal{X} \to \mathcal{X}$ defines the structure of a $\mathcal{Y}$-space on $\mathcal{X}$ then there is a unique morphism of quantum monoids $g : \mathcal{Y} \to \mathcal{M}(\mathcal{X})$ such that the following diagram commutes*

$$\begin{tikzcd}
\mathcal{Y} \perp \mathcal{X} \arrow[d, "g \perp 1_{\mathcal{X}}"'] \arrow[dr, "f"] & \\
\mathcal{M}(\mathcal{X}) \perp \mathcal{X} \arrow[r, "\mu"'] & \mathcal{X}.
\end{tikzcd}$$

Proof. We give the proof for the algebra version of the proposition. Consider the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[d, "\delta"'] & M(A) \otimes A \arrow[d, "\Delta \otimes 1_A"] \\
M(A) \otimes A \arrow[r, "1_{M(A)} \otimes \delta"'] & M(A) \otimes M(A) \otimes A
\end{tikzcd}$$

where the morphism of algebras Δ is defined by the universal property of $M(A)$ with respect to the algebra morphism $(1_{M(A)} \otimes \delta)\delta$. Furthermore there is a unique morphism of algebras $\epsilon : M(A) \to \mathbb{K}$ such that

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[dr, "1_A"'] & M(A) \otimes A \arrow[d, "\epsilon \otimes 1_A"] \\
 & A \cong \mathbb{K} \otimes A
\end{tikzcd}$$

commutes.

The coalgebra axioms arise from the following commutative diagrams

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[d, "\delta"] & M(A) \otimes A \arrow[d, "\Delta \otimes 1_A"] \\
M(A) \otimes A \arrow[r, "1_{M(A)} \otimes \delta"] \arrow[d, shift right, "\Delta \otimes 1_A"'] \arrow[d, shift left, "1_{M(A)} \otimes \delta"] & M(A) \otimes M(A) \otimes A \arrow[d, shift right, "\Delta \otimes 1_{M(A)} \otimes 1_A"'] \arrow[d, shift left, "1_{M(A)} \otimes \Delta \otimes 1_A"] \\
M(A) \otimes M(A) \otimes A \arrow[r, "1_{M(A)} \otimes 1_{M(A)} \otimes \delta"] & M(A) \otimes M(A) \otimes M(A) \otimes A
\end{tikzcd}$$

and

$$
\begin{tikzcd}
A \arrow[rr, "\delta"] \arrow[d, "\delta"] & & M(A)\otimes A \arrow[d, "\Delta\otimes 1_A"'] \arrow[dd, shift left=10ex, "1_{M(A)}\otimes 1_A"] \\
M(A)\otimes A \arrow[r, "1_{M(A)}\otimes\delta"] \arrow[drr, "1_{M(A)}\otimes 1_A"'] & M(A)\otimes M(A)\otimes A \arrow[d, phantom] & M(A)\otimes M(A)\otimes A \arrow[d, "1_{M(A)}\otimes\epsilon\otimes 1_A"'] \\
& & M(A)\otimes A \cong M(A)\otimes \mathbb{K}\otimes A
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
A \arrow[rr, "\delta"] \arrow[d, "\delta"] \arrow[dd, bend right, "1_A"'] & & M(A)\otimes A \arrow[d, "\Delta\otimes 1_A"] \\
M(A)\otimes A \arrow[rr, "1_{M(A)}\otimes\delta"] \arrow[d, "\epsilon\otimes 1_A"] & & M(A)\otimes M(A)\otimes A \arrow[d, "\epsilon\otimes 1_{M(A)}\otimes 1_A"] \\
A \arrow[rr, "\delta"] & & M(A)\otimes A \cong \mathbb{K}\otimes M(A)\otimes A.
\end{tikzcd}
$$

In fact these diagrams imply by the uniqueness of the induced homomorphisms of algebras $(\Delta\otimes 1_{M(A)})\Delta = (1_{M(A)}\otimes\Delta)\Delta$, $(1_{M(A)}\otimes\epsilon)\Delta = 1_{M(A)}$ and $\epsilon\otimes(1_{M(A)})\Delta = 1_{M(A)}$. Finally A is an $M(A)$-comodule algebra by the definition of Δ and ϵ.

Now assume that a structure of a B-comodule algebra on A is given by a bialgebra B and $f : A \to B\otimes A$. Then there is a unique homomorphism of algebras $g : M(A) \to B$ such that the diagram

$$
\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[dr, "f"'] & M(A)\otimes A \arrow[d, "g\otimes 1_A"] \\
& B\otimes A
\end{tikzcd}
$$

commutes. Then the following diagram

$$
\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[dr, "f"'] & M(A)\otimes A \arrow[r, shift left, "\Delta\otimes 1_A"] \arrow[r, shift right, "1_{M(A)}\otimes\delta"'] \arrow[d, "g\otimes 1_A"] & M(A)\otimes M(A)\otimes A \arrow[d, "g\otimes g\otimes 1_A"] \\
& B\otimes A \arrow[r, shift left, "\Delta_B\otimes 1_A"] \arrow[r, shift right, "1_B\otimes f"'] & B\otimes B\otimes A
\end{tikzcd}
$$

implies $((g\otimes g)\Delta\otimes 1_A)\delta = (g\otimes g\otimes 1_A)(\Delta\otimes 1_A)\delta = (g\otimes g\otimes 1_A)(1_{M(A)}\otimes\delta)\delta = (g\otimes(g\otimes 1_A)\delta)\delta = (1_B\otimes(g\otimes 1_A)\delta)(g\otimes 1_A)\delta = (1_B\otimes f)f = (\Delta_B\otimes 1_A)f = (\Delta_B\otimes$

$1_A)(g \otimes 1_A)\delta = (\Delta_B g \otimes 1_A)\delta$ hence $(g \otimes g)\Delta = \Delta_B g$. Furthermore the diagram

$$\begin{tikzcd}
A \arrow[rr, "\delta"] \arrow[drr, "f"] \arrow[ddrr, "1_A"'] & & M(A) \otimes A \arrow[d, "g\otimes 1_A"'] \arrow[dd, shift left=6, "\epsilon\otimes 1_A"] \\
& & B \otimes A \arrow[d, "\epsilon_B\otimes 1_A"'] \\
& & A \cong \mathbb{K} \otimes A
\end{tikzcd}$$

implies $\epsilon_B g = \epsilon$. Thus g is a homomorphism of bialgebras. □

Since universal algebras for algebras A tend to become very big they do not exist in general. But a theorem of Tambara's says that they exist for finite dimensional algebras (over a field $\mathbb{K}$).

Definition 1.3.10. If $\mathcal{X}$ is a quantum space with finite dimensional function algebra then we call $\mathcal{X}$ a *finite quantum space.*

The following theorem is the quantum space version and equivalent to a theorem of Tambara.

Theorem 1.3.11. *Let $\mathcal{X}$ be a finite quantum space. Then there exists a (universal) quantum space $\mathcal{M}(\mathcal{X})$ with morphism of quantum spaces $\mu : \mathcal{M}(\mathcal{X}) \perp \mathcal{X} \to \mathcal{X}$.*

The algebra version of this theorem is

Theorem 1.3.12. (Tambara) *Let A be a finite dimensional $\mathbb{K}$-algebra. Then there exists a (universal) $\mathbb{K}$-algebra $M(A)$ with homomorphism of algebras $\delta : A \to M(A) \otimes A$.*

Proof. We are going to construct the $\mathbb{K}$-algebra $M(A)$ quite explicitly. First we observe that $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ is a coalgebra (cf. [Advanced Algebra] Problem 2.25) with the structural morphism $\Delta : A^* \to (A \otimes A)^* \cong A^* \otimes A^*$. Denote the dual basis by $\sum_{i=1}^n a_i \otimes \bar{a}^i \in A \otimes A^*$. Now let $T(A \otimes A^*)$ be the tensor algebra of the vector space $A \otimes A^*$. Consider elements of the tensor algebra

$$\begin{aligned}
&xy \otimes \zeta \in A \otimes A^*,\\
&x \otimes y \otimes \Delta(\zeta) \in A \otimes A \otimes A^* \otimes A^* \cong A \otimes A^* \otimes A \otimes A^*,\\
&1 \otimes \zeta \in A \otimes A^*,\\
&\zeta(1) \in \mathbb{K}.
\end{aligned}$$

The following elements

$$xy \otimes \zeta - x \otimes y \otimes \Delta(\zeta) \tag{1}$$

and

$$1 \otimes \zeta - \zeta(1) \tag{2}$$

generate a two-sided ideal $I \subseteq T(A \otimes A^*)$. Now we define

$$M(A) := T(A \otimes A^*)/I$$

and the cooperation $\delta : A \ni a \to \sum_{i=1}^n (a \otimes \bar{a}^i) \otimes a_i \in T(A \otimes A^*)/I \otimes A$. This is a well-defined linear map.

To show that this map is a homomorphism of algebras we first describe the multiplication of A by $a_i a_j = \sum_k \alpha_{ij}^k a_k$. Then the comultiplication of A^* is given by $\Delta(\bar{a}^k) = \sum_{ij} \alpha_{ij}^k \bar{a}^i \otimes \bar{a}^j$ since $(\Delta(\bar{a}^k), a_l \otimes a_m) = (\bar{a}^k, a_l a_m) = \sum_r \alpha_{lm}^r (\bar{a}^k, a_r) = \alpha_{lm}^k = \sum_{ij} \alpha_{ij}^k (\bar{a}^i, a_l)(\bar{a}^j, a_m) = (\sum_{ij} \alpha_{ij}^k \bar{a}^i \otimes \bar{a}^j, a_l \otimes a_m)$. Now write $1 = \sum \beta^k a_k$. Then we get $\epsilon(\bar{a}^i) = \beta^i$ since $\epsilon(\bar{a}^i) = (\bar{a}^i, 1) = \sum_j \beta^j (\bar{a}^i, a_j) = \beta^i$. So we have $\delta(a)\delta(b) = (\sum_{i=1}^n (a \otimes \bar{a}^i) \otimes a_i) \cdot (\sum_{j=1}^n (b \otimes \bar{a}^j) \otimes a_j) = \sum_{ij} (a \otimes b \otimes \bar{a}^i \otimes \bar{a}^j) \otimes a_i a_j = \sum_{ijk} \alpha_{ij}^k (a \otimes b \otimes \bar{a}^i \otimes \bar{a}^j) \otimes a_k = \sum_k (a \otimes b \otimes \Delta(\bar{a}^k)) \otimes a_k = \sum_k (ab \otimes \bar{a}^k) \otimes a_k = \delta(ab)$. Furthermore we have $\delta(1) = \sum_i (1 \otimes \bar{a}^i) \otimes a_i = \sum_i \bar{a}^i(1) \otimes a_i = 1 \otimes \sum_i \bar{a}^i(1) a_i = 1 \otimes 1$. Hence δ is a homomorphism of algebras.

Now we have to show that there is a unique g for each f. First of all $f : A \to B \otimes A$ induces uniquely determined linear maps $f_i : A \to B$ with $f(a) = \sum_i f_i(a) \otimes a_i$ since the a_i form a basis. Since f is a homomorphism of algebras we get from $\sum_k f_k(a) \otimes a_k = f(ab) = f(a)f(b) = \sum_{ij} (f_i(a) \otimes a_i)(f_j(b) \otimes a_j) = \sum_{ij} f_i(a) f_j(b) \otimes a_i a_j = \sum_{ijk} \alpha_{ij}^k f_i(a) f_j(b) \otimes a_k$ by comparison of coefficients

$$f_k(ab) = \sum_{ij} \alpha_{ij}^k f_i(a) f_j(b).$$

Furthermore we define $g(a \otimes \bar{a}) := (1 \otimes \bar{a}) f(a) \in B$. Then we get in particular $g(a \otimes \bar{a}^i) = (1 \otimes \bar{a}^i)(\sum_j f_j(a) \otimes a_j) = f_i(a)$. We can extend the map g to a homomorphism of algebras $g : T(A \otimes A^*) \to B$. Applied to the generators of the ideal we get $g(ab \otimes \bar{a}^k - a \otimes b \otimes \Delta(\bar{a}^k)) = (1 \otimes \bar{a}^k) \sum_l f_l(ab) \otimes a_l - \sum_{rsij} \alpha_{ij}^k (1 \otimes \bar{a}^i)(f_r(a) \otimes a_r) \cdot (1 \otimes \bar{a}^j)(f_s(b) \otimes a_s) = f_k(ab) - \sum_{ij} \alpha_{ij}^k f_i(a) f_j(b) = 0$. We have furthermore $g(1 \otimes \zeta - \zeta(1)) = (1 \otimes \zeta) f(1) - \zeta(1) = (1 \otimes \zeta)(1 \otimes 1) - \zeta(1) = 1\zeta(1) - \zeta(1) = 0$. Thus the homomorphism of algebras g vanishes on the ideal I so it may be factored through $M(A) = T(A)/I$. Denote this factorization also by g. Then the diagram commutes since $(g \otimes 1_A)\delta(a) = (g \otimes 1_A)(\sum_i (a \otimes \bar{a}^i) \otimes a_i) = \sum_i (1 \otimes \bar{a}^i) f(a) \otimes a_i = \sum_{ij} f_j(a)(\bar{a}^i, a_j) \otimes a_i = \sum_i f_i(a) \otimes a_i = f(a)$.

We still have to show that g is uniquely determined. Assume that we also have $(h \otimes 1_A)\delta = f$ then $\sum_i h(a \otimes \bar{a}^i) \otimes a_i = (h \otimes 1_A)\delta(a) = f(a) = \sum_i f_i(a) \otimes a_i$ hence $h(a \otimes \bar{a}^i) = f_i(a) = g(a \otimes \bar{a}^i)$, i.e. $g = h$. $\square$

Definition 1.3.13. Let A be a $\mathbb{K}$-algebra. The universal algebra $M(A)$ for A that is a bialgebra is also called the *coendomorphism bialgebra* of A.

Problem 1.3.7. (1) Determine explicitly the dual coalgebra A^* of the algebra $A := \mathbb{K}\langle x \rangle / (x^2)$. (Hint: Find a basis for A.)

(2) Determine and describe the coendomorphism bialgebra of A from problem 1.1. (Hint: Determine first a set of algebra generators of $M(A)$. Then describe the relations.)
(3) Determine explicitly the dual coalgebra A^* of $A := \mathbb{K}\langle x \rangle/(x^3)$.
(4) Determine and describe the coendomorphism bialgebra of A from problem 1.3.
(5) (*) Determine explicitly the dual coalgebra A^* of $A := \mathbb{K}\langle x, y\rangle/I$ where the ideal I is generated as a two-sided ideal by the polynomials

$$xy - q^{-1}yx, x^2, y^2.$$

(6) (*) Determine the coendomorphism bialgebra of A from problem 1.5.
(7) Let A be a finite dimensional $\mathbb{K}$-algebra with universal bialgebra $A \longrightarrow B \otimes A$. Show
   i) that $A^{op} \longrightarrow B^{op} \otimes A^{op}$ is universal (where A^{op} has the multiplication $\nabla\tau : A \otimes A \longrightarrow A \otimes A \longrightarrow A$);
   ii) that $A \cong A^{op}$ implies $B \cong B^{op}$ (as bialgebras);
   iii) that for commutative algebras A the algebra B satisfies $B \cong B^{op}$ but that B need not be commutative.
   iv) Find an isomorphism $B \cong B^{op}$ for the bialgebra $B = \mathbb{K}\langle a, b\rangle/(a^2, ab + ba)$. (compare Problem 1.7 2).
(8) Consider the algebra $K[\epsilon]/(\epsilon^2)$ the so called algebra of dual numbers over a field K. Consider the algebra B with (noncommuting) generators a, b, c, d and relations:

$$\begin{aligned}
&ac = acac \quad ad = acad + adac \quad bc = acbc = bcac\\
&bd = acbd + adbc = bcad + bdac \quad bcbc = bcbd + bdbc = 0\\
&ac = 1 \quad ad = 0
\end{aligned}$$

Show that B together with the cooperation

$$\delta : A \longrightarrow B \otimes A$$

with $\delta(\epsilon) = bc \otimes 1 + bd \otimes \epsilon$ is the coendomorphism bialgebra of A.

Sketch of solution:

1/2. The bialgebra has the form $B = \mathbb{K}\langle a, b\rangle/(a^2, ab+ba)$ with $\Delta(a) = a\otimes 1+b\otimes a$, $\Delta(b) = b \otimes b$ and $\varepsilon(a) = 0$, $\varepsilon(b) = 1$. The coaction is $\delta(x) = a \otimes 1 + b \otimes x$.

3/4. A has the basis $1, x, x^2$. The dual coalgebra has the dual basis e, ζ, ζ_2 with $\Delta(e) = e \otimes e$, $\Delta(\zeta) = \zeta \otimes e + e \otimes \zeta$ and $\Delta(\zeta_2) = \zeta_2 \otimes e + \zeta \otimes \zeta + e \otimes \zeta_2$.

The universal bialgebra $B = T(A \otimes A^*)/I$ satisfies $\delta(x) = x \otimes e \otimes 1 + x \otimes \zeta \otimes x + x \otimes \zeta_2 \otimes x^2 = a \otimes 1 + b \otimes x + c \otimes x^2$. Thus it is generated by the elements $a = x \otimes e$,

$b = x \otimes \zeta$ and $c = x \otimes \zeta_2$. The multiplication table and the relations arise from

$$\begin{aligned}
&1 \otimes e = 1,\\
&1 \otimes \zeta = 1 \otimes \zeta_2 = 0,\\
&x^2 \otimes e = (x \otimes e)(x \otimes e),\\
&x^2 \otimes \zeta = (x \otimes \zeta)(x \otimes e) + (x \otimes e)(x \otimes \zeta),\\
&x^2 \otimes \zeta_2 = (x \otimes \zeta_2)(x \otimes e) + (x \otimes \zeta)(x \otimes \zeta) + (x \otimes e)(x \otimes \zeta_2),
\end{aligned}$$

$$\begin{aligned}
&0 = x^3 \otimes e = (x^2 \otimes e)(x \otimes e),\\
&0 = x^3 \otimes \zeta = (x^2 \otimes \zeta)(x \otimes e) + (x^2 \otimes e)(x \otimes \zeta),\\
&0 = x^3 \otimes \zeta_2 = (x^2 \otimes \zeta_2)(x \otimes e) + (x^2 \otimes \zeta)(x \otimes \zeta) + (x^2 \otimes e)(x \otimes \zeta_2)
\end{aligned}$$

We use the abbreviation $\{u, v\} := u^2v + uvu + vu^2$ and have

$$\begin{aligned}
&a^3 = 0,\\
&\{a, b\} = 0,\\
&\{a, c\} + \{b, a\} = 0.
\end{aligned}$$

The condition $(1 \otimes \delta)\delta = (\Delta \otimes 1)\delta$ implies

$$\begin{aligned}
&\Delta(a) = a \otimes 1 + b \otimes a + c \otimes a^2,\\
&\Delta(b) = b \otimes b + c \otimes (ba + ab),\\
&\Delta(c) = b \otimes c + c \otimes b^2 + c \otimes (ca + ac),\\
&\epsilon(a) = 0,\\
&\epsilon(b) = 1,\\
&\epsilon(c) = 0.
\end{aligned}$$

5/6. A has the basis $1, x, y, xy$. The dual basis of A^* is denoted by e, ξ, η, θ. The diagonal is

$$\begin{aligned}
&\Delta(e) = e \otimes e,\\
&\Delta(\xi) = \xi \otimes e + e \otimes \xi,\\
&\Delta(\eta) = \eta \otimes e + e \otimes \eta,\\
&\Delta(\theta) = \theta \otimes e + e \otimes \theta + \xi \otimes \eta + q\eta \otimes \xi.
\end{aligned}$$

Thus the coendomorphism bialgebra has the algebra generators $a \otimes \zeta$ with $a \in \{1, x, y, xy\}$ and $\zeta \in \{e, \xi, \eta, \theta\}$. The generators of the relations (of I) are given by the equations 1.1 and 1.2. They imply that $1 \otimes e$ is the unit element, that $1 \otimes \xi = 1 \otimes \eta = 1 \otimes \theta = 0$ and that

$$\begin{aligned}
&ab \otimes e = (a \otimes e)(b \otimes e),\\
&ab \otimes \xi = (a \otimes \xi)(b \otimes e) + (a \otimes 1)(b \otimes \xi),\\
&ab \otimes \eta = (a \otimes \eta)(b \otimes e) + (a \otimes 1)(b \otimes \eta),\\
&ab \otimes \theta = (a \otimes \theta)(b \otimes e) + (a \otimes 1)(b \otimes \theta) + (a \otimes \xi)(b \otimes \eta) + q(a \otimes \eta)(b \otimes \xi).
\end{aligned}$$

Furthermore for ab we have to take into account the relations in A.

We define

$$\begin{aligned}
&a := x \otimes e, \quad b := x \otimes \xi, \quad c := x \otimes \eta, \quad d := x \otimes \theta,\\
&e := y \otimes e, \quad f := y \otimes \xi, \quad g := y \otimes \eta, \quad h := x \otimes \theta,
\end{aligned}$$

and get $\delta(x) = a\otimes 1 + b\otimes x + c\otimes y + d\otimes xy$ and $\delta(y) = e\otimes 1 + f\otimes x + g\otimes y + h\otimes xy$. Hence B is generated by $a, \dots, h$ as an algebra. The relations are

$$\begin{aligned}
&a^2 = e^2 = 0,\\
&ab + ba = ac + ca = ef + fe = eg + ge = 0,\\
&ad + da + bc + qcb = eh + he + fg + qgf = 0,\\
&ae = qea,\\
&af + be = q(fa + eb),\\
&ag + ce = q(ga + ec),\\
&ah - qha + de - qed + bg - q^2gb + qcf - qfc = 0.
\end{aligned}$$

The diagonal is

$$\begin{aligned}
\Delta(a) &= a\otimes 1 + b\otimes a + c\otimes e + d\otimes ae,\\
\Delta(b) &= b\otimes b + c\otimes f + d\otimes (af + be),\\
\Delta(c) &= b\otimes c + c\otimes g + d\otimes (ag + ce),\\
\Delta(d) &= b\otimes d + c\otimes h + d\otimes (ah + de + bg + q^{-1}cf) \quad \text{etc.}
\end{aligned}$$

CHAPTER 2

Hopf Algebras, Algebraic, Formal, and Quantum Groups

Introduction

In the first chapter we have encountered quantum monoids and studied their role as monoids operating on quantum spaces. The "elements" of quantum monoids operating on quantum spaces should be understood as endomorphisms of the quantum spaces. In the construction of the multiplication for universal quantum monoids of quantum spaces we have seen that this multiplication is essentially the "composition" of endomorphisms.

We are, however, primarily interested in automorphisms and we know that automorphisms should form a group under composition. This chapter is devoted to finding group structures on quantum monoids, i.e. to define and study quantum groups.

This is easy in the commutative situation, i.e. if the representing algebra of a quantum monoid is commutative. Then we can define a morphism that sends elements of the quantum group to their inverses. This will lead us to the notion of affine algebraic groups.

In the noncommutative situation, however, it will turn out that such an inversion morphism (of quantum spaces) does not exist. It will have to be replaced by a more complicated construction. Thus quantum groups will not be groups in the sense of category theory. Still we will be able to perform one of the most important and most basic constructions in group theory, the formation of the group of invertible elements of a monoid. In the case of a quantum monoid acting universally on a quantum space this will lead to the good definition of a quantum automorphism group of the quantum space.

In order to have the appropriate tools for introducing quantum groups we first introduce Hopf algebras which will be the representing algebras of quantum groups. Furthermore we need the notion of a monoid and of a group in a category. We will see, however, that quantum groups are in general not groups in the category of quantum spaces.

We first study the simple cases of affine algebraic groups and of formal groups. They will have Hopf algebras as representing objects and will indeed be groups in reasonable categories. Then we come to quantum groups, and construct quantum automorphism groups of quantum spaces.

At the end of the chapter you should

- know what a Hopf algebra is,

- know what a group in a category is,
- know the definition and examples of affine algebraic groups and formal groups,
- know the definition and examples of quantum groups and be able to construct quantum automorphism groups for small quantum spaces,
- understand why a Hopf algebra is a reasonable representing algebra for a quantum group.

1. Hopf Algebras

The difference between a monoid and a group lies in the existence of an additional map $S : G \ni g \mapsto g^{-1} \in G$ for a group G that allows forming inverses. This map satisfies the equation $S(g)g = 1$ or in a diagrammatic form

$$\begin{tikzcd} G \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \{1\} \arrow[r, "1"] & G \\ G \times G \arrow[rr, "S\times\mathrm{id}"] & & G \times G \arrow[u, "\mathrm{mult}"'] \end{tikzcd}$$

We want to carry this property over to bialgebras B instead of monoids. An "inverse map" shall be a morphism $S : B \longrightarrow B$ with a similar property. This will be called a Hopf algebra.

Definition 2.1.1. A *left Hopf algebra* H is a bialgebra H together with a *left antipode* $S : H \longrightarrow H$, i.e. a $\mathbb{K}$-module homomorphism S such that the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & H \\ H \otimes H \arrow[rr, "S\otimes\mathrm{id}"] & & H \otimes H \arrow[u, "\nabla"'] \end{tikzcd}$$

Symmetrically we define a *right Hopf algebra* H. A *Hopf algebra* is a left and right Hopf algebra. The map S is called a (left, right, two-sided) *antipode*.

Using the Sweedler notation (2.20) the commutative diagram above can also be expressed by the equation

$$\sum S(a_{(1)})a_{(2)} = \eta\varepsilon(a)$$

for all $a \in H$. Observe that we do not require that $S : H \longrightarrow H$ is an algebra homomorphism.

Problem 2.1.8. (1) Let H be a bialgebra and $S \in \mathrm{Hom}(H, H)$. Then S is an antipode for H (and H is a Hopf algebra) iff S is a two sided inverse for id in the algebra $(\mathrm{Hom}(H, H), *, \eta\varepsilon)$ (see 2.21). In particular S is uniquely determined.

(2) Let H be a Hopf algebra. Then S is an antihomomorphism of algebras and coalgebras i.e. S "inverts the order of the multiplication and the comultiplication".

(3) Let H and K be Hopf algebras and let $f : H \longrightarrow K$ be a homomorphism of bialgebras. Then $fS_H = S_K f$, i.e. f is compatible with the antipode.

Definition 2.1.2. Because of Problem (22) every homomorphism of bialgebras between Hopf algebras is compatible with the antipodes. So we define a *homomorphism of Hopf algebras* to be a homomorphism of bialgebras. The category of Hopf algebras will be denoted by $\mathbb{K}\text{-}\mathcal{H}opf$.

Proposition 2.1.3. *Let H be a bialgebra with an algebra generating set X. Let $S : H \longrightarrow H^{op}$ be an algebra homomorphism such that $\sum S(x_{(1)})x_{(2)} = \eta\varepsilon(x)$ for all $x \in X$. Then S is a left antipode of H.*

PROOF. Assume $a, b \in H$ such that $\sum S(a_{(1)})a_{(2)} = \eta\varepsilon(a)$ and $\sum S(b_{(1)})b_{(2)} = \eta\varepsilon(b)$. Then

$$\begin{aligned} \sum S((ab)_{(1)})(ab)_{(2)} &= \sum S(a_{(1)}b_{(1)})a_{(2)}b_{(2)} = \sum S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)} \\ &= \sum S(b_{(1)})\eta\varepsilon(a)b_{(2)} = \eta\varepsilon(a)\eta\varepsilon(b) = \eta\varepsilon(ab). \end{aligned}$$

Since every element of H is a finite sum of finite products of elements in X, for which the equality holds, this equality extends to all of H by induction. □

Example 2.1.4. (1) Let V be a vector space and $T(V)$ the tensor algebra over V. We have seen in Problem 2.2 that $T(V)$ is a bialgebra and that V generates $T(V)$ as an algebra. Define $S : V \longrightarrow T(V)^{op}$ by $S(v) := -v$ for all $v \in V$. By the universal property of the tensor algebra this map extends to an algebra homomorphism $S : T(V) \longrightarrow T(V)^{op}$. Since $\Delta(v) = v \otimes 1 + 1 \otimes v$ we have $\sum S(v_{(1)})v_{(2)} = \nabla(S \otimes 1)\Delta(v) = -v + v = 0 = \eta\varepsilon(v)$ for all $v \in V$, hence $T(V)$ is a Hopf algebra by the preceding proposition.

(2) Let V be a vector space and $S(V)$ the symmetric algebra over V (that is commutative). We have seen in Problem 2.3 that $S(V)$ is a bialgebra and that V generates $S(V)$ as an algebra. Define $S : V \longrightarrow S(V)$ by $S(v) := -v$ for all $v \in V$. S extends to an algebra homomorphism $S : S(V) \longrightarrow S(V)$. Since $\Delta(v) = v \otimes 1 + 1 \otimes v$ we have $\sum S(v_{(1)})v_{(2)} = \nabla(S \otimes 1)\Delta(v) = -v + v = 0 = \eta\varepsilon(v)$ for all $v \in V$, hence $S(V)$ is a Hopf algebra by the preceding proposition.

Example 2.1.5. **(Group Algebras)** For each algebra A we can form the *group of units* $U(A) := \{a \in A | \exists a^{-1} \in A\}$ with the multiplication of A as composition of the group. Then U is a covariant functor $U : \mathbb{K}\text{-}\mathcal{A}lg \longrightarrow \mathcal{G}r$. This functor leads to the following universal problem.

Let G be a group. An algebra $\mathbb{K}G$ together with a group homomorphism $\iota : G \longrightarrow U(\mathbb{K}G)$ is called a (the) *group algebra of G*, if for every algebra A and for every group homomorphism $f : G \longrightarrow U(A)$ there exists a unique homomorphism of algebras

$g : \mathbb{K}G \to A$ such that the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[dr, "f"'] & U(\mathbb{K}G) \arrow[d, "g"] \\ & U(A). \end{tikzcd}$$

The group algebra $\mathbb{K}G$ is (if it exists) unique up to isomorphism. It is generated as an algebra by the image of G. The map $\iota : G \to U(\mathbb{K}G) \subseteq \mathbb{K}G$ is injective and the image of G in $\mathbb{K}G$ is a basis.

The group algebra can be constructed as the free vector space $\mathbb{K}G$ with basis G and the algebra structure of $\mathbb{K}G$ is given by $\mathbb{K}G \otimes \mathbb{K}G \ni g \otimes h \mapsto gh \in \mathbb{K}G$ and the unit $\eta : \mathbb{K} \ni \alpha \mapsto \alpha e \in \mathbb{K}G$.

The group algebra $\mathbb{K}G$ is a Hopf algebra. The comultiplication is given by the diagram

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{K}G \arrow[d, "\Delta"] \\ & \mathbb{K}G \otimes \mathbb{K}G \end{tikzcd}$$

with $f(g) := g \otimes g$ which defines a group homomorphism $f : G \to U(\mathbb{K}G \otimes \mathbb{K}G)$. The counit is given by

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{K}G \arrow[d, "\varepsilon"] \\ & \mathbb{K} \end{tikzcd}$$

where $f(g) = 1$ for all $g \in G$. One shows easily by using the universal property, that Δ is coassociative and has counit ε. Define an algebra homomorphism $S : \mathbb{K}G \to (\mathbb{K}G)^{op}$ by

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{K}G \arrow[d, "S"] \\ & (\mathbb{K}G)^{op} \end{tikzcd}$$

with $f(g) := g^{-1}$ which is a group homomorphism $f : G \to U((\mathbb{K}G)^{op})$. Then one shows with Proposition 1.3 that $\mathbb{K}G$ is a Hopf algebra.

The example $\mathbb{K}G$ of a Hopf algebra gives rise to the definition of particular elements in arbitrary Hopf algebras, that share certain properties with elements of a group. We will use and study these elements later on in in chapter 4.

Definition 2.1.6. Let H be a Hopf algebra. An element $g \in H, g \neq 0$ is called a *grouplike element* if

$$\Delta(g) = g \otimes g.$$

Observe that $\varepsilon(g) = 1$ for each grouplike element g in a Hopf algebra H. In fact we have $g = \nabla(\varepsilon \otimes 1)\Delta(g) = \varepsilon(g)g \neq 0$ hence $\varepsilon(g) = 1$. If the base ring is not a field then one adds this property to the definition of a grouplike element.

Problem 2.1.9. (1) Let $\mathbb{K}$ be a field. Show that an element $x \in \mathbb{K}G$ satisfies $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$ if and only if $x = g \in G$.
(2) Show that the grouplike elements of a Hopf algebra form a group under multiplication of the Hopf algebra.

Example 2.1.7. (Universal Enveloping Algebras) A *Lie algebra* consists of a vector space $\mathfrak{g}$ together with a (linear) multiplication $\mathfrak{g} \otimes \mathfrak{g} \ni x \otimes y \mapsto [x, y] \in \mathfrak{g}$ such that the following laws hold:

$$\begin{aligned} &[x, x] = 0, \\ &[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \text{(Jacobi identity).} \end{aligned}$$

A *homomorphism of Lie algebras* $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a linear map f such that $f([x, y]) = [f(x), f(y)]$. Thus Lie algebras form a category $\mathbb{K}$-$\mathcal{L}ie$.

An important example is the Lie algebra associated with an associative algebra (with unit). If A is an algebra then the vector space A with the Lie multiplication

$$[x, y] := xy - yx$$

is a Lie algebra denoted by A^L. This construction of a Lie algebra defines a covariant functor $\text{-}^L : \mathbb{K}\text{-}\mathcal{A}lg \longrightarrow \mathbb{K}\text{-}\mathcal{L}ie$. This functor leads to the following universal problem.

Let $\mathfrak{g}$ be a Lie algebra. An algebra $U(\mathfrak{g})$ together with a Lie algebra homomorphism $\iota : \mathfrak{g} \longrightarrow U(\mathfrak{g})^L$ is called a (the) *universal enveloping algebra of* $\mathfrak{g}$, if for every algebra A and for every Lie algebra homomorphism $f : \mathfrak{g} \longrightarrow A^L$ there exists a unique homomorphism of algebras $g : U(\mathfrak{g}) \longrightarrow A$ such that the following diagram commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "f"'] & U(\mathfrak{g})^L \arrow[d, "g"] \\ & A^L. \end{tikzcd}$$

The universal enveloping algebra $U(\mathfrak{g})$ is (if it exists) unique up to isomorphism. It is generated as an algebra by the image of $\mathfrak{g}$.

The universal enveloping algebra can be constructed as $U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y])$ where $T(\mathfrak{g}) = \mathbb{K} \oplus \mathfrak{g} \oplus \mathfrak{g} \otimes \mathfrak{g} \ldots$ is the tensor algebra. The map $\iota : \mathfrak{g} \longrightarrow U(\mathfrak{g})^L$ is injective.

The universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra. The comultiplication is given by the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "f"'] & U(\mathfrak{g}) \arrow[d, "\Delta"] \\ & U(\mathfrak{g}) \otimes U(\mathfrak{g}) \end{tikzcd}$$

with $f(x) := x \otimes 1 + 1 \otimes x$ which defines a Lie algebra homomorphism $f : \mathfrak{g} \to (U(\mathfrak{g}) \otimes U(\mathfrak{g}))^L$. The counit is given by

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "f"'] & U(\mathfrak{g}) \arrow[d, "\varepsilon"] \\ & \mathbb{K} \end{tikzcd}$$

with $f(x) = 0$ for all $x \in \mathfrak{g}$. One shows easily by using the universal property, that Δ is coassociative and has counit ε. Define an algebra homomorphism $S : U(\mathfrak{g}) \to (U(\mathfrak{g}))^{op}$ by

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "f"'] & U(\mathfrak{g}) \arrow[d, "S"] \\ & (U(\mathfrak{g}))^{op} \end{tikzcd}$$

with $f(x) := -x$ which is a Lie algebra homomorphism $f : \mathfrak{g} \to (U(\mathfrak{g})^{op})^L$. Then one shows with Proposition 1.3 that $U(\mathfrak{g})$ is a Hopf algebra.

(Observe, that the meaning of U in this example and the previous example (group of units, universal enveloping algebra) is totally different, in the first case U can be applied to an algebra and gives a group, in the second case U can be applied to a Lie algebra and gives an algebra.)

The preceding example of a Hopf algebra gives rise to the definition of particular elements in arbitrary Hopf algebras, that share certain properties with elements of a Lie algebra. We will use and study these elements later on in chapter 4.

Definition 2.1.8. Let H be a Hopf algebra. An element $x \in H$ is called a *primitive element* if

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$

Let $g \in H$ be a grouplike element. An element $x \in H$ is called a *skew primitive or g-primitive element* if

$$\Delta(x) = x \otimes 1 + g \otimes x.$$

Problem 2.1.10. Show that the set of primitive elements $P(H) = \{x \in H | \Delta(x) = x \otimes 1 + 1 \otimes x\}$ of a Hopf algebra H is a Lie subalgebra of H^L.

Proposition 2.1.9. *Let H be a Hopf algebra with antipode S. The following are equivalent:*
(1) $S^2 = id$.
(2) $\sum S(a_{(2)})a_{(1)} = \eta\varepsilon(a)$ for all $a \in H$.
(3) $\sum a_{(2)}S(a_{(1)}) = \eta\varepsilon(a)$ for all $a \in H$.

PROOF. Let $S^2 =$id. Then

$$\begin{aligned}\sum S(a_{(2)})a_{(1)} &= S^2(\sum S(a_{(2)})a_{(1)}) = S(\sum S(a_{(1)})S^2(a_{(2)})) \\ &= S(\sum S(a_{(1)})a_{(2)}) = S(\eta\varepsilon(a)) = \eta\varepsilon(a)\end{aligned}$$

by using Problem (21).

Conversely assume that (2) holds. Then

$$\begin{aligned}S * S^2(a) &= \sum S(a_{(1)}S^2(a_{(2)}) = S(\sum S(a_{(2)})a_{(1)} \\ &= S(\eta\varepsilon(a)) = \eta\varepsilon(a).\end{aligned}$$

Thus S^2 and id are inverses of S in the convolution algebra $\mathrm{Hom}(H, H)$, hence $S^2 = \mathrm{id}$.

Analogously one shows that (1) and (3) are equivalent. □

Corollary 2.1.10. *If H is a commutative Hopf algebra or a cocommutative Hopf algebra with antipode S, then $S^2 = id$.*

Remark 2.1.11. Kaplansky: Ten conjectures on Hopf algebras

In a set of lecture notes on bialgebras based on a course given at Chicago university in 1973, made public in 1975, Kaplansky formulated a set of conjectures on Hopf algebras that have been the aim of intensive research.

(1) If C is a Hopf subalgebra of the Hopf algebra B then B is a free left C-module.
(Yes, if H is finite dimensional [Nichols-Zoeller]; No for infinite dimensional Hopf algebras [Oberst-Schneider]; $B : C$ is not necessarily faithfully flat [Schauenburg])

(2) Call a coalgebra C *admissible* if it admits an algebra structure making it a Hopf algebra. The conjecture states that C is admissible if and only if every finite subset of C lies in a finite-dimensional admissible subcoalgebra.
(Remarks.
(a) Both implications seem hard.
(b) There is a corresponding conjecture where "Hopf algebra" is replaced by "bialgebra".
(c) There is a dual conjecture for locally finite algebras.)
(No results known.)

(3) A Hopf algebra of characteristic 0 has no non-zero central nilpotent elements.
(First counter example given by [Schmidt-Samoa]. If H is unimodular and not semisimple, e.g. a Drinfel'd double of a not semisimple finite dimensional Hopf algebra, then the integral Λ satisfies $\Lambda \neq 0$, $\Lambda^2 = \varepsilon(\Lambda)\Lambda = 0$ since $D(H)$ is not semisimple, and $a\Lambda = \varepsilon(a)\Lambda = \Lambda\varepsilon(a) = \Lambda a$ since $D(H)$ is unimodular [Sommerhäuser].)

(4) (Nichols). Let x be an element in a Hopf algebra H with antipode S. Assume that for any a in H we have

$$\sum b_i x S(c_i) = \varepsilon(a)x$$

where $\Delta a = \sum b_i \otimes c_i$. Conjecture: x is in the center of H.

$(ax = \sum a_{(1)}x\varepsilon(a_{(2)}) = \sum a_{(1)}xS(a_{(2)})a_{(3)}) = \sum \varepsilon(a_{(1)})xa_{(2)} = xa.)$

In the remaining six conjectures H is a finite-dimensional Hopf algebra over an algebraically closed field.

(5) If H is semisimple on either side (i.e. either H or the dual H^* is semisimple as an algebra) the square of the antipode is the identity.

(Yes if $\mathrm{char}(\mathbb{K}) = 0$ [Larson-Radford], yes if $\mathrm{char}(\mathbb{K})$ is large [Sommerhäuser])

(6) The size of the matrices occurring in any full matrix constituent of H divides the dimension of H.

(Yes is Hopf algebra is defined over $\mathbb{Z}$ [Larson]; in general not known; work by [Montgomery-Witherspoon], [Zhu], [Gelaki])

(7) If H is semisimple on both sides the characteristic does not divide the dimension.

(Larson-Radford)

(8) If the dimension of H is prime then H is commutative and cocommutative.

(Yes in characteristic 0 [Zhu: 1994])

Remark. Kac, Larson, and Sweedler have partial results on 5 – 8.

(Was also proved by [Kac])

In the two final conjectures assume that the characteristic does not divide the dimension of H.

(9) The dimension of the radical is the same on both sides.

(Counterexample by [Nichols]; counterexample in Frobenius-Lusztig kernel of $U_q(sl(2))$ [Schneider])

(10) There are only a finite number (up to isomorphism) of Hopf algebras of a given dimension.

(Yes for semisimple, cosemisimple Hopf algebras: Stefan 1997)

(Counterexamples: [Andruskiewitsch, Schneider], [Beattie, others] 1997)

2. Monoids and Groups in a Category

Before we use Hopf algebras to describe quantum groups and some of the better known groups, such as affine algebraic groups and formal groups, we introduce the concept of a general group (and of a monoid) in an arbitrary category. Usually this concept is defined with respect to a categorical product in the given category. But in some categories there are in general no products. Still, one can define the concept of a group in a very simple fashion. We will start with that definition and then show that it coincides with the usual notion of a group in a category in case that category has finite products.

Definition 2.2.1. Let $\mathcal{C}$ be an arbitrary category. Let $G \in \mathcal{C}$ be an object. We use the notation $G(X) := \mathrm{Mor}_{\mathcal{C}}(X, G)$ for all $X \in \mathcal{C}$, $G(f) := \mathrm{Mor}_{\mathcal{C}}(f, G)$ for all morphisms $f : X \to Y$ in $\mathcal{C}$, and $f(X) := \mathrm{Mor}_{\mathcal{C}}(X, f)$ for all morphisms $f : G \to G'$.

G together with a natural transformation $m : G(\text{-}) \times G(\text{-}) \to G(\text{-})$ is called a *group* (*monoid*) *in the category* $\mathcal{C}$, if the sets $G(X)$ together with the multiplication $m(X) : G(X) \times G(X) \to G(X)$ are groups (monoids) for all $X \in \mathcal{C}$.

Let (G, m) and (G', m') be groups in $\mathcal{C}$. A morphism $f : G \to G'$ in $\mathcal{C}$ is called a *homomorphism of groups* in $\mathcal{C}$, if the diagrams

$$\begin{tikzcd} G(X) \times G(X) \arrow[r, "m(X)"] \arrow[d, "f(X)\times f(X)"'] & G(X) \arrow[d, "f(X)"] \\ G'(X) \times G'(X) \arrow[r, "m'(X)"] & G'(X) \end{tikzcd}$$

commute for all $X \in \mathcal{C}$.

Let (G, m) and (G', m') be monoids in $\mathcal{C}$. A morphism $f : G \to G'$ in $\mathcal{C}$ is called a *homomorphism of monoids* in $\mathcal{C}$, if the diagrams

$$\begin{tikzcd} G(X) \times G(X) \arrow[r, "m(X)"] \arrow[d, "f(X)\times f(X)"'] & G(X) \arrow[d, "f(X)"] \\ G'(X) \times G'(X) \arrow[r, "m'(X)"] & G'(X) \end{tikzcd}$$

and

$$\begin{tikzcd} & \{*\} \arrow[dl, "u"'] \arrow[dr, "u'"] & \\ G(X) \arrow[rr, "f(X)"] & & G'(X) \end{tikzcd}$$

commute for all $X \in \mathcal{C}$.

Problem 2.2.11. (1) Let the set Z together with the multiplication $m : Z \times Z \to Z$ be a monoid. Show that the unit element $e \in Z$ is uniquely determined.

Let (Z, m) be a group. Show that also the inverse $i : Z \to Z$ is uniquely determined.

Show that unit element and inverses of groups are preserved by maps that are compatible with the multiplication.

(2) Find an example of monoids Y and Z and a map $f : Y \to Z$ with $f(y_1y_2) = f(y_1)f(y_2)$ for all $y_1, y_2 \in Y$, but $f(e_Y) \neq e_Z$.

(3) Let (G, m) be a group in $\mathcal{C}$ and $i_X : G(X) \longrightarrow G(X)$ be the inverse for all $X \in \mathcal{C}$. Show that i is a natural transformation.

Show that the Yoneda Lemma provides a morphism $S : G \longrightarrow G$ such that $i_X = \mathrm{Mor}_{\mathcal{C}}(X, S) = S(X)$ for all $X \in \mathcal{C}$.

Formulate and prove properties of S of the type $S * \mathrm{id} = \ldots$.

Proposition 2.2.2. *Let $\mathcal{C}$ be a category with finite (categorical) products. An object G in $\mathcal{C}$ carries the structure $m : G(\text{-}) \times G(\text{-}) \longrightarrow G(\text{-})$ of a group in $\mathcal{C}$ if and only if there are morphisms $m : G \times G \longrightarrow G$, $u : E \longrightarrow G$, and $S : G \longrightarrow G$ such that the diagrams*

$$\begin{tikzcd}
G \times G \times G \arrow[r, "m \times 1"] \arrow[d, "1 \times m"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
E \times G \cong G \cong G \times E \arrow[r, "1 \times u"] \arrow[d, "u \times 1"'] \arrow[dr, "1"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r] \arrow[d, "\Delta"'] & E \arrow[r, "u"] & G \\
G \times G \arrow[rr, "1 \times S"', "S \times 1"] & & G \times G \arrow[u, "m"']
\end{tikzcd}$$

commute where Δ is the morphism defined in [Advanced Algebra] *4.2. The multiplications are related by $m_X = \mathrm{Mor}_{\mathcal{C}}(X, m) = m(X)$.*

An analogous statement holds for monoids.

Proof. The Yoneda Lemma defines a bijection between the set of morphisms $f : X \longrightarrow Y$ and the set of natural transformations $f(\text{-}) : X(\text{-}) \longrightarrow Y(\text{-})$ by $f(Z) = \mathrm{Mor}_{\mathcal{C}}(Z, f)$. In particular we have $m_X = \mathrm{Mor}_{\mathcal{C}}(X, m) = m(X)$. The diagram

$$\begin{tikzcd}
G(\text{-}) \times G(\text{-}) \times G(\text{-}) \arrow[r, "m_{\text{-}} \times 1"] \arrow[d, "1 \times m_{\text{-}}"'] & G(\text{-}) \times G(\text{-}) \arrow[d, "m_{\text{-}}"] \\
G(\text{-}) \times G(\text{-}) \arrow[r, "m_{\text{-}}"] & G(\text{-})
\end{tikzcd}$$

commutes if and only if $\mathrm{Mor}_{\mathcal{C}}(\text{-}, m(m \times 1)) = \mathrm{Mor}_{\mathcal{C}}(\text{-}, m)(\mathrm{Mor}_{\mathcal{C}}(\text{-}, m) \times 1) = m_{\text{-}}(m_{\text{-}} \times 1) = m_{\text{-}}(1 \times m_{\text{-}}) = \mathrm{Mor}_{\mathcal{C}}(\text{-}, m)(1 \times \mathrm{Mor}_{\mathcal{C}}(\text{-}, m)) = \mathrm{Mor}_{\mathcal{C}}(\text{-}, m(1 \times m))$ if and only if $m(m \times 1) = m(1 \times m)$ if and only if the diagram

$$\begin{tikzcd}
G \times G \times G \arrow[r, "m \times 1"] \arrow[d, "1 \times m"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}$$

commutes. In a similar way one shows the equivalence of the other diagram(s). □

Problem 2.2.12. Let $\mathcal{C}$ be a category with finite products. Show that a morphism $f : G \to G'$ in $\mathcal{C}$ is a homomorphism of groups if and only if

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "f\times f"'] & G \arrow[d, "f"] \\ G' \times G' \arrow[r, "m'"] & G' \end{tikzcd}$$

commutes.

Definition 2.2.3. A *cogroup* (*comonoid*) G in $\mathcal{C}$ is a group (monoid) in $\mathcal{C}^{op}$, i.e. an object $G \in \operatorname{Ob}\mathcal{C} = \operatorname{Ob}\mathcal{C}^{op}$ together with a natural transformation $m(X) : G(X) \times G(X) \to G(X)$ where $G(X) = \operatorname{Mor}_{\mathcal{C}^{op}}(X, G) = \operatorname{Mor}_{\mathcal{C}}(G, X)$, such that $(G(X), m(X))$ is a group (monoid) for each $X \in \mathcal{C}$.

Remark 2.2.4. Let $\mathcal{C}$ be a category with finite (categorical) coproducts. An object G in $\mathcal{C}$ carries the structure $m : G(\text{-}) \times G(\text{-}) \to G(\text{-})$ of a cogroup in $\mathcal{C}$ if and only if there are morphisms $\Delta : G \to G \amalg G$, $\varepsilon : G \to I$, and $S : G \to G$ such that the diagrams

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & G \amalg G \arrow[d, "1\amalg\Delta"] \\ G \amalg G \arrow[r, "\Delta\amalg 1"] & G \amalg G \amalg G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] \arrow[drr, "1"] & & G \amalg G \arrow[d, "1\amalg\varepsilon"] \\ G \amalg G \arrow[r, "\varepsilon\amalg 1"] & I \amalg G \cong G & \cong G \amalg I \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\Delta"'] & I \arrow[r, "\varepsilon"] & G \\ G \amalg G \arrow[rr, "1\amalg S", "S\amalg 1"'] & & G \amalg G \arrow[u, "\nabla"'] \end{tikzcd}$$

commute where ∇ is dual to the morphism Δ defined in [Advanced Algebra] 4.2. The multiplications are related by $\Delta_X = \operatorname{Mor}_{\mathcal{C}}(\Delta, X) = \Delta(X)$.

Let $\mathcal{C}$ be a category with finite coproducts and let G and G' be cogroups in $\mathcal{C}$. Then a homomorphism of groups $f : G' \to G$ is a morphism $f : G \to G'$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d, "f"'] & G \times G \arrow[d, "f\times f"] \\ G' \arrow[r, "\Delta'"] & G' \times G' \end{tikzcd}$$

commutes. An analogous result holds for comonoids.

Remark 2.2.5. Obviously similar observations and statements can be made for other algebraic structures in a category $\mathcal{C}$. So one can introduce vector spaces and covector spaces, monoids and comonoids, rings and corings in a category $\mathcal{C}$. The structures can be described by morphisms in $\mathcal{C}$ if $\mathcal{C}$ is a category with finite (co-) products.

Problem 2.2.13. Determine the structure of a covector space on a vector space V from the fact that $\mathrm{Hom}(V, W)$ is a vector space for all vector spaces W.

Proposition 2.2.6. *Let $G \in \mathcal{C}$ be a group with multiplication $a * b$, unit e, and inverse a^{-1} in $G(X)$. Then the morphisms $m : G \times G \longrightarrow G$, $u : E \longrightarrow G$, and $S : G \longrightarrow G$ are given by*

$$m = p_1 * p_2, \qquad u = e_E, \qquad S = \mathrm{id}_G^{-1}.$$

Proof. By the Yoneda Lemma [Advanced Algebra] 5.7 these morphisms can be constructed from the natural transformation as follows. Under $\mathrm{Mor}_{\mathcal{C}}(G\times G, G\times G) = G \times G(G \times G) \cong G(G \times G) \times G(G \times G) \xrightarrow{*} G(G \times G) = \mathrm{Mor}_{\mathcal{C}}(G \times G, G)$ the identity $\mathrm{id}_{G\times G} = (p_1, p_2)$ is mapped to $m = p_1 * p_2$. Under $\mathrm{Mor}_{\mathcal{C}}(E, E) = E(E) \longrightarrow G(E) = \mathrm{Mor}_{\mathcal{C}}(E, G)$ the identity of E is mapped to the neutral element $u = e_E$. Under $\mathrm{Mor}_{\mathcal{C}}(G, G) = G(G) \longrightarrow G(G) = \mathrm{Mor}_{\mathcal{C}}(G, G)$ the identity is mapped to its $*$-inverse $S = \mathrm{id}_G^{-1}$. $\square$

Corollary 2.2.7. *Let $G \in \mathcal{C}$ be a cogroup with multiplication $a * b$, unit e, and inverse a^{-1} in $G(X)$. Then the morphisms $\Delta : G \longrightarrow G \amalg G$, $\varepsilon : G \longrightarrow I$, and $S : G \longrightarrow G$ are given by*

$$\Delta = \iota_1 * \iota_2, \qquad \varepsilon = e_I, \qquad S = \mathrm{id}_G^{-1}.$$

3. Affine Algebraic Groups

We apply the preceding considerations to the categories $\mathbb{K}$-$c\mathcal{A}lg$ and $\mathbb{K}$-$c\mathcal{C}oalg$.

Consider $\mathbb{K}$-$c\mathcal{A}lg$, the category of commutative $\mathbb{K}$-algebras. Let $A, B \in \mathbb{K}$-$c\mathcal{A}lg$. Then $A \otimes B$ is again a commutative $\mathbb{K}$-algebra with componentwise multiplication. In fact this holds also for non-commutative $\mathbb{K}$-algebras ([Advanced Algebra] 2.3), but in $\mathbb{K}$-$c\mathcal{A}lg$ we have

Proposition 2.3.1. *The tensor product in $\mathbb{K}$-$c\mathcal{A}lg$ is the (categorical) coproduct.*

Proof. Let $f \in \mathbb{K}\text{-}c\mathcal{A}lg(A, Z), g \in \mathbb{K}\text{-}c\mathcal{A}lg(B, Z)$. The map $[f, g] : A \otimes B \longrightarrow Z$ defined by $[f, g](a \otimes b) := f(a)g(b)$ is the unique algebra homomorphism such that $[f, g](a \otimes 1) = f(a)$ and $[f, g](1 \otimes b) = g(b)$ or such that the diagram

$$\begin{tikzcd}
A \arrow[r, "\iota_A"] \arrow[dr, "f"'] & A \otimes B \arrow[d, "{[f,g]}"] & B \arrow[l, "\iota_B"'] \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

commutes, where $\iota_A(a) = a \otimes 1$ and $\iota_B(b) = 1 \otimes b$ are algebra homomorphisms. $\square$

So the category $\mathbb{K}\text{-}c\mathcal{A}lg$ has finite coproducts and also an initial object $\mathbb{K}$.

A more general property of the tensor product of arbitrary algebras was already considered in 1.2.13.

Observe that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\iota_1"] \arrow[dr, "1_A"'] & A \otimes A \arrow[d, "\nabla"] & A \arrow[l, "\iota_2"'] \arrow[dl, "1_A"] \\ & A & \end{tikzcd}$$

where ∇ is the multiplication of the algebra and by the diagram the codiagonal of the coproduct.

Definition 2.3.2. An *affine algebraic group* is a group in the category of commutative geometric spaces.

By the duality between the categories of commutative geometric spaces and commutative algebras, an affine algebraic group is represented by a cogroup in the category of $\mathbb{K}\text{-}c\mathcal{A}lg$ of commutative algebras.

For an arbitrary affine algebraic group H we get by Corollary 2.2.7

$$\Delta = \iota_1 * \iota_2 \in \mathbb{K}\text{-}c\mathcal{A}lg(H, H \otimes H),$$

$$\varepsilon = e \in \mathbb{K}\text{-}c\mathcal{A}lg(H, \mathbb{K}), \qquad \text{and } S = (\mathrm{id})^{-1} \in \mathbb{K}\text{-}c\mathcal{A}lg(H, H).$$

These maps and Corollary 2.2.7 lead to

Proposition 2.3.3. *Let $H \in \mathbb{K}\text{-}c\mathcal{A}lg$. H is a representing object for a functor $\mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{G}r$ if and only if H is a Hopf algebra.*

Proof. Both statements are equivalent to the existence of morphisms in $\mathbb{K}\text{-}c\mathcal{A}lg$

$$\Delta : H \to H \otimes H \quad \varepsilon : H \to \mathbb{K} \quad S : H \to H$$

such that the following diagrams commute

(coassociativity)
$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & H \otimes H \arrow[d, "\Delta\otimes 1"] \\ H \otimes H \arrow[r, "1\otimes\Delta"] & H \otimes H \otimes H \end{tikzcd}$$

(counit)
$$\begin{tikzcd} H \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] \arrow[drr, "1"] & & H \otimes H \arrow[d, "1\otimes\varepsilon"] \\ H \otimes H \arrow[r, "\varepsilon\otimes 1"] & \mathbb{K} \otimes H \cong H & \cong H \otimes \mathbb{K} \end{tikzcd}$$

(coinverse)
$$\begin{tikzcd} H \arrow[r, "\epsilon"] \arrow[d, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & H \\ H \otimes H \arrow[rr, "S\otimes\mathrm{id}", "\mathrm{id}\otimes S"'] & & H \otimes H \arrow[u, "\nabla"'] \end{tikzcd}$$

□

This Proposition says two things. First of all each commutative Hopf algebra H defines a functor $\mathbb{K}\text{-}c\mathcal{A}lg(H,\text{-}) : \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{S}et$ that factors through the category of groups or simply a functor $\mathbb{K}\text{-}c\mathcal{A}lg(H,\text{-}) : \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{G}r$. Secondly each representable functor $\mathbb{K}\text{-}c\mathcal{A}lg(H,\text{-}) : \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{S}et$ that factors through the category of groups is represented by a commutative Hopf algebra.

Corollary 2.3.4. *An algebra $H \in \mathbb{K}\text{-}c\mathcal{A}lg$ represents an affine algebraic group if and only if H is a commutative Hopf algebra.*

The category of commutative Hopf algebras is dual to the category of affine algebraic groups.

In the following lemmas we consider functors represented by commutative algebras. They define functors on the category $\mathbb{K}\text{-}c\mathcal{A}lg$ as well as more generally on $\mathbb{K}\text{-}\mathcal{A}lg$. We first study the functors and the representing algebras. Then we use them to construct commutative Hopf algebras.

Lemma 2.3.5. *The functor $\mathbb{G}_a : \mathbb{K}\text{-}\mathcal{A}lg \longrightarrow \mathcal{A}b$ defined by $\mathbb{G}_a(A) := A^+$, the underlying additive group of the algebra A, is a representable functor represented by the algebra $\mathbb{K}[x]$ the polynomial ring in one variable x.*

PROOF. $\mathbb{G}_a$ is an underlying functor that forgets the multiplicative structure of the algebra and only preserves the additive group of the algebra. We have to determine natural isomorphisms (natural in A) $\mathbb{G}_a(A) \cong \mathbb{K}\text{-}\mathcal{A}lg(\mathbb{K}[x], A)$. Each element $a \in A^+$ is mapped to the homomorphism of algebras $a_* : \mathbb{K}[x] \ni p(x) \mapsto p(a) \in A$. This is a homomorphism of algebras since $a_*(p(x) + q(x)) = p(a) + q(a) = a_*(p(x)) + a_*(q(x))$ and $a_*(p(x)q(x)) = p(a)q(a) = a_*(p(x))a_*(q(x))$. Another reason to see this is that $\mathbb{K}[x]$ is the free (commutative) $\mathbb{K}$-algebra over $\{x\}$ i.e. since each map $\{x\} \longrightarrow A$ can be uniquely extended to a homomorphism of algebras $\mathbb{K}[x] \longrightarrow A$. The map $A \ni a \mapsto a_* \in \mathbb{K}\text{-}\mathcal{A}lg(\mathbb{K}[x], A)$ is bijective with the inverse map $\mathbb{K}\text{-}\mathcal{A}lg(\mathbb{K}[x], A) \ni f \mapsto f(x) \in A$. Finally this map is natural in A since

$$\begin{tikzcd}
A \arrow[r, "\text{-}_*"] \arrow[d, "g"'] & \mathbb{K}\text{-}\mathcal{A}lg(\mathbb{K}[x], A) \arrow[d, "\text{K-Alg}(K[x],g)"] \\
B \arrow[r, "\text{-}_*"'] & \mathbb{K}\text{-}\mathcal{A}lg(\mathbb{K}[x], B)
\end{tikzcd}$$

commutes for all $g \in \mathbb{K}\text{-}\mathcal{A}lg(A, B)$. □

Remark 2.3.6. Since A^+ has the structure of an additive group the sets of homomorphisms of algebras $\mathbb{K}\text{-}\mathcal{A}lg(\mathbb{K}[x], A)$ are also additive groups.

Lemma 2.3.7. *The functor $\mathbb{G}_m = U : \mathbb{K}\text{-}\mathcal{A}lg \longrightarrow \mathcal{G}r$ defined by $\mathbb{G}_m(A) := U(A)$, the underlying multiplicative group of units of the algebra A, is a representable functor* *represented by the algebra* $\mathbb{K}[x, x^{-1}] = \mathbb{K}[x, y]/(xy-1)$ *the ring of Laurent polynomials in one variable* x.

Proof. We have to determine natural isomorphisms (natural in A) $\mathbb{G}_m(A) \cong \mathbb{K}\text{-}\mathcal{A}lg(\mathbb{K}[x, x^{-1}], A)$. Each element $a \in \mathbb{G}_m(A)$ is mapped to the homomorphism of algebras $a_* := (\mathbb{K}[x, x^{-1}] \ni x \mapsto a \in A)$. This defines a unique homomorphism of algebras since each homomorphism of algebras f from $\mathbb{K}[x, x^{-1}] = \mathbb{K}[x, y]/(xy - 1)$ to A is completely determined by the images of x and of y but in addition the images have to satisfy $f(x)f(y) = 1$, i.e. $f(x)$ must be invertible and $f(y)$ must be the inverse to $f(x)$. This mapping is bijective. Furthermore it is natural in A since

$$\begin{tikzcd}
A \arrow[r, "-_*"] \arrow[d, "g"'] & \mathbb{K}\text{-}\mathcal{A}lg(\mathbb{K}[x, x^{-1}], A) \arrow[d, "\text{K-Alg}(K[x,x^{-1}],g)"] \\
B \arrow[r, "-_*"'] & \mathbb{K}\text{-}\mathcal{A}lg(\mathbb{K}[x, x^{-1}], B)
\end{tikzcd}$$

for all $g \in \mathbb{K}\text{-}\mathcal{A}lg(A, B)$ commute. $\square$

Remark 2.3.8. Since $U(A)$ has the structure of a (multiplicative) group the sets $\mathbb{K}\text{-}\mathcal{A}lg(\mathbb{K}[x, x^{-1}], A)$ are also groups.

Lemma 2.3.9. *The functor* $\mathbb{M}_n : \mathbb{K}\text{-}\mathcal{A}lg \longrightarrow \mathbb{K}\text{-}\mathcal{A}lg$ *with* $\mathbb{M}_n(A)$ *the algebra of* $n \times n$*-matrices with entries in* A *is representable by the algebra* $\mathbb{K}\langle x_{11}, x_{12}, \dots, x_{nn}\rangle$, *the non commutative polynomialring in the variables* x_{ij}.

Proof. The polynomial ring $\mathbb{K}\langle x_{ij}\rangle$ is free over the set $\{x_{ij}\}$ in the category of (non commutative) algebras, i.e. for each algebra and for each map $f : \{x_{ij}\} \longrightarrow A$ there exists a unique homomorphism of algebras $g : \mathbb{K}\langle x_{11}, x_{12}, \dots, x_{nn}\rangle \longrightarrow A$ such that the diagram

$$\begin{tikzcd}
\{x_{ij}\} \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{K}\langle x_{ij}\rangle \arrow[d, "g"] \\
 & A
\end{tikzcd}$$

commutes. So each matrix in $M_n(A)$ defines a unique a homomorphism of algebras $\mathbb{K}\langle x_{11}, x_{12}, \dots, x_{nn}\rangle \longrightarrow A$ and conversely. $\square$

Example 2.3.10. 1. The affine algebraic group called *additive group*

$$\mathbb{G}_a : \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{A}b$$

with $\mathbb{G}_a(A) := A^+$ from Lemma 2.3.5 is represented by the Hopf algebra $\mathbb{K}[x]$. We determine comultiplication, counit, and antipode.

By Corollary 2.2.7 the comultiplication is $\Delta = \iota_1 * \iota_2 \in \mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}[x], \mathbb{K}[x] \otimes \mathbb{K}[x]) \cong \mathbb{G}_a(\mathbb{K}[x] \otimes \mathbb{K}[x])$. Hence

$$\Delta(x) = \iota_1(x) + \iota_2(x) = x \otimes 1 + 1 \otimes x.$$

The counit is $\varepsilon = e_{\mathbb{K}} = 0 \in \mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}[x], \mathbb{K}) \cong \mathbb{G}_a(\mathbb{K})$ hence

$$\varepsilon(x) = 0.$$

The antipode is $S = \mathrm{id}_{\mathbb{K}[x]}^{-1} \in \mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}[x], \mathbb{K}[x]) \cong \mathbb{G}_a(\mathbb{K}[x])$ hence

$$S(x) = -x.$$

2. The affine algebraic group called *multiplicative group*

$$\mathbb{G}_m : \mathbb{K}\text{-}c\mathcal{A}lg \to \mathcal{A}b$$

with $\mathbb{G}_m(A) := A^* = U(A)$ from Lemma 2.3.7 is represented by the Hopf algebra $\mathbb{K}[x, x^{-1}] = \mathbb{K}[x, y]/(xy - 1)$. We determine comultiplication, counit, and antipode.

By Corollary 2.2.7 the comultiplication is

$$\Delta = \iota_1 * \iota_2 \in \mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}[x, x^{-1}], \mathbb{K}[x, x^{-1}] \otimes \mathbb{K}[x, x^{-1}]) \cong \mathbb{G}_m(\mathbb{K}[x, x^{-1}] \otimes \mathbb{K}[x, x^{-1}]).$$

Hence

$$\Delta(x) = \iota_1(x) \cdot \iota_2(x) = x \otimes x.$$

The counit is $\varepsilon = e_{\mathbb{K}} = 1 \in \mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}[x, x^{-1}], \mathbb{K}) \cong \mathbb{G}_m(\mathbb{K})$ hence

$$\varepsilon(x) = 1.$$

The antipode is $S = \mathrm{id}_{\mathbb{K}[x,x^{-1}]}^{-1} \in \mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}[x, x^{-1}], \mathbb{K}[x, x^{-1}]) \cong \mathbb{G}_a(\mathbb{K}[x, x^{-1}])$ hence

$$S(x) = x^{-1}.$$

3. The affine algebraic group called *additive matrix group*

$$\mathbb{M}_n^+ : \mathbb{K}\text{-}c\mathcal{A}lg \to \mathcal{A}b,$$

with $\mathbb{M}_n^+(A)$ the additive group of $n \times n$-matrices with coefficients in A is represented by the commutative algebra $M_n^+ = \mathbb{K}[x_{ij} | 1 \leq i, j \leq n]$ (Lemma 2.3.9). This algebra must be a Hopf algebra.

The comultiplication is $\Delta = \iota_1 * \iota_2 \in \mathbb{K}\text{-}c\mathcal{A}lg(M_n^+, M_n^+ \otimes M_n^+) \cong \mathbb{M}_n^+(M_n^+ \otimes M_n^+)$. Hence

$$\Delta(x_{ij}) = \iota_1(x_{ij}) + \iota_2(x_{ij}) = x_{ij} \otimes 1 + 1 \otimes x_{ij}.$$

The counit is $\varepsilon = e_{\mathbb{K}} = (0) \in \mathbb{K}\text{-}c\mathcal{A}lg(M_n^+, \mathbb{K}) \cong \mathbb{M}_n^+(\mathbb{K})$ hence

$$\varepsilon(x_{ij}) = 0.$$

The antipode is $S = \mathrm{id}_{M_n^+}^{-1} \in \mathbb{K}\text{-}c\mathcal{A}lg(M_n^+, M_n^+) \cong \mathbb{M}_n^+(M_n^+)$ hence

$$S(x_{ij}) = -x_{ij}.$$

4. The matrix algebra $\mathbb{M}_n(A)$ also has a noncommutative multiplication, the matrix multiplication, defining a monoid structure $\mathbb{M}_n^\times(A)$. Thus $\mathbb{K}[x_{ij}]$ carries another

coalgebra structure which defines a bialgebra $M_n^\times = \mathbb{K}[x_{ij}]$. Obviously there is no antipode.

The comultiplication is $\Delta = \iota_1 * \iota_2 \in \mathbb{K}\text{-}c\mathcal{A}lg(M_n^\times, M_n^\times \otimes M_n^\times) \cong \mathbb{M}_n^\times(M_n^\times \otimes M_n^\times)$. Hence $\Delta((x_{ij})) = \iota_1((x_{ij})) \cdot \iota_2((x_{ij})) = (x_{ij}) \otimes (x_{ij})$ or

$$\Delta(x_{ik}) = \sum_j x_{ij} \otimes x_{jk}.$$

The counit is $\varepsilon = e_\mathbb{K} = E \in \mathbb{K}\text{-}c\mathcal{A}lg(M_n^\times, \mathbb{K}) \cong \mathbb{M}_n^\times(\mathbb{K})$ hence

$$\varepsilon(x_{ij}) = \delta_{ij}.$$

5. Let $\mathbb{K}$ be a field of characteristic p. The algebra $H = \mathbb{K}[x]/(x^p)$ carries the structure of a Hopf algebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$, and $S(x) = -x$. To show that Δ is well defined we have to show $\Delta(x)^p = 0$. But this is obvious by the rules for computing p-th powers in characteristic p. We have $(x \otimes 1 + 1 \otimes x)^p = x^p \otimes 1 + 1 \otimes x^p = 0$.

Thus the algebra H represents an affine algebraic group:

$$\alpha_p(A) := \mathbb{K}\text{-}c\mathcal{A}lg(H, A) \cong \{a \in A | a^p = 0\}.$$

The group multiplication is the addition of p-nilpotent elements. So we have the *group of p-nilpotent elements.*

6. The algebra $H = \mathbb{K}[x]/(x^n - 1)$ is a Hopf algebra with the comultiplication $\Delta(x) = x \otimes x$, the counit $\varepsilon(x) = 1$, and the antipode $S(x) = x^{n-1}$. These maps are well defined since we have for example $\Delta(x)^n = (x \otimes x)^n = x^n \otimes x^n = 1 \otimes 1$. Observe that this Hopf algebra is isomorphic to the group algebra $\mathbb{K}C_n$ of the cyclic group C_n of order n.

Thus the algebra H represents an affine algebraic group:

$$\mu_n(A) := \mathbb{K}\text{-}c\mathcal{A}lg(H, A) \cong \{a \in A | a^n = 1\},$$

that is the *group of n-th roots of unity.* The group multiplication is the ordinary multiplication of roots of unity.

7. The linear groups or matrix groups $\mathbb{GL}(n)(A)$, $\mathbb{SL}(n)(A)$ and other such groups are further examples of affine algebraic groups. We will discuss them in the section on quantum groups.

Problem 2.3.14. (1) The construction of the general linear group

$$\mathbb{GL}(n)(A) = \{(a_{ij}) \in \mathbb{M}_n(A) | (a_{ij}) \text{ invertible}\}$$

defines an affine algebraic group. Describe the representing Hopf algebra.

(2) The special linear group $\mathbb{SL}(n)(A)$ is an affine algebraic group. What is the representing Hopf algebra?

(3) The real unit circle $\mathbb{S}^1(\mathbb{R})$ carry the structure of a group by the addition of angles. Is it possible to make $\mathbb{S}^1$ with the affine algebra $\mathbb{K}[c, s]/(s^2 + c^2 - 1)$ into an affine algebraic group? (Hint: How can you add two points (x_1, y_1)

and (x_2, y_2) on the unit circle, such that you get the addition of the associated angles?)

(4) Find a group structure on the torus $\mathcal{T}$.

4. Formal Groups

Consider now $\mathbb{K}$-$c\mathcal{C}oalg$ the category of cocommutative $\mathbb{K}$-coalgebras. Let $C, D \in \mathbb{K}$-$c\mathcal{C}oalg$. Then $C \otimes D$ is again a cocommutative $\mathbb{K}$-coalgebra by [Advanced Algebra] Problem 2.7.4. In fact this holds also for non-commutative $\mathbb{K}$-algebras, but in $\mathbb{K}$-$c\mathcal{C}oalg$ we have

Proposition 2.4.1. *The tensor product in $\mathbb{K}$-$c\mathcal{C}oalg$ is the (categorical) product.*

Proof. Let $f \in \mathbb{K}$-$c\mathcal{C}oalg(Z, C), g \in \mathbb{K}$-$c\mathcal{C}oalg(Z, D)$. The map $(f, g) : Z \to C \otimes D$ defined by $(f, g)(z) := \sum f(z_{(1)}) \otimes g(z_{(2)})$ is the unique homomorphism of coalgebras such that $(1 \otimes \varepsilon_D)(f, g)(z) = f(z)$ and $(\varepsilon_C \otimes 1)(f, g)(z) = g(z)$ or such that the diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, "{(f,g)}"] \arrow[dr, "g"] & \\ C & C \otimes D \arrow[l, "p_C"'] \arrow[r, "p_D"] & D \end{tikzcd}$$

commutes, where $p_C(c \otimes d) = (1 \otimes \varepsilon)(c \otimes d) = c\varepsilon(d)$ and $p_D(c \otimes d) = (\varepsilon \otimes 1)(c \otimes d) = \varepsilon(c)d$ are homomorphisms of coalgebras. $\square$

So the category $\mathbb{K}$-$c\mathcal{C}oalg$ has finite products and also a final object $\mathbb{K}$.

Definition 2.4.2. A *formal group* is a group in the category of $\mathbb{K}$-$c\mathcal{C}oalg$ of cocommutative coalgebras.

A formal group G defines a contravariant representable functor from $\mathbb{K}$-$c\mathcal{C}oalg$ to $\mathcal{G}r$, the category of groups.

Proposition 2.4.3. *Let $H \in \mathbb{K}$-$c\mathcal{C}oalg$. H a represents a formal group if and only if there are given morphisms in $\mathbb{K}$-$c\mathcal{C}oalg$*

$$\nabla : H \otimes H \to H, \quad u : \mathbb{K} \to H, \quad S : H \to H$$

such that the following diagrams commute

(associativity)
$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "1 \otimes \nabla"] \arrow[d, "\nabla \otimes 1"'] & H \otimes H \arrow[d, "\nabla"] \\ H \otimes H \arrow[r, "\nabla"] & H \end{tikzcd}$$

(unit)
$$\begin{tikzcd} \mathbb{K} \otimes H \cong H \cong H \otimes \mathbb{K} \arrow[r, "u \otimes 1"] \arrow[d, "1 \otimes u"'] \arrow[dr, "1"] & H \otimes H \arrow[d, "\nabla"] \\ H \otimes H \arrow[r, "\nabla"] & H \end{tikzcd}$$

$$\begin{tikzcd} H \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & H \\ H \otimes H \arrow[rr, "S\otimes \mathrm{id}", "\mathrm{id}\otimes S"'] & & H \otimes H \arrow[u, "\nabla"'] \end{tikzcd} \quad \textit{(inverse)}$$

PROOF. For an arbitrary formal group H we get $\nabla = p_1 * p_2 \in \mathbb{K}\text{-}c\mathcal{C}oalg(H \otimes H, H)$, $u = e \in \mathbb{K}\text{-}c\mathcal{C}oalg(\mathbb{K}, H)$, and $S = (\mathrm{id})^{-1} \in \mathbb{K}\text{-}c\mathcal{C}oalg(H,H)$. These maps, the Yoneda Lemma and Remark 2.2.6 lead to the proof of the proposition. □

Remark 2.4.4. In particular the representing object $(H, \nabla, u, \Delta, \varepsilon, S)$ of a formal group G is a cocommutative Hopf algebra and every such Hopf algebra represents a formal group. Hence the category of formal groups is equivalent to the category of cocommutative Hopf algebras.

Corollary 2.4.5. *A coalgebra $H \in \mathbb{K}\text{-}c\mathcal{C}oalg$ represents a formal group if and only if H is a cocommutative Hopf algebra.*

The category of cocommutative Hopf algebras is equivalent to the category of formal groups.

Corollary 2.4.6. *The following categories are equivalent:*

(1) *The category of commutative, cocommutative Hopf algebras.*
(2) *The category of commutative formal groups.*
(3) *The dual of the category of commutative affine algebraic groups.*

Example 2.4.7. (1) Group algebras $\mathbb{K}G$ are formal groups.

(2) Universal enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$ are formal groups.
(3) The tensor algebra $T(V)$ and the symmetric algebra $S(V)$ are formal groups.
(4) Let C be a cocommutative coalgebra and G be a group. Then the group $\mathbb{K}G(C) = \mathbb{K}\text{-}c\mathcal{C}oalg(C, \mathbb{K}G)$ is isomorphic to the set of families $(h_g^*|g \in G)$ of decompositions of the unit of C^* into a sum of orthogonal idempotents $h_g^* \in C^*$ that are locally finite.

    To see this embed $\mathbb{K}\text{-}c\mathcal{C}oalg(C, \mathbb{K}G) \subseteq \mathrm{Hom}(C, \mathbb{K}G)$ and embedthe set $\mathrm{Hom}(C, \mathbb{K}G)$ into the set $(C^*)^G$ of G-families of elements in the algebra C^* by $h \mapsto (h_g^*)$ with $h(c) = \sum_{g\in G} h_g^*(c)g$. The linear map h is a homomorphism of coalgebras iff $(h \otimes h)\Delta = \Delta h$ and $\varepsilon h = \varepsilon$ iff $\sum h(c_{(1)}) \otimes h(c_{(2)}) = \sum h(c)_{(1)} \otimes h(c)_{(2)}$ and $\varepsilon(h(c)) = \varepsilon(c)$ for all $c \in C$ iff $\sum h_g^*(c_{(1)})g \otimes h_l^*(c_{(2)})l = \sum h_g^*(c)g \otimes g$ and $\sum h_g^*(c) = \varepsilon(c)$ iff $\sum h_g^*(c_{(1)})h_l^*(c_{(2)}) = \delta_{gl}h_g^*(c)$ and $\sum h_g^* = \varepsilon$ iff $h_g^* * h_l^* = \delta_{gl}h_g^*$ and $\sum h_g^* = 1_{C^*}$. Furthermore the families must be locally finite, i.e. for each $c \in C$ only finitely many of them give non-zero values.
(5) Let C be a cocommutative coalgebra and $\mathbb{K}[x]$ be the Hopf algebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$ (the symmetric algebra of the one dimensional vector space $\mathbb{K}x$). We embed as before $\mathbb{K}\text{-}c\mathcal{C}oalg(C, \mathbb{K}[x]) \subseteq \mathrm{Hom}(C, \mathbb{K}[x]) = (C^*)^{\{\mathbf{N}_0\}}$, the set of locally finite $\mathbf{N}_0$-families in C^* by $h(c) = \sum_{i=0}^{\infty} h_i^*(c)x^i$. The map h is a homomorphism of coalgebras iff $\Delta(h(c)) = \sum h_i^*(c)(x \otimes 1 +$

$1 \otimes x)^i = \sum h_i^*(c)\binom{i}{l}x^l \otimes x^{i-l} = (h \otimes h)\Delta(c) = \sum h_i^*(c_{(1)})h_j^*(c_{(2)})x^i \otimes x^j$ and $\varepsilon(\sum h_i^*(c)x^i) = \varepsilon(c)$ iff $h_i^* * h_j^* = \binom{i+j}{i}h_{i+j}^*$ and $h_0^* = \varepsilon = 1_{C^*}$.

Now let $\mathbb{K}$ be a field of characteristic zero. Let $p_i := h_i^*/i!$. Then the conditions simplify to $p_i p_j = p_{i+j}$ and $p_0 = 1$. Hence the series for h is completely determined by the term $p := p_1$ since $p_n = p_1^n$. Since the series must be locally finite we get that for each $c \in C$ there must be an $n \in \mathbf{N}_0$ such that $p^m(c) = 0$ for all $m \geq n$. Hence the element p is *topologically nilpotent* and

$$\mathbb{K}\text{-}c\mathcal{C}oalg(C, \mathbb{K}[x]) \cong \mathrm{rad}_t(C^*)$$

the radial of topologically nilpotent elements of C^*.

It is easy to see that $\mathrm{rad}_t(C^*)$ is a group under addition and that this group structure coincides with the one on $\mathbb{K}\text{-}c\mathcal{C}oalg(C, \mathbb{K}[x])$.

Remark 2.4.8. Let H be a finite dimensional Hopf algebra. Then by [Advanced Algebra] 2.23 and 2.25 we get that H^* is an algebra and a coalgebra. The commutative diagrams defining the bialgebra property and the antipode can be transferred easily, so H^* is again a Hopf algebra. Hence the functor $\text{-}^* : vec \longrightarrow vec$ from finite dimensional vector spaces to itself induces a duality $\text{-}^* : \mathbb{K}\text{-}hopf \longrightarrow \mathbb{K}\text{-}hopf$ from the category of finite dimensional Hopf algebras to itself.

An affine algebraic group is called *finite* if the representing Hopf algebra is finite dimensional. A formal group is called *finite* if the representing Hopf algebra is finite dimensional.

Thus the category of finite affine algebraic groups is equivalent to the category of finite formal groups.

The category of finite commutative affine algebraic groups is self dual. The category of finite commutative affine algebraic groups is equivalent (and dual) to the category of finite commutative formal groups.

5. Quantum Groups

Definition 2.5.1. (Drinfel'd) A *quantum group* is a noncommutative noncocommutative Hopf algebra.

Remark 2.5.2. We shall consider all Hopf algebras as quantum groups. Observe, however, that the commutative Hopf algebras may be considered as affine algebraic groups and that the cocommutative Hopf algebras may be considered as formal groups. Their property as a quantum space or as a quantum monoid will play some role. But often the (possibly nonexisting) dual Hopf algebra will have the geometrical meaning. The following examples $\mathbb{SL}_q(2)$ and $\mathbb{GL}_q(2)$ will have a geometrical meaning.

Example 2.5.3. The smallest proper quantum group, i.e. the smallest noncommutative noncocommutative Hopf algebra, is the 4-dimensional algebra

$$H_4 := \mathbb{K}\langle g, x\rangle/(g^2 - 1, x^2, xg + gx)$$

which was first described by M. Sweedler. The coalgebra structure is given by

$$\begin{array}{rr} \Delta(g) = g \otimes g, & \Delta(x) = x \otimes 1 + g \otimes x, \\ \varepsilon(g) = 1, & \varepsilon(x) = 0, \\ S(g) = g^{-1}(= g), & S(x) = -gx. \end{array}$$

Since it is finite dimensional its linear dual H_4^* is also a noncommutative noncocommutative Hopf algebra. It is isomorphic as a Hopf algebra to H_4. In fact H_4 is up to isomorphism the only noncommutative noncocommutative Hopf algebra of dimension 4.

Example 2.5.4. The affine algebraic group $\mathbb{SL}(n) : \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{G}r$ defined by $\mathbb{SL}(n)(A)$, the group of $n \times n$-matrices with coefficients in the commutative algebra A and with determinant 1, is represented by the algebra $\mathcal{O}(\mathbb{SL}(n)) = SL(n) = \mathbb{K}[x_{ij}]/(\det(x_{ij}) - 1)$ i.e.

$$\mathbb{SL}(n)(A) \cong \mathbb{K}\text{-}c\mathcal{A}lg\,(\mathbb{K}[x_{ij}]/(\det(x_{ij}) - 1), A).$$

Since $\mathbb{SL}(n)(A)$ has a group structure by the multiplication of matrices, the representing commutative algebra has a Hopf algebra structure with the diagonal $\Delta = \iota_1 * \iota_2$ hence

$$\Delta(x_{ik}) = \sum x_{ij} \otimes x_{jk},$$

the counit $\varepsilon(x_{ij}) = \delta_{ij}$ and the antipode $S(x_{ij}) = adj(X)_{ij}$ where $adj(X)$ is the adjoint matrix of $X = (x_{ij})$. We leave the verification of these facts to the reader.

We consider $\mathbb{SL}(n) \subseteq \mathcal{M}_n = \mathbb{A}^{n^2}$ as a subspace of the n^2-dimensional affine space.

Example 2.5.5. Let $M_q(2) = \mathbb{K}\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle /I$ as in 1.3.6 with I the ideal generated by

$$ab - q^{-1}ba, ac - q^{-1}ca, bd - q^{-1}db, cd - q^{-1}dc, (ad - q^{-1}bc) - (da - qcb), bc - cb.$$

We first define the *quantum determinant* $\det_q = ad - q^{-1}bc = da - qcb$ in $M_q(2)$. It is a central element. To show this, it suffices to show that $\det_q$ commutes with the generators a, b, c, d:

$$\begin{array}{rr} (ad - q^{-1}bc)a = a(da - qbc), & (ad - q^{-1}bc)b = b(ad - q^{-1}bc), \\ (ad - q^{-1}bc)c = c(ad - q^{-1}bc), & (da - qbc)d = d(ad - q^{-1}bc). \end{array}$$

We can form the quantum determinant of an arbitrary quantum matrix in A by

$$\det_q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} := a'd' - q^{-1}b'c' = d'a' - qc'b' = \varphi(\det_q)$$

if $\varphi : M_q(2) \longrightarrow A$ is the algebra homomorphism associated with the quantum matrix $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$.

Given two commuting quantum 2×2-matrices $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, $\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$. The quantum determinant preserves the product, since

$$
\begin{aligned}
\det_q(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}) &= \det_q \begin{pmatrix} a'a''+b'c'' & a'b''+b'd'' \\ c'a''+d'c'' & c'b''+d'd'' \end{pmatrix} \\
&= (a'a''+b'c'')(c'b''+d'd'') - q^{-1}(a'b''+b'd'')(c'a''+d'c'') \\
&= a'c'a''b'' + b'c'c''b'' + a'd'a''d'' + b'd'c''d'' \\
&\quad -q^{-1}(a'c'b''a'' + b'c'd''a'' + a'd'b''c'' + b'd'd''c'') \\
&= b'c'c''b'' + a'd'a''d'' - q^{-1}b'c'd''a'' - q^{-1}a'd'b''c'' \\
&= b'c'c''b'' + a'd'a''d'' - q^{-1}b'c'd''a'' - q^{-1}a'd'b''c'' \\
&\quad -q^{-1}b'c'(a''d'' - d''a'' - q^{-1}b''c'' + qc''b'') \\
&= a'd'a''d'' - q^{-1}a'd'b''c'' - q^{-1}b'c'(a''d'' - q^{-1}b''c'') \\
&= (a'd' - q^{-1}b'c')(a''d'' - q^{-1}b''c'') \\
&= \det_q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \det_q \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}.
\end{aligned} \tag{3}
$$

In particular we have $\Delta(\det_q) = \det_q \otimes \det_q$ and $\varepsilon(\det_q) = 1$. The quantum determinant is a grouplike element (see 2.1.6).

Now we define an algebra

$$SL_q(2) := M_q(2)/(\det_q - 1).$$

The algebra $SL_q(2)$ represents the functor

$$\mathbb{SL}_q(2)(A) = \{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}_q(2)(A) | \det_q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = 1\}.$$

There is a surjective homomorphism of algebras $M_q(2) \longrightarrow SL_q(2)$ and $\mathbb{SL}_q(2)$ is a subfunctor of $\mathcal{M}_q(2)$.

Let X, Y be commuting quantum matrices satisfying $\det_q(X) = 1 = \det_q(Y)$. Since $\det_q(X)\det_q(Y) = \det_q(XY)$ for commuting quantum matrices we get $\det_q(XY) = 1$, hence $\mathbb{SL}_q(2)$ is a quantum submonoid of $\mathcal{M}_q(2)$ and $SL_q(2)$ is a bialgebra with diagonal

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and

$$\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

To show that $SL_q(2)$ has an antipode we first define a homomorphism of algebras $T : M_q(2) \longrightarrow M_q(2)^{op}$ by

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

We check that $T : \mathbb{K}\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle \longrightarrow M_q(2)^{op}$ vanishes on the ideal I.

$$T(ab - q^{-1}ba) = T(b)T(a) - q^{-1}T(a)T(b) = -qbd + q^{-1}qdb = 0.$$

We leave the check of the other defining relations to the reader. Furthermore T restricts to a homomorphism of algebras $S : SL_q(2) \longrightarrow SL_q(2)^{op}$ since $T(\det_q) = T(ad - q^{-1}bc) = T(d)T(a) - q^{-1}T(c)T(b) = ad - q^{-1}(-q^{-1}c)(-qb) = \det_q$ hence $T(\det_q - 1) = \det_q - 1 = 0$ in $SL_q(2)$.

One verifies easily that S satisfies $\sum S(x_{(1)})x_{(2)} = \varepsilon(x)$ for all given generators of $SL_q(2)$, hence S is a left antipode by 2.1.3. Symmetrically S is a right antipode. Thus the bialgebra $SL_q(2)$ is a Hopf algebra or a quantum group.

Example 2.5.6. The affine algebraic group $\mathbb{GL}(n) : \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{G}r$ defined by $\mathbb{GL}(n)(A)$, the group of invertible $n \times n$-matrices with coefficients in the commutative algebra A, is represented by the algebra $\mathcal{O}(\mathbb{GL}(n)) = GL(n) = \mathbb{K}[x_{ij}, t]/(\det(x_{ij})t - 1)$ i.e.

$$\mathbb{GL}(n)(A) \cong \mathbb{K}\text{-}c\mathcal{A}lg\,(\mathbb{K}[x_{ij}, t]/(\det(x_{ij})t - 1), A)).$$

Since $\mathbb{GL}(n)(A)$ has a group structure by the multiplication of matrices, the representing commutative algebra has a Hopf algebra structure with the diagonal $\Delta = \iota_1 * \iota_2$ hence

$$\Delta(x_{ik}) = \sum x_{ij} \otimes x_{jk},$$

the counit $\varepsilon(x_{ij}) = \delta_{ij}$ and the antipode $S(x_{ij}) = t \cdot adj(X)_{ij}$ where $adj(X)$ is the adjoint matrix of $X = (x_{ij})$. We leave the verification of these facts from linear algebra to the reader. The diagonal applied to t gives

$$\Delta(t) = t \otimes t.$$

Hence $t (= \det(X)^{-1})$ is a grouplike element in $GL(n)$. This reflects the rule $\det(AB) = \det(A)\det(B)$ hence $\det(AB)^{-1} = \det(A)^{-1}\det(B)^{-1}$.

Example 2.5.7. Let $M_q(2)$ be as in the example 2.5.5. We define

$$GL_q(2) := M_q(2)[t]/J$$

with J generated by the elements $t \cdot (ad - q^{-1}bc) - 1$. The algebra $GL_q(2)$ represents the functor

$$\mathbb{GL}_q(2)(A) = \{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}_q(2)(A) | \det_q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \text{ invertible in } A\}.$$

In fact there is a canonical homomorphism of algebras $M_q(2) \longrightarrow GL_q(2)$. A homomorphism of algebras $\varphi : M_q(2) \longrightarrow A$ has a unique continuation to $GL_q(2)$ iff

$\det_q(\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix})$ is invertible:

$$\begin{tikzcd} M_q(2) \arrow[r] \arrow[dr] & M_q(2)[t] \arrow[r] \arrow[d] & G_q(2) \arrow[dl] \\ & A & \end{tikzcd}$$

with $t \mapsto \det_q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^{-1}$. Thus $\mathbb{GL}_q(2)(A)$ is a subset of $\mathcal{M}_q(2)(A)$. Observe that $M_q(2) \longrightarrow GL_q(2)$ is *not* surjective.

Since the quantum determinant preserves products and the product of invertible elements is again invertible we get $\mathbb{GL}_q(2)$ is a quantum submonoid of $\mathcal{M}_q(2)$, hence $\Delta : GL_q(2) \longrightarrow GL_q(2) \otimes GL_q(2)$ with $\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\Delta(t) = t \otimes t$.

We construct the antipode for $GL_q(2)$. We define $T : M_q(2)[t] \longrightarrow M_q(2)[t]^{op}$ by

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} := t \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} \qquad \text{and} \qquad T(t) := \det_q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - q^{-1}bc.$$

As in 2.5.5 T defines a homomorphism of algebras. We obtain an induced homomorphism of algebras $S : GL_q(2) \longrightarrow GL_q(2)^{op}$ or a $GL_q(2)^{op}$-point in $GL_q(2)$ since $S(t(ad - q^{-1}bc) - 1) = (S(d)S(a) - q^{-1}S(c)S(b))S(t) - S(1) = (t^2ad - q^{-1}t^2cb)(ad - q^{-1}bc) - 1 = t^2(ad - q^{-1}bc)^2 - 1 = 0$.

Since S satisfies $\sum S(x_{(1)})x_{(2)} = \varepsilon(x)$ for all given generators, S is a left antipode by 2.1.3. Symmetrically S is a right antipode. Thus the bialgebra $GL_q(2)$ is a Hopf algebra or a quantum group.

Example 2.5.8. Let $sl(2)$ be the 3-dimensional vector space generated by the matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $sl(2)$ is a subspace of the algebra $M(2)$ of 2×2-matrices over $\mathbb{K}$. We easily verify $[X, Y] = XY - YX = H$, $[H, X] = HX - XH = 2X$, and $[H, Y] = HY - YH = -2Y$, so that $sl(2)$ becomes a Lie subalgebra of $M(2)^L$, which is the Lie algebra of matrices of trace zero. The universal enveloping algebra $U(sl(2))$ is a Hopf algebra generated as an algebra by the elements X, Y, H with the relations

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y.$$

As a consequence of the Poincaré-Birkhoff-Witt Theorem (that we don't prove) the Hopf algebra $U(sl(2))$ has the basis $\{X^i Y^j H^k | i, j, k \in \mathbb{N}\}$. Furthermore one can prove that all finite dimensional $U(sl(2))$-modules are semisimple.

Example 2.5.9. We define the so-called q-deformed version $U_q(sl(2))$ of $U(sl(2))$ for any $q \in \mathbb{K}$, $q \neq 1, -1$ and q invertible. Let $U_q(sl(2))$ be the algebra generated by the elements E, F, K, K' with the relations

$$\begin{gathered} KK' = K'K = 1, \\ KEK' = q^2 E, \qquad KFK' = q^{-2} F, \\ EF - FE = \frac{K - K'}{q - q^{-1}}. \end{gathered}$$

Since K' is the inverse of K in $U_q(sl(2))$ we write $K^{-1} = K'$. The representation theory of this algebra is fundamentally different depending on whether q is a root of unity or not.

We show that $U_q(sl(2))$ is a Hopf algebra or quantum group. We define

$$\begin{gathered} \Delta(E) = 1 \otimes E + E \otimes K, \qquad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \\ \Delta(K) = K \otimes K, \\ \varepsilon(E) = \varepsilon(F) = 0, \qquad \varepsilon(K) = 1, \\ S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}. \end{gathered}$$

.

First we show that Δ can be expanded in a unique way to an algebra homomorphism $\Delta : U_q(sl(2)) \to U_q(sl(2)) \otimes U_q(sl(2))$. Write $U_q(sl(2))$ as the residue class algebra $\mathbb{K}\langle E, F, K, K^{-1}\rangle / I$ where I is generated by

$$\begin{gathered} KK^{-1} - 1, \qquad K^{-1}K - 1, \\ KEK^{-1} - q^2 E, \qquad KFK^{-1} - q^{-2} F, \\ EF - FE - \frac{K - K^{-1}}{q - q^{-1}}. \end{gathered}$$

Since K^{-1} must be mapped to the inverse of $\Delta(K)$ we must have $\Delta(K^{-1}) = K^{-1} \otimes K^{-1}$. Now Δ can be expanded in a unique way to the free algebra $\Delta : K\langle E, F, K, K^{-1}\rangle \to U_q(sl(2)) \otimes U_q(sl(2))$. We have $\Delta(KK^{-1}) = \Delta(K)\Delta(K^{-1}) = 1$ and similarly $\Delta(K^{-1}K) = 1$. Furthermore we have $\Delta(KEK^{-1}) = \Delta(K)\Delta(E)\Delta(K^{-1}) = (K \otimes K)(1 \otimes E + E \otimes K)(K^{-1} \otimes K^{-1}) = KK^{-1} \otimes KEK^{-1} + KEK^{-1} \otimes K^2K^{-1} = q^2(1 \otimes E + E \otimes K) = q^2\Delta(E) = \Delta(q^2 E)$ and similarly $\Delta(KFK^{-1}) = \Delta(q^{-2}F)$. Finally we have

$$\begin{aligned} \Delta(EF - FE) &= (1 \otimes E + E \otimes K)(K' \otimes F + F \otimes 1) \\ &\quad -(K' \otimes F + F \otimes 1)(1 \otimes E + E \otimes K) \\ &= K' \otimes EF + F \otimes E + EK' \otimes KF + EF \otimes K \\ &\quad -K' \otimes FE - K'E \otimes FK - F \otimes E - FE \otimes K \\ &= K' \otimes (EF - FE) + (EF - FE) \otimes K \\ &= \frac{K' \otimes (K - K') + (K - K') \otimes K}{q - q^{-1}} \\ &= \Delta\left(\frac{K - K'}{q - q^{-1}}\right) \end{aligned}$$

hence Δ vanishes on I and can be factorized through a unique algebra homomorphism

$$\Delta : U_q(sl(2)) \longrightarrow U_q(sl(2)) \otimes U_q(sl(2)).$$

In a similar way, actually much simpler, one gets an algebra homomorphism

$$\varepsilon : U_q(sl(2)) \longrightarrow \mathbb{K}.$$

To check that Δ is coassociative it suffices to check this for the generators of the algebra. We have $(\Delta \otimes 1)\Delta(E) = (\Delta \otimes 1)(1 \otimes E + E \otimes K) = 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K = (1 \otimes \Delta)(1 \otimes E + E \otimes K) = (1 \otimes \Delta)\Delta(E)$. Similarly we get $(\Delta \otimes 1)\Delta(F) = (1 \otimes \Delta)\Delta(F)$. For K the claim is obvious. The counit axiom is easily checked on the generators.

Now we show that S is an antipode for $U_q(sl(2))$. First define $S : \mathbb{K}\langle E, F, K, K^{-1}\rangle \to U_q(sl(2))^{op}$ by the definition of S on the generators. We have

$$\begin{gathered}
S(KK^{-1}) = 1 = S(K^{-1}K),\\
S(KEK^{-1}) = -KEK^{-1}K^{-1} = -q^2EK^{-1} = S(q^2E),\\
S(KFK^{-1}) = -KKFK^{-1} = -q^{-2}KF = S(q^{-2}F),\\
S(EF - FE) = KFEK^{-1} - EK^{-1}KF = KFK^{-1}KEK - EF\\
= \frac{K^{-1} - K}{q - q^{-1}} = S\left(\frac{K - K^{-1}}{q - q^{-1}}\right).
\end{gathered}$$

So S defines a homomorphism of algebras $S : U_q(sl(2)) \longrightarrow U_q(sl(2))$. Since S satisfies $\sum S(x_{(1)})x_{(2)} = \varepsilon(x)$ for all given generators, S is a left antipode by 2.1.3. Symmetrically S is a right antipode. Thus the bialgebra $U_q(sl(2))$ is a Hopf algebra or a quantum group.

This quantum group is of central interest in theoretical physics. Its representation theory is well understood. If q is not a root of unity then the finite dimensional $U_q(sl(2))$-modules are semisimple. Many more properties can be found in [Kassel: Quantum Groups].

6. Quantum Automorphism Groups

Lemma 2.6.1. *The category $\mathbb{K}$-$\mathcal{A}lg$ of $\mathbb{K}$-algebras has arbitrary coproducts.*

PROOF. This is a well known fact from universal algebra. In fact all equationally defined algebraic categories are complete and cocomplete. We indicate the construction of the coproduct of a family $(A_i | i \in I)$ of $\mathbb{K}$-algebras.

Define $\coprod_{i\in I} A_i := T(\oplus_{i\in I} A_i)/L$ where T denotes the tensor algebra and where L is the two sided ideal in $T(\oplus_{i\in I} A_i)$ generated by the set

$$J := \{\iota j_k(x_k y_k) - \iota(j_k(x_k))\iota(j_k(y_k)), 1_{T(\bigoplus A_i)} - \iota j_k(1_{A_k}) | x_k, y_k \in A_k, k \in I\}.$$

Then one checks easily for a family of algebra homomorphisms $(f_k : A_k \longrightarrow B | k \in I)$ that the following diagram gives the required universal property

$$\begin{tikzcd}
A_k \arrow[r, "j_k"] \arrow[drrr, "f_k"'] & \oplus A_i \arrow[r, "\iota"] \arrow[drr, "f"'] & T(\oplus A_i) \arrow[r, "\nu"] \arrow[dr, "f'"'] & T(\oplus A_i)/L \arrow[d, "\bar{f}"] \\
& & & B
\end{tikzcd}$$

□

Corollary 2.6.2. *The category of bialgebras has finite coproducts.*

Proof. The coproduct $\coprod B_i$ of bialgebras $(B_i | i \in I)$ in $\mathbb{K}$-$\mathcal{A}lg$ is an algebra. For the diagonal and the counit we obtain the following commutative diagrams

$$\begin{tikzcd}
B_k \arrow[r, "j_k"] \arrow[d, "\Delta_k"'] & \coprod B_i \arrow[d, "\exists_1 \Delta"] \\
B_k \otimes B_k \arrow[r, "j_k \otimes j_k"'] & \coprod B_i \otimes \coprod B_i
\end{tikzcd}$$

$$\begin{tikzcd}
B_k \arrow[r, "j_k"] \arrow[dr, "\varepsilon_k"'] & \coprod B_i \arrow[d, "\exists_1 \varepsilon"] \\
& \mathbb{K}
\end{tikzcd}$$

since in both cases $\coprod B_i$ is a coproduct in $\mathbb{K}$-$\mathcal{A}lg$. Then it is easy to show that these homomorphisms define a bialgebra structure on $\coprod B_i$ and that $\coprod B_i$ satisfies the universal property for bialgebras. □

Theorem 2.6.3. *Let B be a bialgebra. Then there exists a Hopf algebra $H(B)$ and a homomorphism of bialgebras $\iota : B \longrightarrow H(B)$ such that for every Hopf algebra H and for every homomorphism of bialgebras $f : B \longrightarrow H$ there is a unique homomorphism of Hopf algebras $g : H(B) \longrightarrow H$ such that the diagram*

$$\begin{tikzcd}
B \arrow[r, "\iota"] \arrow[dr, "f"'] & H(B) \arrow[d, "g"] \\
& H
\end{tikzcd}$$

commutes.

PROOF. Define a sequence of bialgebras $(B_i|i \in \mathbb{N})$ by

$$B_0 := B,$$
$$B_{i+1} := B_i^{opcop}, i \in \mathbb{N}.$$

Let B' be the coproduct of the family $(B_i|i \in \mathbb{N})$ with injections $\iota_i : B_i \to B'$. Because B' is a coproduct of bialgebras there is a unique homomorphism of bialgebras $S' : B' \to B'^{opcop}$ such that the diagrams

$$\begin{tikzcd} B_i \arrow[r, "\iota_i"] \arrow[d, "\mathrm{id}"'] & B' \arrow[d, "S'"] \\ B_{i+1}^{opcop} \arrow[r, "\iota_{i+1}"'] & B'^{opcop} \end{tikzcd}$$

commute.

Now let I be the two sided ideal in B' generated by

$$\{(S' * 1 - u\varepsilon)(x_i), (1 * S' - u\varepsilon)(x_i)|x_i \in \iota_i(B_i), i \in \mathbb{N}\}.$$

I is a coideal, i.e. $\varepsilon_{B'}(I) = 0$ and $\Delta_{B'}(I) \subseteq I \otimes B' + B' \otimes I$.

Since $\varepsilon_{B'}$ and $\Delta_{B'}$ are homomorphisms of algebras it suffices to check this for the generating elements of I. Let $x \in B_i$ be given. Then we have $\varepsilon((1 * S')\iota_i(x)) = \varepsilon(\nabla(1 \otimes S')\Delta\iota_i(x)) = \nabla_{\mathbb{K}}(\varepsilon \otimes \varepsilon S')(\iota_i \otimes \iota_i)\Delta_i(x) = (\varepsilon\iota_i \otimes \varepsilon\iota_i)\Delta_i(x) = \varepsilon_i(x) = \varepsilon(u\varepsilon\iota_i(x))$. Symmetrically we have $\varepsilon((S' * 1)\iota_i(x)) = \varepsilon(u\varepsilon\iota_i(x))$. Furthermore we have

$$\begin{aligned}
&\Delta((1 * S')\iota_i(x)) \\
&= \Delta\nabla(1 \otimes S')\Delta\iota_i(x) \\
&= (\nabla \otimes \nabla)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta)(1 \otimes S')(\iota_i \otimes \iota_i)\Delta_i(x) \\
&= (\nabla \otimes \nabla)(1 \otimes \tau \otimes 1)(\Delta \otimes \tau(S' \otimes S')\Delta)(\iota_i \otimes \iota_i)\Delta_i(x) \\
&= \sum (\nabla \otimes \nabla)(1 \otimes \tau \otimes 1)(\iota_i(x_{(1)}) \otimes \iota_i(x_{(2)}) \otimes S'\iota_i(x_{(4)}) \otimes S'\iota_i(x_{(3)})) \\
&= \sum \iota_i(x_{(1)})S'\iota_i(x_{(4)}) \otimes \iota_i(x_{(2)})S'\iota_i(x_{(3)}) \\
&= \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes (1 * S')\iota_i(x_{(2)}).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\Delta((1 * S' - u\varepsilon)\iota_i(x)) \\
&= \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes (1 * S')\iota_i(x_{(2)}) - \Delta u\varepsilon\iota_i(x) \\
&= \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes ((1 * S') - u\varepsilon)\iota_i(x_{(2)}) \\
&\quad + \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes u\varepsilon\iota_i(x_{(2)}) - \Delta u\varepsilon\iota_i(x) \\
&= \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes (1 * S' - u\varepsilon)\iota_i(x_{(2)}) \\
&\quad + \sum \iota_i(x_{(1)})S'\iota_i(x_{(2)}) \otimes 1_{B'} - u\varepsilon\iota_i(x) \otimes 1_{B'} \\
&= \sum \iota_i(x_{(1)})S'\iota_i(x_{(3)}) \otimes (1 * S' - u\varepsilon)\iota_i(x_{(2)}) \\
&\quad + (1 * S' - u\varepsilon)\iota_i(x) \otimes 1_{B'} \\
&\in B' \otimes I + I \otimes B'.
\end{aligned}$$

Thus I is a coideal and a biideal of B'.

Now let $H(B) := B'/I$ and let $\nu : B' \to H(B)$ be the residue class homomorphism. We show that $H(B)$ is a bialgebra and ν is a homomorphism of bialgebras. $H(B)$ is an algebra and ν is a homomorphism of algebras since I is a two sided ideal. Since $I \subseteq \mathrm{Ker}(\varepsilon)$ there is a unique factorization

$$\begin{tikzcd} B' \arrow[r, "\nu"] \arrow[dr, "\varepsilon'"'] & B'/I \arrow[d, "\varepsilon"] \\ & \mathbb{K} \end{tikzcd}$$

where $\varepsilon : B'/I \to \mathbb{K}$ is a homomorphism of algebras. Since $\Delta(I) \subseteq B' \otimes I + I \otimes B' \subseteq \mathrm{Ker}(\nu \otimes \nu : B' \otimes B' \to B'/I \otimes B'/I)$ and thus $I \subseteq \mathrm{Ker}(\Delta(\nu \otimes \nu))$ we have a unique factorization

$$\begin{tikzcd} B' \arrow[r, "\nu"] \arrow[d, "\Delta_{B'}"'] & B'/I \arrow[d, "\Delta"] \\ B' \otimes B' \arrow[r, "\nu\otimes\nu"'] & B'/I \otimes B'/I \end{tikzcd}$$

by an algebra homomorphism $\Delta : B'/I \to B'/I \otimes B'/I$. Now it is easy to verify that B'/I becomes a bialgebra and ν a bialgebra homomorphism.

We show that the map $\nu S' : B' \to B'/I$ can be factorized through B'/I in the commutative diagram

$$\begin{tikzcd} B' \arrow[r, "\nu"] \arrow[dr, "\nu S'"'] & B'/I \arrow[d, "S"] \\ & B'/I \end{tikzcd}$$

This holds if $I \subseteq \mathrm{Ker}(\nu S')$. Since $\mathrm{Ker}(\nu) = I$ it suffices to show $S'(I) \subseteq I$. We have

$$\begin{aligned} & S'((S' * 1)\iota_i(x)) = \\ & = \nabla\tau({S'}^2\iota_i \otimes S'\iota_i)\Delta_i(x) \\ & = \nabla\tau(S' \otimes 1)(\iota_{i+1} \otimes \iota_{i+1})\Delta_i(x) \\ & = \nabla(1 \otimes S')(\iota_{i+1} \otimes \iota_{i+1})\tau\Delta_i(x) \\ & = \nabla(1 \otimes S')(\iota_{i+1} \otimes \iota_{i+1})\Delta_{i+1}(x) \\ & = (1 * S')\iota_{i+1}(x) \end{aligned}$$

and

$$S'(u\varepsilon\iota_i(x)) = S'(1)\varepsilon_i(x) = S'(1)\varepsilon_{i+1}(x) = S'(u\varepsilon\iota_{i+1}(x))$$

hence we get

$$S'((S' * 1 - u\varepsilon)\iota_i(x)) = (1 * S' - u\varepsilon)\iota_{i+1}(x) \in I.$$

This shows $S'(I) \subseteq I$. So there is a unique homomorphism of bialgebras $S : H(B) \to H(B)^{opcop}$ such that the diagram

$$\begin{tikzcd} B' \arrow[r, "\nu"] \arrow[d, "S'"'] & H(B) \arrow[d, "S"] \\ B'^{opcop} \arrow[r, "\nu"'] & H(B)^{opcop} \end{tikzcd}$$

commutes.

Now we show that $H(B)$ is a Hopf algebra with antipode S. By Proposition 2.1.3 it suffices to test on generators of $H(B)$ hence on images $\nu\iota_i(x)$ of elements $x \in B_i$. We have

$$\begin{aligned}(1 * S)\nu\iota_i(x) = \nabla(\nu \otimes S\nu)\Delta\iota_i(x) = \nabla(\nu \otimes \nu)(1 \otimes S')\Delta\iota_i(x) = \\ = \nu(1 * S')\iota_i(x) = \nu u\varepsilon\iota_i(x) = u\varepsilon\nu\iota_i(x).\end{aligned}$$

By Proposition 2.1.3 S is an antipode for $H(B)$.

We prove now that $H(B)$ together with $\iota := \nu\iota_0 : B \to H(B)$ is a free Hopf algebra over B. Let H be a Hopf algebra and let $f : B \to H$ be a homomorphism of bialgebras. We will show that there is a unique homomorphism $\bar{f} : H(B) \to H$ such that

$$\begin{tikzcd} B \arrow[r, "\iota"] \arrow[dr, "f"'] & H(B) \arrow[d, "\bar{f}"] \\ & H \end{tikzcd}$$

commutes.

We define a family of homomorphisms of bialgebras $f_i : B_i \to H$ by

$$\begin{aligned} &f_0 := f, \\ &f_{i+1} := S_H f_i, i \in \mathbb{N}. \end{aligned}$$

We have in particular $f_i = S_H^i f$ for all $i \in \mathbb{N}$. Thus there is a unique homomorphism of bialgebras $f' : B' = \coprod B_i \to H$ such that $f'\iota_i = f_i$ for all $i \in \mathbb{N}$.

We show that $f'(I) = 0$. Let $x \in B_i$. Then

$$\begin{aligned} f'((1 * S')\iota_i(x)) &= f'(\nabla(1 \otimes S')(\iota_i \otimes \iota_i)\Delta_i(x)) \\ &= \sum f'\iota_i(x_{(1)})f'S'\iota_i(x_{(2)}) \\ &= \sum f'\iota_i(x_{(1)})f'\iota_{i+1}(x_{(2)}) \\ &= \sum f_i(x_{(1)})f_{i+1}(x_{(2)}) \\ &= \sum f_i(x_{(1)})Sf_i(x_{(2)}) \\ &= (1 * S)f_i(x) = u\varepsilon f_i(x) = u\varepsilon_i(x) \\ &= f'(u\varepsilon\iota_i(x)). \end{aligned}$$

This together with the symmetric statement gives $f'(I) = 0$. Hence there is a unique factorization through a homomorphism of algebras $\bar{f} : H(B) \to H$ such that $f' = \bar{f}\nu$.

The homomorphism $\bar{f} : H(B) \longrightarrow H$ is a homomorphism of bialgebras since the diagram

$$\begin{tikzcd}
B' \arrow[r, "\nu"'] \arrow[rr, bend left, "f'"] \arrow[d, "\Delta"'] & B'/I \arrow[r, "\bar{f}"'] \arrow[d, "\Delta'"] & H \arrow[d, "\Delta_H"] \\
B' \otimes B' \arrow[r, "\nu\otimes\nu"] \arrow[rr, bend right, "f'\otimes f'"'] & B'/I \otimes B'/I \arrow[r, "\bar{f}\otimes\bar{f}"] & H \otimes H
\end{tikzcd}$$

commutes with the possible exception of the right hand square $\Delta\bar{f}$ and $(\bar{f}\otimes\bar{f})\Delta'$. But ν is surjective so also the last square commutes. Similarly we get $\varepsilon_H\bar{f} = \varepsilon_{H(B)}$. Thus $\bar{f}$ is a homomorphism of bialgebras and hence a homomorphism of Hopf algebras. $\square$

Remark 2.6.4. In chapter 1 we have constructed universal bialgebras $M(A)$ with coaction $\delta : A \longrightarrow M(A) \otimes A$ for certain algebras A (see 1.3.12). This induces a homomorphism of algebras

$$\delta' : A \longrightarrow H(M(A)) \otimes A$$

such that A is a comodule-algebra over the Hopf algebra $H(M(A))$. If H is a Hopf algebra and A is an H-comodule algebra by $\partial : A \longrightarrow H \otimes A$ then there is a unique homomorphism of bialgebras $f : M(A) \longrightarrow H$ such that

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[dr, "\partial"'] & M(A) \otimes A \arrow[d, "f\otimes 1"] \\
& H \otimes A
\end{tikzcd}$$

commutes. Since the $f : M(A) \longrightarrow H$ factorizes uniquely through $\bar{f} : H(M(A)) \longrightarrow H$ we get a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\delta'"] \arrow[dr, "\partial"'] & H(M(A)) \otimes A \arrow[d, "\bar{f}\otimes 1"] \\
& H \otimes A
\end{tikzcd}$$

with a unique homomorphism of Hopf algebras $\bar{f} : H(M(A)) \longrightarrow H$.

This proof depends only on the existence of a universal algebra $M(A)$ for the algebra A. Hence we have

Corollary 2.6.5. *Let $\mathcal{X}$ be a quantum space with universal quantum space (and quantum monoid) $\mathcal{M}(\mathcal{X})$. Then there is a unique (up to isomorphism) quantum group $\mathcal{H}(\mathcal{M}(\mathcal{X}))$ acting universally on $\mathcal{X}$.*

This quantum group $\mathcal{H}(\mathcal{M}(\mathcal{X}))$ can be considered as the "quantum subgroup of invertible elements" of $\mathcal{M}(\mathcal{X})$ or the quantum group of "quantum automorphisms" of $\mathcal{X}$.

7. Duality of Hopf Algebras

In 2.4.8 we have seen that the dual Hopf algebra H^* of a finite dimensional Hopf algebra H satisfies certain relations w.r.t. the evaluation map. The multiplication of H^* is derived from the comultiplication of H and the comultiplication of H^* is derived from the multiplication of H.

This kind of duality is restricted to the finite-dimensional situation. Nevertheless one wants to have a process that is close to the finite-dimensional situation. This short section is devoted to several approaches of duality for Hopf algebras.

First we use the relations of the finite-dimensional situation to give a general definition.

Definition 2.7.1. Let H and L be Hopf algebras. Let

$$\mathrm{ev} : L \otimes H \ni a \otimes h \mapsto \langle a, h\rangle \in \mathbb{K}$$

be a bilinear form satisfying

$$\langle a \otimes b, \sum h_{(1)} \otimes h_{(2)}\rangle = \langle ab, h\rangle, \quad \langle 1, h\rangle = \varepsilon(h) \tag{4}$$

$$\langle \sum a_{(1)} \otimes a_{(2)}, h \otimes j\rangle = \langle a, hj\rangle, \quad \langle a, 1\rangle = \varepsilon(a) \tag{5}$$

$$\langle a, S(h)\rangle = \langle S(a), h\rangle \tag{6}$$

Such a map is called a *weak duality of Hopf algebras*. The bilinear form is called *left* (*right*) *nondegenerate* if $\langle a, H\rangle = 0$ implies $a = 0$ ($\langle L, h\rangle = 0$ implies $h = 0$). A *duality of Hopf algebras* is a weak duality that is left and right nondegenerate.

Remark 2.7.2. If H is a finite dimensional Hopf algebra then the usual evaluation $\mathrm{ev} : H^* \otimes H \longrightarrow \mathbb{K}$ defines a duality of Hopf algebras.

Remark 2.7.3. Assume that $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ defines a weak duality. By [Advanced Algebra] 1.22 we have isomorphisms $\mathrm{Hom}(L\otimes H, \mathbb{K}) \cong \mathrm{Hom}(L, \mathrm{Hom}(H, \mathbb{K}))$ and $\mathrm{Hom}(L \otimes H, \mathbb{K}) \cong \mathrm{Hom}(H, \mathrm{Hom}(L, \mathbb{K}))$. Denote the homomorphisms associated with $\mathrm{ev} : L \otimes K \longrightarrow \mathbb{K}$ by $\varphi : L \longrightarrow \mathrm{Hom}(H, \mathbb{K})$ resp. $\psi : H \longrightarrow \mathrm{Hom}(L, \mathbb{K})$. They satisfy $\varphi(a)(h) = \mathrm{ev}(a \otimes h) = \psi(h)(a)$.

$\mathrm{ev} : L \otimes K \longrightarrow \mathbb{K}$ is left nondegenerate iff $\varphi : L \longrightarrow \mathrm{Hom}(H, \mathbb{K})$ is injective. $\mathrm{ev} : L \otimes K \longrightarrow \mathbb{K}$ is right nondegenerate iff $\psi : H \longrightarrow \mathrm{Hom}(L, \mathbb{K})$ is injective.

Lemma 2.7.4. 1. *The bilinear form* $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ *satisfies (2.4) if and only if* $\varphi : L \longrightarrow \mathrm{Hom}(H, \mathbb{K})$ *is a homomorphism of algebras.*

2. *The bilinear form* $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ *satisfies (2.5) if and only if* $\psi : H \longrightarrow \mathrm{Hom}(L, \mathbb{K})$ *is a homomorphism of algebras.*

PROOF. $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ satisfies the right equation of (4) iff $\varphi(ab)(h) = \langle ab, h\rangle = \langle a \otimes b, \sum h_{(1)} \otimes h_{(2)}\rangle = \sum \langle a, h_{(1)}\rangle\langle b, h_{(2)}\rangle = \sum \varphi(a)(h_{(1)})\varphi(b)(h_{(2)}) = (\varphi(a) * \varphi(b))(h)$ by the definition of the algebra structure on $\mathrm{Hom}(H, \mathbb{K})$.

$\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ satisfies the left equation of (4) iff $\varphi(1)(h) = \langle 1, h\rangle = \varepsilon(h)$.

The second part of the Lemma follows by symmetry. □

Example 2.7.5. There is a weak duality between the quantum groups $\mathbb{SL}_q(2)$ and $U_q(sl(2))$. (Kassel: Chapter VII.4).

Proposition 2.7.6. *Let* $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ *be a weak duality of Hopf algebras. Let* $I := \mathrm{Ker}(\varphi : L \longrightarrow \mathrm{Hom}(H, \mathbb{K}))$ *and* $J := \mathrm{Ker}(\psi : H \longrightarrow \mathrm{Hom}(L, \mathbb{K}))$. *Let* $\overline{L} := L/I$ *and* $\overline{H} := H/J$. *Then* $\overline{L}$ *and* $\overline{H}$ *are Hopf algebras and the induced bilinear form* $\overline{\mathrm{ev}} : \overline{L} \otimes \overline{H} \longrightarrow \mathbb{K}$ *is a duality.*

PROOF. First observe that I and J are two sided ideals hence $\overline{L}$ and $\overline{H}$ are algebras. Then $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ can be factored through $\overline{\mathrm{ev}} : \overline{L} \otimes \overline{H} \longrightarrow \mathbb{K}$ and the equations (4) and (5) are still satisfied for the residue classes.

The ideals I and J are biideals. In fact, let $x \in I$ then $\langle \Delta(x), a \otimes b\rangle = \langle x, ab\rangle = 0$ hence $\Delta(x) \in \mathrm{Ker}(\varphi \otimes \varphi : L \otimes L \longrightarrow \mathrm{Hom}(H \otimes H, \mathbb{K}) = I \otimes L + L \otimes I$ (the last equality is an easy exercise in linear algebra) and $\varepsilon(x) = \langle x, 1\rangle = 0$. Hence as in the proof of Theorem 2.6.3 we get that $\overline{L} = L/I$ and $\overline{H} = H/J$ are bialgebras. Since $\langle S(x), a\rangle = \langle x, S(a)\rangle = 0$ we have an induced homomorphism $\overline{S} : \overline{L} \longrightarrow \overline{L}$. The identities satisfied in L hold also for the residue classes in $\overline{L}$ so that L and similarly $\overline{H}$ become Hopf algebras. Finally we have by definition of I that $\langle \overline{x}, \overline{a}\rangle = \langle x, a\rangle = 0$ for all $a \in H$ iff $a \in I$ or $\overline{a} = 0$. Thus the bilinear form $\overline{\mathrm{ev}} : \overline{L} \otimes \overline{H} \longrightarrow K$ defines a duality. □

Problem 2.7.15. (in Linear Algebra)

(1) For $U \subseteq V$ define $U^{\perp} := \{f \in V^* | f(U) = 0\}$. For $Z \subseteq V^*$ define $Z^{\perp} := \{v \in V | Z(v) = 0\}$. Show that the following hold:
   (a) $U \subseteq V \Longrightarrow U = U^{\perp\perp}$;
   (b) $Z \subseteq V^*$ and $\dim Z < \infty \Longrightarrow Z = Z^{\perp\perp}$;
   (c) $\{U \subseteq V | \dim V/U < \infty\} \cong \{Z \subseteq V^* | \dim Z < \infty\}$ under the maps $U \mapsto U^{\perp}$ and $Z \mapsto Z^{\perp}$.
(2) Let $V = \oplus_{i=1}^{\infty} \mathbb{K} x_i$ be an infinite-dimensional vector space. Find an element $g \in (V \otimes V)^*$ that is not in $V^* \otimes V^* \quad (\subseteq (V \otimes V)^*)$.

Definition 2.7.7. Let A be an algebra. We define $A^o := \{f \in A^* | \exists \text{ ideal } {}_AI_A \subseteq A : \dim(A/I) < \infty \text{ and } f(I) = 0\}$.

Lemma 2.7.8. *Let A be an algebra and $f \in A^*$. The following are equivalent:*

(1) $f \in A^o$;
(2) *there exists* $I_A \subseteq A$ *such that* $\dim A/I < \infty$ *and* $f(I) = 0$;
(3) $A \cdot f \subseteq {}_A\mathrm{Hom}_{\mathbb{K}}(.A_A, .\mathbb{K})$ *is finite dimensional;*
(4) $A \cdot f \cdot A$ *is finite dimensional;*

(5) $\nabla^*(f) \in A^* \otimes A^*$.

PROOF. 1. $\Longrightarrow$ 2. and 4. $\Longrightarrow$ 3. are trivial.

2. $\Longrightarrow$ 3. Let $I_A \subseteq A$ with $f(I) = 0$ and $\dim A/I < \infty$. Write $A^* \otimes A \to \mathbb{K}$ as $\langle g, a \rangle$. Then $\langle af, i \rangle = \langle f, ia \rangle = 0$ hence $Af \subset I^\perp$ and $\dim Af < \infty$.

3. $\Longrightarrow$ 2. Let $\dim Af < \infty$. Then $I_A := (Af)^\perp$ is an ideal of finite codimension in A and $f(I) = 0$ holds.

2. $\Longrightarrow$ 1. Let $I_A \subset A$ with $\dim A/I_A < \infty$ and $f(I) = 0$ be given. Then right multiplication induces $\varphi : A \to \mathrm{Hom}_{\mathbb{K}}(A/I., A/I.)$ and $\dim \mathrm{End}_{\mathbb{K}}(A/I) < \infty$. Thus $J = \mathrm{Ker}(\varphi) \subseteq A$ is a two sided ideal of finite codimension and $J \subset I$ (since $\varphi(j)(\bar{1}) = 0 = \bar{1} \cdot j = \bar{j}$ implies $j \in I$). Furthermore we have $f(J) \subseteq f(I) = 0$.

1. $\Longrightarrow$ 4. $\langle afb, i \rangle = \langle f, bia \rangle = 0$ implies $A \cdot f \cdot A \subseteq {}_A I_A^\perp$ hence $\dim AfA < \infty$.

3. $\Longrightarrow$ 5. We observe that $\nabla^*(f) = f\nabla \in (A \otimes A)^*$. We want to show that $\nabla^*(f) \in A^* \otimes A^*$. Let $g_1, \ldots, g_n$ be a basis of Af. Then there exist $h_1, \ldots, h_n \in A^*$ such that $bf = \sum h_i(b) g_i$. Let $a, b \in A$. Then $\langle \nabla^*(f), a \otimes b \rangle = \langle f, ab \rangle = \langle bf, a \rangle = \sum h_i(b) g_i(a) = \langle \sum g_i \otimes h_i, a \otimes b \rangle$ so that $\nabla^*(f) = \sum g_i \otimes h_i \in A^* \otimes A^*$.

5. $\Longrightarrow$ 3. Let $\nabla^*(f) = \sum g_i \otimes h_i \in A^* \otimes A^*$. Then $bf = \sum h_i(b) g_i$ for all $b \in A$ as before. Thus Af is generated by the $g_1, \ldots, g_n$. $\square$

Proposition 2.7.9. *Let (A, m, u) be an algebra. Then we have $m^*(A^o) \subseteq A^o \otimes A^o$. Furthermore $(A^o, \Delta, \varepsilon)$ is a coalgebra with $\Delta = m^*$ and $\varepsilon = u^*$.*

PROOF. Let $f \in A^o$ and let $g_1, \ldots, g_n$ be a basis for Af. Then we have $m^*(f) = \sum g_i \otimes h_i$ for suitable $h_i \in A^*$ as in the proof of the previous proposition. Since $g_i \in Af$ we get $Ag_i \subseteq Af$ and $\dim(Ag_i) < \infty$ and hence $g_i \in A^o$. Choose $a_1, \ldots, a_n \in A$ such that $g_i(a_j) = \delta_{ij}$. Then $(fa_j)(a) = f(a_j a) = \langle m^*(f), a_j \otimes a \rangle = \sum g_i(a_j) h_i(a) = h_j(a)$ implies $fa_j = h_j \in fA$. Observe that $\dim(fA) < \infty$ hence $\dim(h_j A) < \infty$, so that $h_j \in A^o$. This proves $m^*(f) \in A^o \otimes A^o$.

One checks easily that counit law and coassociativity hold. $\square$

Theorem 2.7.10. (The Sweedler dual:) *Let $(B, m, u, \Delta, \varepsilon)$ be a bialgebra. Then $(B^o, \Delta^*, \varepsilon^*, m^*, u^*)$ again is a bialgebra. If $B = H$ is a Hopf algebra with antipode S, then S^* is an antipode for $B^o = H^o$.*

PROOF. We know that $(B^*, \Delta^*, \varepsilon^*)$ is an algebra and that (B^o, m^*, u^*) is a coalgebra. We show now that $B^o \subseteq B^*$ is a subalgebra. Let $f, g \in B^o$ with $\dim(Bf) < \infty$ and $\dim(Bg) < \infty$. Let $a \in B$. Then we have $(a(fg))(b) = (fg)(ba) = \sum f(b_{(1)} a_{(1)}) g(b_{(2)} a_{(2)}) = \sum (a_{(1)} f)(b_{(1)}) (a_{(2)} g)(b_{(2)}) = \sum ((a_{(1)} f)(a_{(2)} g))(b)$ hence $a(fg) = \sum (a_{(1)} f)(a_{(2)} g) \in (Bf)(Bg)$. Since $\dim (Bf)(Bg) < \infty$ we have $\dim(B(fg)) < \infty$ so that $fg \in B^o$. Furthermore we have $\varepsilon \in B^o$, since $\mathrm{Ker}(\varepsilon)$ has codimension 1. Thus $B^o \subseteq B^*$ is a subalgebra. It is now easy to see that B^o is a bialgebra.

Now let S be the antipode of H. We show $S^*(H^o) \subseteq H^o$. Let $a \in H$, $f \in H^o$. Then $\langle aS^*(f), b \rangle = \langle S^*(f), ba \rangle = \langle f, S(ba) \rangle = \langle f, S(a)S(b) \rangle = \langle fS(a), S(b) \rangle =$

$\langle S^*(fS(a)), b\rangle$. This implies $aS^*(f) = S^*(fS(a))$ and $HS^*(f) = S^*(fS(H)) \subseteq S^*(fH)$. Since $f \in H^o$ we get $\dim(fH) < \infty$ so that $\dim(S^*(fH)) < \infty$ and $\dim(HS^*(f)) < \infty$. This shows $S^*(f) \in H^o$. The rest of the proof is now trivial. $\square$

Definition 2.7.11. Let $G = \mathbb{K}\text{-}c\mathcal{A}lg(H, \text{-})$ be an affine group and $R \in \mathbb{K}\text{-}c\mathcal{A}lg$. We define $G \otimes_{\mathbb{K}} R := G|_{R\text{-}c\mathcal{A}lg}$ to be the restriction to commutative R-algebras. The functor $G \otimes_{\mathbb{K}} R$ is represented by $H \otimes R \in R\text{-}c\mathcal{A}lg$:

$$G|_{R\text{-}c\mathcal{A}lg}(A) = \mathbb{K}\text{-}c\mathcal{A}lg(H, A) \cong R\text{-}c\mathcal{A}lg(H \otimes R, A).$$

Theorem 2.7.12. (The Cartier dual:) *Let H be a finite dimensional commutative cocommutative Hopf algebra. Let $G = \mathbb{K}\text{-}c\mathcal{A}lg(H, \text{-})$ be the associated affine group and let $D(G) := \mathbb{K}\text{-}c\mathcal{A}lg(H^*, \text{-})$ be the dual group. Then we have*

$$D(G) = \mathcal{G}r(G, G_m)$$

where $\mathcal{G}r(G, G_m)(R) = \mathrm{Gr}(G \otimes_{\mathbb{K}} R, G_m \otimes_{\mathbb{K}} R)$ is the set of group (-functor) homomorphisms and G_m is the multiplicative group.

Proof. We have $\mathcal{G}r(G, G_m)(R) = \mathrm{Gr}(G \otimes_{\mathbb{K}} R, G_M \otimes_{\mathbb{K}} R) \cong R\text{-Hopf-Alg}(\mathbb{K}[t, t^{-1}] \otimes R, H \otimes R) \cong R\text{-Hopf-Alg}(R[t, t^{-1}], H \otimes R) \cong \{x \in U(H \otimes R) | \Delta(x) = x \otimes x, \varepsilon(x) = 1\}$, since $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$ imply $xS(x) = \varepsilon(x) = 1$.

Consider $x \in \mathrm{Hom}_R((H \otimes R)^*, R) = \mathrm{Hom}_R(H^* \otimes R, R)$. Then $\Delta(x) = x \otimes x$ iff $x(v^*w^*) = \langle x, v^*w^*\rangle = \langle \Delta(x), v^* \otimes w^*\rangle = x(v^*)x(w^*)$ and $\varepsilon(x) = 1$ iff $\langle x, \varepsilon\rangle = 1$. Hence $x \in R\text{-}c\mathcal{A}lg((H \otimes R)^*, R) \cong \mathbb{K}\text{-}c\mathcal{A}lg(H^*, R) = D(G)(R)$. $\square$

CHAPTER 3

Representation Theory, Reconstruction and Tannaka Duality

Introduction

One of the most interesting properties of quantum groups is their representation theory. It has deep applications in theoretical physics. The mathematical side has to distinguish between the representation theory of quantum groups and the representation theory of Hopf algebras. In both cases the particular structure allows to form tensor products of representations such that the category of representations becomes a monoidal (or tensor) category.

The problem we want to study in this chapter is, how much structure of the quantum group or Hopf algebra can be found in the category of representations. We will show that a quantum monoid can be uniquely reconstructed (up to isomorphism) from its representations. The additional structure given by the antipode is itimitely connected with a certain duality of representations. We will also generalize this process of reconstruction.

On the other hand we will show that the process of reconstruction can also be used to obtain the Tambara construction of the universal quantum monoid of a noncommutative geometrical space (from chapter 1.). Thus we will get another perspective for this theorem.

At the end of the chapter you should

- understand representations of Hopf algebras and of quantum groups,
- know the definition and first fundamental properties of monoidal or tensor categories,
- be familiar with the monoidal structure on the category of representations of Hopf algebras and of quantum groups,
- understand why the category of representations contains the full information about the quantum group resp. the Hopf algebra (Theorem of Tannaka-Krein),
- know the process of reconstruction and examples of bialgebras reconstructed from certain diagrams of finite dimensional vector spaces,
- understand better the Tambara construction of a universal algebra for a finite dimensional algebra.

1. Representations of Hopf Algebras

Let A be an algebra over a commutative ring $\mathbb{K}$. Let A-$\mathcal{M}od$ be the category of A-modules. An A-module is also called a *representation* of A.

Observe that the action $A \otimes M \longrightarrow M$ satisfying the module axioms and an algebra homomorphism $A \longrightarrow \mathrm{End}(M)$ are equivalent descriptions of an A-module structure on the $\mathbb{K}$-module M.

The functor $\mathcal{U} : A\text{-}\mathcal{M}od \longrightarrow \mathbb{K}\text{-}\mathcal{M}od$ with $\mathcal{U}({}_AM) = M$ and $\mathcal{U}(f) = f$ is called the *forgetful functor* or the *underlying functor*.

If B is a bialgebra then a *representation of* B is also defined to be a B-module. It will turn out that the property of being a bialgebra leads to the possibility of building tensor products of representations in a canonical way.

Let C be a coalgebra over a commutative ring $\mathbb{K}$. Let C-$\mathcal{C}omod$ be the category of C-comodules. A C-comodule is also called a *corepresentation* of C.

The functor $\mathcal{U} : C\text{-}\mathcal{C}omod \longrightarrow \mathbb{K}\text{-}\mathcal{M}od$ with $\mathcal{U}({}^CM) = M$ and $\mathcal{U}(f) = f$ is called the *forgetful functor* or the *underlying functor*.

If B is a bialgebra then a *corepresentation of* B is also defined to be a B-comodule. It will turn out that the property of being a bialgebra leads to the possibility of building tensor products of corepresentations in a canonical way.

Usually representations of a ring are considered to be modules over the given ring. The role of comodules certainly arises in the context of coalgebras. But it is not quite clear what the good definition of a representation of a quantum group or its representing Hopf algebra is.

For this purpose consider representations M of an ordinary group G. Assume for the simplicity of the argument that G is finite. Representations of G are vector spaces together with a group action $G \times M \longrightarrow M$. Equivalently they are vector spaces together with a group homomorphism $G \longrightarrow \mathrm{Aut}(M)$ or modules over the group algebra: $\mathbb{K}[G] \otimes M \longrightarrow M$. In the situation of quantum groups we consider the representing Hopf algebra H as algebra of functions on the quantum group G.

Then the algebra of functions on G is the Hopf algebra $\mathbb{K}^G$, the dual of the group algebra $\mathbb{K}[G]$. An easy exercise shows that the module structure $\mathbb{K}[G] \otimes M \longrightarrow M$ translates to the structure of a comodule $M \longrightarrow \mathbb{K}^G \otimes M$ and conversely. (Observe that G is finite.) So we should define representations of a quantum group as comodules over the representing Hopf algebra.

Problem 3.1.16. Let G be a finite group and $\mathbb{K}^G := \mathbb{K}[G]^*$ the dual of the group algebra. Show that $\mathbb{K}^G$ is a Hopf algebra and that each module structure $\mathbb{K}[G] \otimes M \longrightarrow M$ translates to the structure of a comodule $M \longrightarrow \mathbb{K}^G \otimes M$ and conversely. Show that this defines a monoidal equivalence of categories.

Describe the group valued functor $\mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}^G, -)$ in terms of sets and their group structure.

Definition 3.1.1. Let G be a quantum group with representing Hopf algebra H. A *representation* of G is a comodule over the representing Hopf algebra H.

From this definition we obtain immediately that we may form tensor products of representations of quantum groups since the representing algebra is a bialgebra.

We come now to the canonical construction of tensor products of (co-)representations.

Lemma 3.1.2. *Let B be a bialgebra. Let $M, N \in B\text{-}\mathcal{M}od$ be two B-modules. Then $M \otimes N$ is a B-module by the action $b(m \otimes n) = \sum b_{(1)}m \otimes b_{(2)}n$. If $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$ are homomorphisms of B-modules in $B\text{-}\mathcal{M}od$ then $f \otimes g : M \otimes N \longrightarrow M' \otimes N'$ is a homomorphism of B-modules.*

Proof. We have homomorphisms of $\mathbb{K}$-algebras $\alpha : B \longrightarrow \mathrm{End}(M)$ and $\beta : B \longrightarrow \mathrm{End}(N)$ defining the B-module structure on M and N. Thus we get a homomorphism of algebras $\mathrm{can}(\alpha \otimes \beta)\Delta : B \longrightarrow B \otimes B \longrightarrow \mathrm{End}(M) \otimes \mathrm{End}(N) \longrightarrow \mathrm{End}(M \otimes N)$. Thus $M \otimes N$ is a B-module. The structure is $b(m \otimes n) = \mathrm{can}(\alpha \otimes \beta)(\sum b_{(1)} \otimes b_{(2)})(m \otimes n) = \mathrm{can}(\sum \alpha(b_{(1)}) \otimes \beta(b_{(2)}))(m \otimes n) = \sum \alpha(b_{(1)})(m) \otimes \beta(b_{(2)})(n) = \sum b_{(1)}m \otimes b_{(2)}n$.

Furthermore we have $1(m \otimes n) = 1m \otimes 1m = m \otimes n$.

If f, g are homomorphisms of B-modules, then we have $(f \otimes g)(b(m \otimes n)) = (f \otimes g)(\sum b_{(1)}m \otimes b_{(2)}n) = \sum f(b_{(1)}m) \otimes g(b_{(2)}n) = \sum b_{(1)}f(m) \otimes b_{(2)}g(n) = b(f(m) \otimes g(n)) = b(f \otimes g)(m \otimes n)$. Thus $f \otimes g$ is a homomorphism of B-modules. □

Corollary 3.1.3. *Let B be a bialgebra. Then $\otimes : B\text{-}\mathcal{M}od \times B\text{-}\mathcal{M}od \longrightarrow B\text{-}\mathcal{M}od$ with $\otimes(M, N) = M \otimes N$ and $\otimes(f, g) = f \otimes g$ is a functor.*

Proof. The following are obvious from the ordinary properties of the tensor product over $\mathbb{K}$. $1_M \otimes 1_N = 1_{M \otimes N}$ and $(f \otimes g)(f' \otimes g') = ff' \otimes gg'$ for $M, N, f, f', g, g' \in B\text{-}\mathcal{M}od$. □

Lemma 3.1.4. *Let B be a bialgebra. Let $M, N \in B\text{-}\mathcal{C}omod$ be two B-comodules. Then $M \otimes N$ is a B-comodule by the coaction $\delta_{M \otimes N}(m \otimes n) = \sum m_{(1)}n_{(1)} \otimes m_{(M)} \otimes n_{(N)}$.*

If $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$ are homomorphisms of B-comodules in $B\text{-}\mathcal{C}omod$ then $f \otimes g : M \otimes N \longrightarrow M' \otimes N'$ is a homomorphism of B-comodules.

Proof. The coaction on $M \otimes N$ may also be described by $(\nabla_B \otimes 1_M \otimes 1_N)(1_B \otimes \tau \otimes 1_N)(\delta_M \otimes \delta_N) : M \otimes N \longrightarrow B \otimes M \otimes B \otimes N \longrightarrow B \otimes B \otimes M \otimes N \longrightarrow B \otimes M \otimes N$. Although a diagrammatic proof of the coassociativity of the coaction and the law of the counit is quite involved it allows generalization so we give it here.

Consider the next diagram.

Square (1) commutes since M and N are comodules.

Squares (2) and (3) commute since $\tau : M \otimes N \longrightarrow N \otimes M$ for $\mathbb{K}$-modules M and N is a natural transformation.

Square (4) represents an interesting property of τ namely

$$(1 \otimes 1 \otimes \tau)(\tau_{B \otimes M, B} \otimes 1) = (1 \otimes 1 \otimes \tau)(\tau \otimes 1 \otimes 1)(1 \otimes \tau \otimes 1) =$$
$$(\tau \otimes 1 \otimes 1)(1 \otimes 1 \otimes \tau)(1 \otimes \tau \otimes 1) = (\tau \otimes 1 \otimes 1)(1 \otimes \tau_{M, B \otimes B})$$

that uses the fact that $(1\otimes g)(f\otimes 1) = (f\otimes 1)(1\otimes g)$ holds and that $\tau_{B\otimes M,B} = (\tau\otimes 1)(1\otimes\tau)$ and $\tau_{M,B\otimes B} = (1\otimes\tau)(\tau\otimes 1)$.

Square (5) and (6) commute by the properties of the tensor product.

Square (7) commutes since B is a bialgebra.

$$
\begin{tikzcd}
M\otimes N \arrow[r, "\delta\otimes\delta"] \arrow[d, "\delta\otimes\delta"] \arrow[dr, phantom, "(1)"] & B\otimes M\otimes B\otimes N \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "\Delta\otimes 1\otimes\Delta\otimes 1"] \arrow[dr, phantom, "(2)"] & B\otimes B\otimes M\otimes N \arrow[r, "\nabla\otimes 1\otimes 1"] \arrow[d, "\Delta\otimes\Delta\otimes 1\otimes 1"] & B\otimes M\otimes N \arrow[ddd, "\Delta\otimes 1\otimes 1"] \\
B\otimes M\otimes B\otimes N \arrow[r, "1\otimes\delta\otimes 1\otimes\delta"] \arrow[d, "1\otimes\tau\otimes 1"] \arrow[dr, phantom, "(3)"] & B\otimes B\otimes M\otimes B\otimes B\otimes N \arrow[r, "1^2\otimes\tau_{M,B\otimes B}\otimes 1"] \arrow[d, "1\otimes\tau_{B\otimes M,B}\otimes 1\otimes 1"] \arrow[dr, phantom, "(4)"] & B\otimes B\otimes B\otimes B\otimes M\otimes N \arrow[d, "1\otimes\tau\otimes 1\otimes 1\otimes 1"] \arrow[r, phantom, "(7)"] & \\
B\otimes B\otimes M\otimes N \arrow[r, "1\otimes 1\otimes\delta\otimes\delta"] \arrow[d, "\nabla\otimes 1\otimes 1"] \arrow[dr, phantom, "(5)"] & B\otimes B\otimes B\otimes M\otimes B\otimes N \arrow[r, "1\otimes 1\otimes 1\otimes\tau\otimes 1"] \arrow[d, "\nabla\otimes 1\otimes 1\otimes 1\otimes 1"] \arrow[dr, phantom, "(6)"] & B\otimes B\otimes B\otimes B\otimes M\otimes N \arrow[d, "\nabla\otimes 1\otimes 1\otimes 1\otimes 1"] & \\
B\otimes M\otimes N \arrow[r, "1\otimes\delta\otimes\delta"] & B\otimes B\otimes M\otimes B\otimes N \arrow[r, "1\otimes 1\otimes\tau\otimes 1"] & B\otimes B\otimes B\otimes M\otimes N \arrow[r, "1\otimes\nabla\otimes 1\otimes 1"] & B\otimes B\otimes M\otimes N
\end{tikzcd}
$$

The law of the counit is

$$
\begin{tikzcd}
M\otimes N \arrow[r, "\delta\otimes\delta"] \arrow[dr, "1"] & B\otimes M\otimes B\otimes N \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "\varepsilon\otimes 1\otimes\varepsilon\otimes 1"] & B\otimes B\otimes M\otimes N \arrow[r, "\nabla\otimes 1\otimes 1"] \arrow[d, "\varepsilon\otimes\varepsilon\otimes 1\otimes 1"] & B\otimes M\otimes N \arrow[d, "\varepsilon\otimes 1\otimes 1"] \\
& M\otimes N \arrow[r, "1"] & M\otimes N \arrow[r, "1"] & M\otimes N
\end{tikzcd}
$$

where the last square commutes since ε is a homomorphism of algebras.

Now let f and g be homomorphisms of B-comodules. Then the diagram

$$
\begin{tikzcd}
M\otimes N \arrow[r, "\delta\otimes\delta"] \arrow[d, "f\otimes g"] & B\otimes M\otimes B\otimes N \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "1\otimes f\otimes 1\otimes g"] & B\otimes B\otimes M\otimes N \arrow[r, "\nabla\otimes 1\otimes 1"] \arrow[d, "1\otimes 1\otimes f\otimes g"] & B\otimes M\otimes N \arrow[d, "1\otimes f\otimes g"] \\
M'\otimes N' \arrow[r, "\delta\otimes\delta"] & B\otimes M'\otimes B\otimes N' \arrow[r, "1\otimes\tau\otimes 1"] & B\otimes B\otimes M'\otimes N' \arrow[r, "\nabla\otimes 1\otimes 1"] & B\otimes M'\otimes N'
\end{tikzcd}
$$

commutes. Thus $f\otimes g$ is a homomorphism of B-comodules. $\square$

Corollary 3.1.5. *Let B be a bialgebra. Then $\otimes : B\text{-}\mathcal{C}omod \times B\text{-}\mathcal{C}omod \longrightarrow B\text{-}\mathcal{C}omod$ with $\otimes(M,N) = M\otimes N$ and $\otimes(f,g) = f\otimes g$ is a functor.*

Proposition 3.1.6. *Let B be a bialgebra. Then the tensor product $\otimes : B\text{-}\mathcal{M}od \times B\text{-}\mathcal{M}od \longrightarrow B\text{-}\mathcal{M}od$ satisfies the following properties:*

(1) *The associativity isomorphism* $\alpha : (M_1 \otimes M_2) \otimes M_3 \longrightarrow M_1 \otimes (M_2 \otimes M_3)$ *with* $\alpha((m \otimes n) \otimes p) = m \otimes (n \otimes p)$ *is a natural transformation from the functor* $\otimes \circ (\otimes \times \mathrm{Id})$ *to the functor* $\otimes \circ (\mathrm{Id} \times \otimes)$ *in the variables* M_1, M_2, *and* M_3 *in* B-$\mathcal{M}od$.
(2) *The counit isomorphisms* $\lambda : \mathbb{K} \otimes M \longrightarrow M$ *with* $\lambda(\kappa \otimes m) = \kappa m$ *and* $\rho : M \otimes \mathbb{K} \longrightarrow M$ *with* $\rho(m \otimes \kappa) = \kappa m$ *are natural transformations in the variable* M *in* B-$\mathcal{M}od$ *from the functor* $\mathbb{K} \otimes$ - *resp.* - $\otimes \mathbb{K}$ *to the identity functor* Id.
(3) *The following diagrams of natural transformations are commutative*

$$\begin{tikzcd}
((M_1 \otimes M_2) \otimes M_3) \otimes M_4 \arrow[r, "\alpha(M_1,M_2,M_3)\otimes 1"] \arrow[d, "\alpha(M_1 \otimes M_2, M_3, M_4)"] & (M_1 \otimes (M_2 \otimes M_3)) \otimes M_4 \arrow[r, "\alpha(M_1, M_2 \otimes M_3, M_4)"] & M_1 \otimes ((M_2 \otimes M_3) \otimes M_4) \arrow[d, "1 \otimes \alpha(M_2,M_3,M_4)"'] \\
(M_1 \otimes M_2) \otimes (M_3 \otimes M_4) \arrow[rr, "\alpha(M_1,M_2,M_3 \otimes M_4)"] & & M_1 \otimes (M_2 \otimes (M_3 \otimes M_4))
\end{tikzcd}$$

$$\begin{tikzcd}
(M_1 \otimes \mathbb{K}) \otimes M_2 \arrow[rr, "\alpha(M_1,\mathbb{K},M_2)"] \arrow[dr, "\rho(M_1)\otimes 1"'] & & M_1 \otimes (\mathbb{K} \otimes M_2) \arrow[dl, "1 \otimes \lambda(M_2)"] \\
& M_1 \otimes M_2 &
\end{tikzcd}$$

Proof. The homomorphisms α, λ, and ρ are already defined in the category $\mathbb{K}$-$\mathcal{M}od$ and satisfy the claimed properties. So we have to show, that these are homomorphisms in B-$\mathcal{M}od$ and that $\mathbb{K}$ is a B-module. $\mathbb{K}$ is a B-module by $\varepsilon \otimes 1_{\mathbb{K}} : B \otimes \mathbb{K} \longrightarrow \mathbb{K}$. The easy verification uses the coassociativity and the counital property of B. $\square$

Similarly we get

Proposition 3.1.7. *Let* B *be a bialgebra. Then the tensor product*

$$\otimes : B\text{-}\mathcal{C}omod \times B\text{-}\mathcal{C}omod \longrightarrow B\text{-}\mathcal{C}omod$$

satisfies the following properties:

(1) *The associativity isomorphism* $\alpha : (M_1 \otimes M_2) \otimes M_3 \longrightarrow M_1 \otimes (M_2 \otimes M_3)$ *with* $\alpha((m \otimes n) \otimes p) = m \otimes (n \otimes p)$ *is a natural transformation from the functor* $\otimes \circ (\otimes \times \mathrm{Id})$ *to the functor* $\otimes \circ (\mathrm{Id} \times \otimes)$ *in the variables* M_1, M_2, *and* M_3 *in* B-$\mathcal{C}omod$.
(2) *The counit isomorphisms* $\lambda : \mathbb{K} \otimes M \longrightarrow M$ *with* $\lambda(\kappa \otimes m) = \kappa m$ *and* $\rho : M \otimes \mathbb{K} \longrightarrow M$ *with* $\rho(m \otimes \kappa) = \kappa m$ *are natural transformations in the variable* M *in* B-$\mathcal{C}omod$ *from the functor* $\mathbb{K} \otimes$ - *resp.* - $\otimes \mathbb{K}$ *to the identity functor* Id.

(3) *The following diagrams of natural transformations are commutative*

$$\begin{tikzcd}
((M_1 \otimes M_2) \otimes M_3) \otimes M_4 \arrow[r, "\alpha(M_1,M_2,M_3)\otimes 1"] \arrow[d, "\alpha(M_1\otimes M_2,M_3,M_4)"] & (M_1 \otimes (M_2 \otimes M_3)) \otimes M_4 \arrow[r, "\alpha(M_1,M_2\otimes M_3,M_4)"] & M_1 \otimes ((M_2 \otimes M_3) \otimes M_4) \arrow[d, "1\otimes\alpha(M_2,M_3,M_4)"] \\
(M_1 \otimes M_2) \otimes (M_3 \otimes M_4) \arrow[rr, "\alpha(M_1,M_2,M_3\otimes M_4)"] & & M_1 \otimes (M_2 \otimes (M_3 \otimes M_4))
\end{tikzcd}$$

$$\begin{tikzcd}
(M_1 \otimes \mathbb{K}) \otimes M_2 \arrow[rr, "\alpha(M_1,\mathbb{K},M_2)"] \arrow[dr, "\rho(M_1)\otimes 1"'] & & M_1 \otimes (\mathbb{K} \otimes M_2) \arrow[dl, "1\otimes\lambda(M_2)"] \\
& M_1 \otimes M_2 &
\end{tikzcd}$$

Remark 3.1.8. We now get some simple properties of the underlying functors $\mathcal{U} : B\text{-}\mathcal{M}od \to \mathbb{K}\text{-}\mathcal{M}od$ resp. $\mathcal{U} : B\text{-}\mathcal{C}omod \to \mathbb{K}\text{-}\mathcal{M}od$ that are easily verified.

$$\begin{aligned}
&\mathcal{U}(M \otimes N) = \mathcal{U}(M) \otimes \mathcal{U}(N), \\
&\mathcal{U}(f \otimes g) = f \otimes g, \\
&\mathcal{U}(\mathbb{K}) = \mathbb{K}, \\
&\mathcal{U}(\alpha) = \alpha,\ \mathcal{U}(\lambda) = \lambda,\ \mathcal{U}(\rho) = \rho.
\end{aligned}$$

Problem 3.1.17. We have seen that in representation theory and in corepresentation theory of quantum groups such as $\mathbb{K}G$, $U(\mathfrak{g})$, $SL_q(2)$, $U_q(sl(2))$ the ordinary tensor product (in $\mathbb{K}$-$\mathcal{M}od$) of two (co-)reprensentations is in a canonical way again a (co-)reprensentation. For two H-modules M and N describe the module structure on $M \otimes N$ if

(1) $H = \mathbb{K}G$: $g(m \otimes n) = \ldots$ for $g \in G$;
(2) $H = U(\mathfrak{g})$: $g(m \otimes n) = \ldots$ for $g \in \mathfrak{g}$;
(3) $H = U_q(sl(2))$:
  (a) $E(m \otimes n) = \ldots$,
  (b) $F(m \otimes n) = \ldots$,
  (c) $K(m \otimes n) = \ldots$
  for the elements $E, F, K \in U_q(sl(2))$.

For two $\mathbb{K}G$-modules M and N the structure is $g(m \otimes n) = gm \otimes gn$ for $g \in G$. For $U(\mathfrak{g})$-modules it is $g(m \otimes n) = gm \otimes n + m \otimes gn$ for $g \in \mathfrak{g}$. For $U_q(sl(2))$-modules it is $E(m \otimes n) = m \otimes En + Em \otimes Kn$, $F(m \otimes n) = K^{-1}m \otimes Fn + Fm \otimes n$, $K(m \otimes n) = Km \otimes Kn$.

Remark 3.1.9. Let A and B be algebras over a commutative ring $\mathbb{K}$. Let $f : A \to B$ be a homomorphism of algebras. Then we have a functor $\mathcal{U}_f : B\text{-}\mathcal{M}od \to A\text{-}\mathcal{M}od$ with $\mathcal{U}_f({}_BM) = {}_AM$ and $\mathcal{U}_f(g) = g$ where $am := f(a)m$ for $a \in A$ and $m \in M$. The functor $\mathcal{U}_f$ is also called *forgetful* or *underlying functor*.

The action of A on a B-module M can also be seen as the homomorphism $A \to B \to \mathrm{End}(M)$.

We denote the underlying functors previously discussed by

$$\mathcal{U}_A : A\text{-}\mathcal{M}od \longrightarrow \mathbb{K}\text{-}\mathcal{M}od \text{ resp. } \mathcal{U}_B : B\text{-}\mathcal{M}od \longrightarrow \mathbb{K}\text{-}\mathcal{M}od.$$

Proposition 3.1.10. *Let $f : B \longrightarrow C$ be a homomorphism of bialgebras. Then $\mathcal{U}_f$ satisfies the following properties:*

$$\begin{aligned}
&\mathcal{U}_f(M \otimes N) = \mathcal{U}_f(M) \otimes \mathcal{U}_f(N),\\
&\mathcal{U}_f(g \otimes h) = g \otimes h,\\
&\mathcal{U}_f(\mathbb{K}) = \mathbb{K},\\
&\mathcal{U}_f(\alpha) = \alpha,\ \mathcal{U}_f(\lambda) = \lambda,\ \mathcal{U}_f(\rho) = \rho,\\
&\mathcal{U}_B\mathcal{U}_f(M) = \mathcal{U}_C(M),\\
&\mathcal{U}_B\mathcal{U}_f(g) = \mathcal{U}_C(g).
\end{aligned}$$

PROOF. This is clear since the underlying $\mathbb{K}$-modules and the $\mathbb{K}$-linear maps stay unchanged. The only thing to check is that U_f generates the correct B-module structure on the tensor product. For $U_f(M \otimes N) = M \otimes N$ we have $b(m \otimes n) = f(b)(m \otimes n) = \sum f(b)_{(1)}m \otimes f(b)_{(2)}n = \sum f(b_{(1)})m \otimes f(b_{(2)})n = \sum b_{(1)}m \otimes b_{(2)}n$. □

Remark 3.1.11. Let C and D be coalgebras over a commutative ring $\mathbb{K}$. Let $f : C \longrightarrow D$ be a homomorphism of coalgebras. Then we have a functor $\mathcal{U}_f : C\text{-}\mathcal{C}omod \longrightarrow D\text{-}\mathcal{C}omod$ with $\mathcal{U}_f({}^CM) = {}^DM$ and $\mathcal{U}_f(g) = g$ where $\delta_D = (f \otimes 1)\delta_C : M \longrightarrow C \otimes M \longrightarrow D \otimes M$. Again the functor $\mathcal{U}_f$ is called *forgetful* or *underlying functor*.
We denote the underlying functors previously discussed by

$$\mathcal{U}_C : C\text{-}\mathcal{C}omod \longrightarrow \mathbb{K}\text{-}\mathcal{M}od \text{ resp. } \mathcal{U}_D : D\text{-}\mathcal{C}omod \longrightarrow \mathbb{K}\text{-}\mathcal{M}od.$$

Proposition 3.1.12. *Let $f : B \longrightarrow C$ be a homomorphism of bialgebras. Then $\mathcal{U}_f : C\text{-}\mathcal{C}omod \longrightarrow D\text{-}\mathcal{C}omod$ satisfies the following properties:*

$$\begin{aligned}
&\mathcal{U}_f(M \otimes N) = \mathcal{U}_f(M) \otimes \mathcal{U}_f(N),\\
&\mathcal{U}_f(g \otimes h) = g \otimes h,\\
&\mathcal{U}_f(\mathbb{K}) = \mathbb{K},\\
&\mathcal{U}_f(\alpha) = \alpha,\ \mathcal{U}_f(\lambda) = \lambda,\ \mathcal{U}_f(\rho) = \rho,\\
&\mathcal{U}_C\mathcal{U}_f(M) = \mathcal{U}_B(M),\\
&\mathcal{U}_C\mathcal{U}_f(g) = \mathcal{U}_B(g).
\end{aligned}$$

PROOF. We leave the proof to the reader. □

Proposition 3.1.13. *Let H be a Hopf algebra. Let M and N be be H-modules. Then* $\mathrm{Hom}(M, N)$*, the set $\mathbb{K}$-linear maps from M to N, becomes an H-module by $(hf)(m) = \sum h_{(1)}f(S(h_{(2)}m)$. This structure makes*

$$\mathrm{Hom} : H\text{-}\mathcal{M}od \times H\text{-}\mathcal{M}od \longrightarrow H\text{-}\mathcal{M}od$$

a functor contravariant in the first variable and covariant in the second variable.

PROOF. The main part to be proved is that the action $H \otimes \mathrm{Hom}(M,N) \longrightarrow \mathrm{Hom}(M,N)$ satisfies the associativity law. Let $f \in \mathrm{Hom}(M,N)$, $h,k \in H$, and $m \in M$. Then $((hk)f)(m) = \sum (hk)_{(1)} f(S((hk)_{(2)}) = \sum h_{(1)}k_{(1)}f(S(k_{(2)})S(h_{(2)})m) = \sum h_{(1)}(kf)(S(h_{(2)})m) = (h(kf))(m)$.

We leave the proof of the other properties, in particular the functorial properties, to the reader. □

Corollary 3.1.14. *Let M be an H-module. Then the dual $\mathbb{K}$-module $M^* = \mathrm{Hom}(M,\mathbb{K})$ becomes an H-module by $(hf)(m) = f(S(h)m)$.*

PROOF. The space $\mathbb{K}$ is an H-module via $\varepsilon : H \longrightarrow \mathbb{K}$. Hence we have $(hf)(m) = \sum h_{(1)} f(S(h_{(2)}m) = \sum \varepsilon(h_{(1)}) f(S(h_{(2)}m) = f(S(h)m)$. □

2. Monoidal Categories

For our further investigations we need a generalized version of the tensor product that we are going to introduce in this section. This will give us the possibility to study more general versions of the notion of algebras and representations.

Definition 3.2.1. A *monoidal category* (or *tensor category*) consists of
a category $\mathcal{C}$,
a covariant functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, called the *tensor product*,
an object $I \in \mathcal{C}$, called the *unit*,
natural isomorphisms

$$\begin{aligned} &\alpha(A,B,C) : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C), \\ &\lambda(A) : I \otimes A \longrightarrow A, \\ &\rho(A) : A \otimes I \longrightarrow A, \end{aligned}$$

called *associativity, left unit* and *right unit*, such that the following diagrams commute:

$$\begin{tikzcd} ((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha(A,B,C)\otimes 1"] \arrow[d, "\alpha(A\otimes B,C,D)"] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha(A,B\otimes C,D)"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "1\otimes\alpha(B,C,D)"'] \\ (A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha(A,B,C\otimes D)"] & & A \otimes (B \otimes (C \otimes D)) \end{tikzcd}$$

$$\begin{tikzcd} (A \otimes I) \otimes B \arrow[rr, "\alpha(A,I,B)"] \arrow[dr, "\rho(A)\otimes 1"'] & & A \otimes (I \otimes B) \arrow[dl, "1\otimes\lambda(B)"] \\ & A \otimes B & \end{tikzcd}$$

These diagrams are called *coherence diagrams* or *constraints.*

A monoidal category is called a *strict monoidal category*, if the morphisms α, λ, ρ are the identity morphisms.

Remark 3.2.2. We define $A_1 \otimes \ldots \otimes A_n := (\ldots(A_1 \otimes A_2) \otimes \ldots) \otimes A_n$.

There is an important theorem of S. MacLane that says that all diagrams whose morphisms are constructed by using copies of α, λ, ρ, identities, inverses, tensor products and compositions of such commute. We will only prove this theorem for some very special cases (3.2.4). It implies that each monoidal category can be replaced by (is monoidally equivalent to) a strict monoidal category. That means that we may omit in diagrams the morphisms α, λ, ρ or replace them by identities. In particular there is only one automorphism of $A_1 \otimes \ldots \otimes A_n$ formed by coherence morphisms namely the identity.

Remark 3.2.3. For each monoidal category $\mathcal{C}$ we can construct the monoidal category $\mathcal{C}^{symm}$ symmetric to $\mathcal{C}$ that coincides with $\mathcal{C}$ as a category and has tensor product $A \boxtimes B := B \otimes A$ and the coherence morphisms

$$\begin{aligned}
&\alpha(C,B,A)^{-1} : (A \boxtimes B) \boxtimes C \longrightarrow A \boxtimes (B \boxtimes C),\\
&\rho(A) : I \boxtimes A \longrightarrow A,\\
&\lambda(A) : A \boxtimes I \longrightarrow A.
\end{aligned}$$

Then the coherence diagrams are commutative again, so that $\mathcal{C}^{symm}$ is a monoidal category.

Lemma 3.2.4. *Let $\mathcal{C}$ be a monoidal category. Then the following diagrams commute*

$$\begin{tikzcd}
(I \otimes A) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\lambda(A)\otimes 1_B"'] & & I \otimes (A \otimes B) \arrow[dl, "\lambda(A\otimes B)"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes B) \otimes I \arrow[rr, "\alpha"] \arrow[dr, "\rho(A\otimes B)"'] & & A \otimes (B \otimes I) \arrow[dl, "1_A\otimes\rho(B)"] \\
& A \otimes B &
\end{tikzcd}$$

and we have $\lambda(I) = \rho(I)$.

Proof. First we observe that the identity functor $\mathrm{Id}_{\mathcal{C}}$ and the functor $I \otimes$ - are isomorphic by the natural isomorphism λ. Thus we have $I \otimes f = I \otimes g \Longrightarrow f = g$. In the following diagram

$$\begin{tikzcd}
((I \otimes I) \otimes A) \otimes B \arrow[rr, "\alpha\otimes 1"] \arrow[dr, "(\rho\otimes 1)\otimes 1"] \arrow[ddd, "\alpha"'] & & (I \otimes (I \otimes A)) \otimes B \arrow[r, "\alpha"] \arrow[dl, "(1\otimes\lambda)\otimes 1"'] & I \otimes ((I \otimes A) \otimes B) \arrow[dl, "1\otimes(\lambda\otimes 1)"'] \arrow[ddd, "1\otimes\alpha"] \\
& (I \otimes A) \otimes B \arrow[r, "\alpha"] \arrow[d, "\alpha"] & I \otimes (A \otimes B) \arrow[d, "1"'] & \\
& I \otimes (A \otimes B) \arrow[r, "1"] & I \otimes (A \otimes B) \arrow[dr, "1\otimes\lambda"'] & \\
(I \otimes I) \otimes (A \otimes B) \arrow[ur, "\rho\otimes(1\otimes 1)"] \arrow[rrr, "\alpha"] & & & I \otimes (I \otimes (A \otimes B))
\end{tikzcd}$$

all subdiagrams commute except for the right hand trapezoid. Since all morphisms are isomorphisms the right hand trapezoid must commute also. Hence the first diagram of the Lemma commutes.

In a similar way one shows the commutativity of the second diagram.

Furthermore the following diagram commutes

$$\begin{tikzcd}
I \otimes (I \otimes I) \arrow[dr, "1\otimes\rho"'] & & (I \otimes I) \otimes I \arrow[ll, "\alpha"'] \arrow[rr, "\alpha"] \arrow[dl, "\rho"] \arrow[dr, "\rho\otimes 1"'] & & I \otimes (I \otimes I) \arrow[dl, "1\otimes\lambda"] \\
& I \otimes I \arrow[dr, "\rho"'] & & I \otimes I \arrow[dl, "\rho"] & \\
& & I & &
\end{tikzcd}$$

Here the left hand triangle commutes by the previous property. The commutativity of the right hand diagram is given by the axiom. The lower square commutes since ρ is a natural transformation. In particular $\rho(1 \otimes \rho) = \rho(1 \otimes \lambda)$. Since ρ is an isomorphism and $I \otimes \text{-} \cong \mathrm{Id}_{\mathcal{C}}$ we get $\rho = \lambda$. $\square$

Problem 3.2.18. For morphisms $f : I \longrightarrow M$ and $g : I \longrightarrow N$ in a monoidal category we define $(f \otimes 1 : N \longrightarrow M \otimes N) := (f \otimes 1_I)\rho(I)^{-1}$ and $(1 \otimes g : M \longrightarrow M \otimes N) := (1 \otimes g)\lambda(I)^{-1}$. Show that the diagram

$$\begin{tikzcd}
I \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[d, "1\otimes g"] \\
N \arrow[r, "f\otimes 1"'] & M \otimes N
\end{tikzcd}$$

commutes.

We continue with a series of examples of monoidal categories.

Example 3.2.5. (1) Let R be an arbitrary ring. The category $R\text{-}\mathcal{M}od\text{-}R$ of R-R-bimodules with the tensor product $M \otimes_R N$ is a monoidal category. In particular the $\mathbb{K}$-modules form a monoidal category. This is our most important example of a monoidal category.

(2) Let B be a bialgebra and $B\text{-}\mathcal{M}od$ be the category of left B-modules. We define the structure of a B-module on the tensor product $M \otimes N = M \otimes_{\mathbb{K}} N$ by

$$B \otimes M \otimes N \stackrel{\Delta\otimes 1_M \otimes 1_N}{\longrightarrow} B \otimes B \otimes M \otimes N \stackrel{1_B \otimes \tau \otimes 1_N}{\longrightarrow} B \otimes M \otimes B \otimes N \stackrel{\rho_M \otimes \rho_N}{\longrightarrow} M \otimes N$$

as in the previous section. So $B\text{-}\mathcal{M}od$ is a monoidal category by 3.1.6

(3) Let B be a bialgebra and $B\text{-}\mathcal{C}omod$ be the category of B-comodules. The tensor product $M \otimes N = M \otimes_{\mathbb{K}} N$ carries the structure of a B-comodule by

$$M \otimes N \stackrel{\delta_M \otimes \delta_N}{\longrightarrow} B \otimes M \otimes B \otimes N \stackrel{1_B \otimes \tau \otimes 1_N}{\longrightarrow} B \otimes B \otimes M \otimes N \stackrel{\nabla \otimes 1_M \otimes 1_N}{\longrightarrow} B \otimes M \otimes N.$$

as in the previous section. So B-$\mathcal{C}omod$ is a monoidal category by 3.1.7

Definition 3.2.6. Let G be a set. The *category of G-families of vector spaces* $\mathcal{M}^G = \prod_{g \in G} \mathcal{V}ec$ has families of vector spaces $(V_g | g \in G)$ as objects and families of linear maps $(f_g : V_g \longrightarrow W_g | g \in G)$ as morphisms. The composition is $(f_g | g \in G) \circ (h_g | g \in G) = (f_g \circ h_g | g \in G)$.

Lemma 3.2.7. *Let G be a monoid. Then $\mathcal{M}^G$ is a monoidal category with the tensor product*

$$(V_g | g \in G) \otimes (W_g | g \in G) := (\bigoplus_{h,k \in G, hk=g} V_h \otimes W_k | g \in G).$$

PROOF. This is an easy exercise. $\square$

Problem 3.2.19. Let G be a monoid. Show that $\mathcal{M}^G$ is a monoidal category. Where do unit and associativity laws of G enter the proof?

Definition 3.2.8. Let G be a set. A vector space V together with a family of subspaces $(V_g \subseteq V | g \in G)$ is called *G-graded*, if $V = \oplus_{g \in G} V_g$ holds.

Let $(V, (V_g | g \in G))$ and $(W, (W_g | g \in G))$ be G-graded vector spaces. A linear map $f : V \longrightarrow W$ is called *G-graded*, if $f(V_g) \subseteq W_g$ for all $g \in G$.

The G-graded vector spaces and G-graded linear maps form the category $\mathcal{M}^{[G]}$ of *G-graded vector spaces*.

Lemma 3.2.9. *Let G be a monoid. Then $\mathcal{M}^{[G]}$ is a monoidal category with the tensor product $V \otimes W$, where the subspaces $(V \otimes W)_g$ are defined by*

$$(V \otimes W)_g := \sum_{h,k \in G, hk=g} V_h \otimes W_k.$$

PROOF. This is an easy exercise. $\square$

Problem 3.2.20. Let G be a monoid. Show that $\mathcal{M}^{[G]}$ is a monoidal category.

Definition 3.2.10. (1) A *chain complex* of $\mathbb{K}$-modules

$$M = (\ldots \xrightarrow{\partial_3} M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0)$$

consists of a family of $\mathbb{K}$-modules M_i and a family of homomorphisms $\partial_n : M_n \longrightarrow M_{n-1}$ with $\partial_{n-1}\partial_n = 0$. This chain complex is indexed by the monoid $\mathbb{N}_0$. One may also consider more general chain complexes indexed by an arbitrary cyclic monoid. Chain complexes indexed by $\mathbb{N}_0 \times \mathbb{N}_0$ are called double complexes. So much more general chain complexes may be considered. We restrict ourselves to chain complexes over $\mathbb{N}_0$.

Let M and N be chain complexes. A *homomorphism of chain complexes* $f : M \longrightarrow N$ consists of a family of homomorphisms of $\mathbb{K}$-modules $f_n : M_n \longrightarrow N_n$ such that $f_n \partial_{n+1} = \partial_{n+1} f_{n+1}$ for all $n \in \mathbb{N}_0$.

The chain complexes with these homomorphisms form the category of chain complexes $\mathbb{K}$-$\mathcal{C}omp$.

If M and N are chain complexes then we form a new chain complex $M \otimes N$ with $(M \otimes N)_n := \oplus_{i=0}^{n} M_i \otimes N_{n-i}$ and $\partial : (M \otimes N)_n \longrightarrow (M \otimes N)_{n-1}$ given by $\partial(m_i \otimes n_{n-i}) := (-1)^i \partial_M(m_i) \otimes n_{n-i} + m_i \otimes \partial(n_{n-i})$. This is often called the total complex associated with the double complex of the tensor product of M and N. Then it is easily checked that $\mathbb{K}$-$\mathcal{C}omp$ is a monoidal category with this tensor product.

(2) A cochain complex has the form

$$M = (M_0 \xrightarrow{\partial_0} M_1 \xrightarrow{\partial_1} M_2 \xrightarrow{\partial_2} \dots)$$

with $\partial_{i+1}\partial_i = 0$. The cochain complexes form a monoidal category of cochain complexes $\mathbb{K}$-$\mathcal{C}ocomp$.

Problem 3.2.21. Show that the cochain complexes form a monoidal category $\mathbb{K}$-$\mathcal{C}ocomp$.

Definition 3.2.11. Let $(\mathcal{C}, \otimes)$ and $(\mathcal{D}, \otimes)$ be monoidal categories. A functor

$$\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$$

together with a natural transformation

$$\xi(M, N) : \mathcal{F}(M) \otimes \mathcal{F}(N) \longrightarrow \mathcal{F}(M \otimes N)$$

and a morphism

$$\xi_0 : I_{\mathcal{D}} \longrightarrow \mathcal{F}(I_{\mathcal{C}})$$

is called *weakly monoidal* if the following diagrams commute

$$\begin{tikzcd}
(\mathcal{F}(M) \otimes \mathcal{F}(N)) \otimes \mathcal{F}(P) \arrow[r, "\xi\otimes 1"] \arrow[d, "\alpha"] & \mathcal{F}(M \otimes N) \otimes \mathcal{F}(P) \arrow[r, "\xi"] & \mathcal{F}((M \otimes N) \otimes P) \arrow[d, "\mathcal{F}(\alpha)"] \\
\mathcal{F}(M) \otimes (\mathcal{F}(N) \otimes \mathcal{F}(P)) \arrow[r, "1\otimes\xi"] & \mathcal{F}(M) \otimes \mathcal{F}(N \otimes P) \arrow[r, "\xi"] & \mathcal{F}(M \otimes (N \otimes P))
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes \mathcal{F}(M) \arrow[r, "\xi_0\otimes 1"] \arrow[dr, "\lambda"'] & \mathcal{F}(I) \otimes \mathcal{F}(M) \arrow[r, "\xi"] & \mathcal{F}(I \otimes M) \arrow[dl, "\mathcal{F}(\lambda)"] \\
& \mathcal{F}(M) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(M) \otimes I \arrow[r, "1\otimes\xi_0"] \arrow[dr, "\rho"'] & \mathcal{F}(M) \otimes \mathcal{F}(I) \arrow[r, "\xi"] & \mathcal{F}(M \otimes I) \arrow[dl, "\mathcal{F}(\rho)"] \\
& \mathcal{F}(M). &
\end{tikzcd}$$

If, in addition, the morphisms ξ and ξ_0 are isomorphisms then the functor is called a *monoidal functor*. The functor is called a *strict monoidal functor* if ξ and ξ_0 are the identity morphisms.

A natural transformation $\zeta : \mathcal{F} \longrightarrow \mathcal{F}'$ between weakly monoidal functors is called a *monoidal natural transformation* if the diagrams

$$\begin{tikzcd} \mathcal{F}(M) \otimes \mathcal{F}(N) \arrow[r, "\xi"] \arrow[d, "\zeta\otimes\zeta"'] & \mathcal{F}(M \otimes N) \arrow[d, "\zeta"] \\ \mathcal{F}'(M) \otimes \mathcal{F}'(N) \arrow[r, "\xi'"'] & \mathcal{F}'(M \otimes N) \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{F}(I) \arrow[dd, "\zeta"] \\ I \arrow[ur, "\xi_0"] \arrow[dr, "\xi_0'"'] & \\ & \mathcal{F}'(I) \end{tikzcd}$$

commute.

Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A monoidal functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is called a *monoidal equivalence* if there exists a monoidal functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ and monoidal natural isomorphisms $\varphi : \mathcal{GF} \cong \mathrm{Id}_{\mathcal{C}}$ and $\psi : \mathcal{FG} \cong \mathrm{Id}_{\mathcal{D}}$.

Proposition 3.2.12. *Let G be a monoid.*

(1) *The monoidal category $\mathcal{M}^G$ of G-families of vector spaces is monoidally equivalent to the monoidal category $\mathcal{M}^{[G]}$ of G-graded vector spaces.*
(2) *The monoidal category $\mathcal{M}^{[G]}$ of G-graded vector spaces is monoidally equivalent to the monoidal category of $\mathbb{K}G$-comodules $\mathcal{C}omod$-$\mathbb{K}G$.*

Proof. We use the following constructions.

(1) For a G-family $(V_g|g \in G)$ we construct a G-graded vector space $\hat{V} := \oplus_{g\in G} V_g$ (exterior direct sum) with the subspaces $\hat{V}_g := \mathrm{Im}(V_g)$ in the direct sum. Conversely if $(V, (V_g|g \in G))$ is a G-graded vector space then $(V_g|g \in G)$ is a G-family of vector spaces. Similar constructions are used for morphisms. It is easy to see that the categories $\mathcal{M}^G$ and $\mathcal{M}^{[G]}$ are equivalent monoidal categories.

(2) For a G-graded vector space $(V, (V_g|g \in G))$ we construct the $\mathbb{K}G$-comodule V with the structure map $\delta : V \longrightarrow V \otimes \mathbb{K}G$, $\delta(v) := v \otimes g$ for all (homogeneous elements) $v \in V_g$ and for all $g \in G$. Conversely let $(V, \delta : V \longrightarrow V \otimes \mathbb{K}G)$ be a $\mathbb{K}G$-comodule. Then one constructs the vector space V with den graded (homogenous) components $V_g := \{v \in V|\delta(v) = v \otimes g\}$. It is easy to verify, that this is an equivalence of categories.

Since $\mathbb{K}G$ is a bialgebra, the category of $\mathbb{K}G$-comodules is a monoidal category by Proposition 3.1.7. One then checks that under the equivalence between $\mathcal{M}^{[G]}$ and $\mathcal{C}omod$-$\mathbb{K}G$ tensor products are mapped into corresponding tensor products so that we have a monoidal equivalence. $\square$

Problem 3.2.22. Give a detailed proof of Proposition 3.2.12.

Example 3.2.13. The following is a bialgebra $B = \mathbb{K}\langle x, y\rangle/I$, where I is generated by $x^2, xy + yx$. The diagonal is $\Delta(y) = y \otimes y$, $\Delta(x) = x \otimes y + 1 \otimes x$ and the counit is $\epsilon(y) = 1, \epsilon(x) = 0$.

Proposition 3.2.14. *The monoidal category $\mathcal{C}omp$-$\mathbb{K}$ of chain complexes over $\mathbb{K}$ is monoidally equivalent to the category of B-comodules $\mathcal{C}omod$-B with B as in the preceding example.*

Proof. We use the following construction. A chain complex M is mapped to the B-comodule $M = \oplus_{i\in\mathbb{N}} M_i$ with the structure map $\delta : M \longrightarrow M \otimes B$, $\delta(m) := \sum m \otimes y^i + \partial_i(m) \otimes xy^{i-1}$ for all $m \in M_i$ and for all $i \in \mathbb{N}$ resp. $\delta(m) := m \otimes 1$ for $m \in M_0$. Conversely if $M, \delta : M \longrightarrow M \otimes B$ is a B-comodule, then one associates with it the vector spaces $M_i := \{m \in M | \exists m' \in M[\delta(m) = m \otimes y^i + m' \otimes xy^{i-1}\}$ and the linear maps $\partial_i : M_i \longrightarrow M_{i-1}$ with $\partial_i(m) := m'$ for $\delta(m) = m \otimes y^i + m' \otimes xy^{i-1}$. It is checked easily that this is an equivalence of categories. By Problem (41) this is a monoidal equivalence. □

Problem 3.2.23. (1) Give a detailed proof that $\mathcal{C}omp$-$\mathbb{K}$ and $\mathcal{C}omod$-B with B as in the preceding Proposition 3.2.14 are monoidally equivalent categories.

(You may use the following arguments:
Let $M \in \mathcal{C}omod$-B. Define

$$M_i := \{m \in M | \exists m' \in M : \delta(m) = m \otimes y^i + m' \otimes xy^{i-1}\}.$$

Let $m \in M$. Since y^i, xy^i form a basis of B we have $\delta(m) = \sum_i m_i \otimes y^i + \sum_i m'_i \otimes xy^i$. Apply $(\delta \otimes 1)\delta = (1 \otimes \Delta)\delta$ to this equation and compare coefficients to get

$$\delta(m_i) = m_i \otimes y^i + m'_{i-1} \otimes xy^{i-1}, \quad \delta(m'_i) = m'_i \otimes y^i$$

for all $i \in \mathbb{N}_0$ (with $m'_{-1} = 0$). Consequently for each $m_i \in M_i$ there is exactly one $\partial(m_i) = m'_{i-1} \in M$ such that

$$\delta(m_i) = m_i \otimes y^i + \partial(m_i) \otimes xy^{i-1}.$$

Since $\delta(m'_{i-1}) = m'_{i-1} \otimes y^{i-1}$ for all $i \in \mathbb{N}$ we see that $\partial(m_i) \in M_{i-1}$. So we have defined $\partial : M_i \longrightarrow M_{i-1}$. Furthermore we see from this equation that $\partial^2(m_i) = 0$ for all $i \in \mathbb{N}$. Hence we have obtained a chain complex from (M, δ).

If we apply $(1 \otimes \epsilon)\delta(m) = m$ then we get $m = \sum m_i$ with $m_i \in M_i$ hence $M = \oplus_{i\in\mathbb{N}} M_i$. This together with the inverse construction leads to the required equivalence.)

(2) Show that the category $\mathbb{K}$-$\mathcal{C}ocomp$ of cochain complexes is monoidally equivalent to B-$\mathcal{C}omod$, where B is chosen as in Example 3.2.13.

(3) Show that the bialgebra B from Example 3.2.13 is a Hopf algebra.

We can generalize the notions of an algebra or of a coalgebra in the context of a monoidal category. We define

Definition 3.2.15. Let $\mathcal{C}$ be a monoidal category. An *algebra* or a *monoid* in $\mathcal{C}$ consists of an object A together with a multiplication $\nabla : A \otimes A \longrightarrow A$ that is associative

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id} \otimes \nabla"] \arrow[d, "\nabla \otimes 1"'] & A \otimes A \arrow[d, "\nabla"] \\ A \otimes A \arrow[r, "\nabla"'] & A \end{tikzcd}$$

or more precisely

$$\begin{tikzcd} (A \otimes A) \otimes A \arrow[r, "\alpha"] \arrow[d, "\nabla \otimes \mathrm{id}"'] & A \otimes (A \otimes A) \arrow[r, "\mathrm{id} \otimes \nabla"] & A \otimes A \arrow[d, "\nabla"] \\ A \otimes A \arrow[rr, "\nabla"] & & A \end{tikzcd}$$

and has a unit $\eta : I \longrightarrow A$ such that the following diagram commutes

$$\begin{tikzcd} I \otimes A \cong A \cong A \otimes I \arrow[r, "\mathrm{id} \otimes \eta"] \arrow[d, "\eta \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & A \otimes A \arrow[d, "\nabla"] \\ A \otimes A \arrow[r, "\nabla"'] & A. \end{tikzcd}$$

Let A and B be algebras in $\mathcal{C}$. A *morphism of algebras* $f : A \longrightarrow B$ is a morphism in $\mathcal{C}$ such that

$$\begin{tikzcd} A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\nabla_A"'] & B \otimes B \arrow[d, "\nabla_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & I \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_B"] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

commute.

Remark 3.2.16. It is obvious that the composition of two morphisms of algebras is again a morphism of algebras. The identity also is a morphism of algebras. Thus we obtain the category $\mathcal{A}lg(\mathcal{C})$ of algebras in $\mathcal{C}$.

Definition 3.2.17. Let $\mathcal{C}$ be a monoidal category. A *coalgebra* or a *comonoid* in $\mathcal{C}$ consists of an object C together with a comultiplication $\Delta : A \longrightarrow A \otimes A$ that is coassociative

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes \mathrm{id}"] \\ C \otimes C \arrow[r, "\mathrm{id} \otimes \Delta"'] & C \otimes C \otimes C \end{tikzcd}$$

or more precisely

$$\begin{tikzcd}
C \arrow[rr, "\Delta"] \arrow[d, "\Delta"] & & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"] \\
C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"] & (C \otimes C) \otimes C \arrow[r, "\alpha"] & C \otimes (C \otimes C)
\end{tikzcd}$$

and has a counit $\varepsilon : C \to I$ such that the following diagram commutes

$$\begin{tikzcd}
C \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] \arrow[drr, "\mathrm{id}"] & & C \otimes C \arrow[d, "\mathrm{id} \otimes \epsilon"] \\
C \otimes C \arrow[r, "\epsilon \otimes \mathrm{id}"'] & I \otimes C \cong C \cong C \otimes I. &
\end{tikzcd}$$

Let C and D be coalgebras in $\mathcal{C}$. A *morphism of coalgebras* $f : C \to D$ is a morphism in $\mathcal{C}$ such that

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\
C \otimes C \arrow[r, "f \otimes f"'] & D \otimes D
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
C \arrow[rr, "f"] \arrow[dr, "\varepsilon_C"'] & & D \arrow[dl, "\varepsilon_D"] \\
& I &
\end{tikzcd}$$

commute.

Remark 3.2.18. It is obvious that the composition of two morphisms of coalgebras is again a morphism of coalgebras. The identity also is a morphism of coalgebras. Thus we obtain the category $\mathcal{C}oalg(\mathcal{C})$ of coalgebras in $\mathcal{C}$.

Remark 3.2.19. Observe that the notions of bialgebra, Hopf algebra, and comodule algebra cannot be generalized to an arbitrary monoidal category since we need to have an algebra structure on the tensor product of two algebras and this requires us to interchange the middle tensor factors. These interchanges or flips are known under the name symmetry, quasisymmetry or braiding and will be discussed later on.

3. Dual Objects

At the end of the first section in Corollary 3.1.14 we saw that the dual of an H-module can be constructed. We did not show the corresponding result for comodules. In fact such a construction for comodules needs some finiteness conditions. With this restriction the notion of a dual object can be introduced in an arbitrary monoidal category.

Definition 3.3.1. Let $(\mathcal{C}, \otimes)$ be a monoidal category $M \in \mathcal{C}$ be an object. An object $M^* \in \mathcal{C}$ together with a morphism $\mathrm{ev} : M^* \otimes M \longrightarrow I$ is called a *left dual* for M if there exists a morphism $\mathrm{db} : I \longrightarrow M \otimes M^*$ in $\mathcal{C}$ such that

$$(M \xrightarrow{\mathrm{db}\otimes 1} M \otimes M^* \otimes M \xrightarrow{1\otimes \mathrm{ev}} M) = 1_M$$
$$(M^* \xrightarrow{1\otimes \mathrm{db}} M^* \otimes M \otimes M^* \xrightarrow{\mathrm{ev}\otimes 1} M^*) = 1_{M^*}.$$

A monoidal category is called *left rigid* if each object $M \in \mathcal{C}$ has a left dual.

Symmetrically we define: an object ${}^*M \in \mathcal{C}$ together with a morphism $\mathrm{ev} : M \otimes {}^*M \longrightarrow I$ is called a *right dual* for M if there exists a morphism $\mathrm{db} : I \longrightarrow {}^*M \otimes M$ in $\mathcal{C}$ such that

$$(M \xrightarrow{1\otimes \mathrm{db}} M \otimes {}^*M \otimes M \xrightarrow{\mathrm{ev}\otimes 1} M) = 1_M$$
$$({}^*M \xrightarrow{\mathrm{db}\otimes 1} {}^*M \otimes M \otimes {}^*M \xrightarrow{1\otimes \mathrm{ev}} {}^*M) = 1_{{}^*M}.$$

A monoidal category is called *right rigid* if each object $M \in \mathcal{C}$ has a left dual.

The morphisms ev and db are called the *evaluation* respectively the *dual basis*.

Remark 3.3.2. If (M^*, ev) is a left dual for M then obviously (M, ev) is a right dual for M^* and conversely. One uses the same morphism $\mathrm{db} : I \longrightarrow M \otimes M^*$.

Lemma 3.3.3. *Let (M^*, ev) be a left dual for M. Then there is a natural isomorphism*

$$\mathrm{Mor}_{\mathcal{C}}(\text{-} \otimes M, \text{-}) \cong \mathrm{Mor}_{\mathcal{C}}(\text{-}, \text{-} \otimes M^*),$$

i. e. the functor $\text{-} \otimes M : \mathcal{C} \longrightarrow \mathcal{C}$ is left adjoint to the functor $\text{-} \otimes M^* : \mathcal{C} \longrightarrow \mathcal{C}$.

Proof. We give the unit and the counit of the pair of adjoint functors. We define $\Phi(A) := 1_A \otimes \mathrm{db} : A \longrightarrow A \otimes M \otimes M^*$ and $\Psi(B) := 1_B \otimes \mathrm{ev} : B \otimes M^* \otimes M \longrightarrow B$. These are obviously natural transformations. We have commutative diagrams

$$(A \otimes M \xrightarrow[1_A \otimes \mathrm{db} \otimes 1_M]{\mathcal{F}\Phi(A)=} A \otimes M \otimes M^* \otimes M \xrightarrow[1_A \otimes 1_M \otimes \mathrm{ev}]{\Psi\mathcal{F}(A)=} A \otimes M) = 1_{A\otimes M}$$

and

$$(B \otimes M^* \xrightarrow[1_B \otimes 1_{M^*} \otimes \mathrm{db}]{\Phi\mathcal{G}(B)=} B \otimes M^* \otimes M \otimes M^* \xrightarrow[1_B \otimes \mathrm{ev} \otimes 1_{M^*}]{\mathcal{G}\Psi(B)=} B \otimes M^*) = 1_{B\otimes M^*}$$

thus the Lemma has been proved by [Advanced Algebra] Corollary 5.17. $\square$

The converse holds as well. If $\text{-} \otimes M$ is left adjoint to $\text{-} \otimes M^*$ then the unit Φ gives a morphism $\mathrm{db} := \Phi(I) : I \longrightarrow M \otimes M^*$ and the counit Ψ gives a morphism $\mathrm{ev} := \Psi(I) : M^* \otimes M \longrightarrow I$ satisfying the required properties. Thus we have

Corollary 3.3.4. *If $\text{-} \otimes M : \mathcal{C} \longrightarrow \mathcal{C}$ is left adjoint to $\text{-} \otimes M^* : \mathcal{C} \longrightarrow \mathcal{C}$ then M^* is a left dual for M.*

Corollary 3.3.5. (M^*, ev) *is a left dual for M if and only if there is a natural isomorphism*

$$\mathrm{Mor}_{\mathcal{C}}(M^* \otimes \text{-}, \text{-}) \cong \mathrm{Mor}_{\mathcal{C}}(\text{-}, M \otimes \text{-}),$$

i. e. the functor $M^* \otimes \text{-} : \mathcal{C} \longrightarrow \mathcal{C}$ *is left adjoint to the functor* $M \otimes \text{-} : \mathcal{C} \longrightarrow \mathcal{C}$*. The natural isomorphism if given by*

$$(f : M^* \otimes N \longrightarrow P) \mapsto ((1_M \otimes f)(\mathrm{db} \otimes 1_N) : N \longrightarrow M \otimes M^* \otimes N \longrightarrow M \otimes P)$$

and

$$(g : N \longrightarrow M \otimes P) \mapsto ((\mathrm{ev} \otimes 1_P)(1_{M^*} \otimes g) : M^* \otimes N \longrightarrow M^* \otimes M \otimes P \longrightarrow P).$$

PROOF. We have a natural isomorphism

$$\mathrm{Mor}_{\mathcal{C}}(M^* \otimes \text{-}, \text{-}) \cong \mathrm{Mor}_{\mathcal{C}}(\text{-}, M \otimes \text{-}),$$

iff (M, ev) is a right dual for M^* (as a symmetric statement to Lemma 3.3.3) iff (M^*, ev) is a left dual for M. □

Corollary 3.3.6. *If* M *has a left dual then this is unique up to isomorphism.*

PROOF. Let (M^*, ev) and $(M^!, \mathrm{ev}^!)$ be left duals for M. Then the functors $\text{-} \otimes M^*$ and $\text{-} \otimes M^!$ are isomorphic by [Advanced Algebra] Lemma 5.13. In particular we have $M^* \cong I \otimes M^* \cong I \otimes M^! \cong M^!$. If we consider the construction of the isomorphism then we get in particular that $(\mathrm{ev}^! \otimes 1)(1 \otimes \mathrm{db}) : M^! \longrightarrow M^! \otimes M \otimes M^* \longrightarrow M^*$ is the given isomorphism. □

Problem 3.3.24. Let (M^*, ev) be a left dual for M. Then there is a *unique* morphism $\mathrm{db} : I \longrightarrow M \otimes M^*$ satisfying the conditions of Definition 3.3.1. $\mathrm{db}' = (1_M \otimes 1_{M^*})\,\mathrm{db}' = (1_M \otimes \mathrm{ev} \otimes 1_{M^*})(1_M \otimes 1_{M^*} \otimes \mathrm{db})\,\mathrm{db}' = (1_M \otimes \mathrm{ev} \otimes 1_{M^*})(\mathrm{db}' \otimes \mathrm{db}) = (1_M \otimes \mathrm{ev} \otimes 1_{M^*})(\mathrm{db}' \otimes 1_M \otimes 1_{M^*})\,\mathrm{db} = (1_M \otimes 1_{M^*})\,\mathrm{db} = \mathrm{db}$

Definition 3.3.7. Let (M^*, ev_M) and (N^*, ev_N) be left duals for M resp. N. For each morphism $f : M \longrightarrow N$ we define the *transposed morphism*

$$(f^* : N^* \to M^*) := (N^* \xrightarrow{1 \otimes \mathrm{db}_M} N^* \otimes M \otimes M^* \xrightarrow{1 \otimes f \otimes 1} N^* \otimes N \otimes M^* \xrightarrow{\mathrm{ev}_N \otimes 1} M^*).$$

With this definition we get that the left dual is a contravariant functor, since we have

Lemma 3.3.8. *Let* (M^*, ev_M)*,* (N^*, ev_N)*, and* (P^*, ev_P) *be left duals for* M*,* N *and* P *respectively.*

1. We have $(1_M)^* = 1_{M^*}$*.*

2. If $f : M \longrightarrow N$ *and* $g : N \longrightarrow P$ *are given then* $(gf)^* = f^* g^*$ *holds.*

PROOF. 1. $(1_M)^* = (\mathrm{ev} \otimes 1)(1 \otimes 1 \otimes 1)(1 \otimes \mathrm{db}) = 1_{M^*}$.

2. The following diagram commutes

$$\begin{tikzcd}
M \arrow[r, "\mathrm{db}_N \otimes 1"] \arrow[d, "f"] & N \otimes N^* \otimes M \arrow[d, "1\otimes 1\otimes f"] & \\
N \arrow[r, "\mathrm{db}_N \otimes 1"] & N \otimes N^* \otimes N \arrow[r, "1\otimes \mathrm{ev}_N"] \arrow[d, "g\otimes 1\otimes 1"] & N \arrow[d, "g"] \\
& P \otimes N^* \otimes N \arrow[r, "1\otimes \mathrm{ev}_N"] & P
\end{tikzcd}$$

Hence we have $gf = (1 \otimes \mathrm{ev}_N)(g \otimes 1 \otimes f)(\mathrm{db}_N \otimes 1)$. Thus the following diagram commutes

$$\begin{tikzcd}
P^* \arrow[r, "1\otimes \mathrm{db}"] \arrow[d, "1\otimes \mathrm{db}"] & P^* \otimes N \otimes N^* \arrow[r, "1\otimes g\otimes 1"] \arrow[d, "1\otimes \mathrm{db}"] & P^* \otimes P \otimes N^* \arrow[dd, "1\otimes \mathrm{db}"] \arrow[ddr, "\mathrm{ev} \otimes 1"] & \\
P^* \otimes M \otimes M^* \arrow[r, "1\otimes \mathrm{db} \otimes 1"] \arrow[dd, "1\otimes gf\otimes 1"] & P^* \otimes N \otimes N^* \otimes M \otimes M^* \arrow[dr, "1\otimes g\otimes 1"] \arrow[dd, "1\otimes g\otimes 1\otimes f\otimes 1"] & & \\
& & P^* \otimes P \otimes N^* \otimes M \otimes M^* \arrow[dl, "1\otimes f\otimes 1"] \arrow[dd, "\mathrm{ev} \otimes 1"] & N^* \arrow[ddl, "1\otimes \mathrm{db}"] \\
P^* \otimes P \otimes M^* \arrow[d, "\mathrm{ev} \otimes 1"] & P^* \otimes P \otimes N^* \otimes N \otimes M^* \arrow[l, "1\otimes \mathrm{ev} \otimes 1"'] \arrow[d, "\mathrm{ev} \otimes 1"] & & \\
M^* & N^* \otimes N \otimes M^* \arrow[l, "1\otimes \mathrm{ev}"'] & N^* \otimes M \otimes M^* \arrow[l, "1\otimes f\otimes 1"'] &
\end{tikzcd}$$

□

Problem 3.3.25. (1) In the category of $\mathbb{N}$-graded vector spaces determine all objects M that have a left dual.

(2) In the category of chain complexes $\mathbb{K}$-$\mathcal{C}omp$ determine all objects M that have a left dual.
(3) In the category of cochain complexes $\mathbb{K}$-$\mathcal{C}ocomp$ determine all objects M that have a left dual.
(4) Let (M^*, ev) be a left dual for M. Show that $\mathrm{db} : I \longrightarrow M \otimes M^*$ is uniquely determined by M, M^*, and ev. (Uniqueness of the dual basis.)
(5) Let (M^*, ev) be a left dual for M. Show that $\mathrm{ev} : M^* \otimes M \longrightarrow I$ is uniquely determined by M, M^*, and db.

Corollary 3.3.9. *Let M, N have the left duals (M^*, ev_M) and (N^*, ev_N) and let $f : M \longrightarrow N$ be a morphism in $\mathcal{C}$. Then the following diagram commutes*

$$\begin{tikzcd} I \arrow[r, "\mathrm{db}_M"] \arrow[d, "\mathrm{db}_N"'] & M \otimes M^* \arrow[d, "f\otimes 1"] \\ N \otimes N^* \arrow[r, "1\otimes f^*"'] & N \otimes M^*. \end{tikzcd}$$

Proof. The following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\mathrm{db}\otimes 1"] \arrow[d, "f"'] & N \otimes N^* \otimes M \arrow[d, "1\otimes 1\otimes f"] \\ N \arrow[r, "\mathrm{db}\otimes 1"] \arrow[dr, "1"'] & N \otimes N^* \otimes N \arrow[d, "1\otimes \mathrm{ev}"] \\ & N \end{tikzcd}$$

This implies $(f \otimes 1_{M^*}) \mathrm{db}_M = ((1_N \otimes \mathrm{ev}_N)(1_N \otimes 1_{N^*} \otimes f)(\mathrm{db}_N \otimes 1_M) \otimes 1_{M^*}) \mathrm{db}_M = (1_N \otimes \mathrm{ev}_N \otimes 1_{M^*})(1_N \otimes 1_{N^*} \otimes f \otimes 1_{M^*})(\mathrm{db}_N \otimes 1_M \otimes 1_{M^*}) \mathrm{db}_M = (1_N \otimes \mathrm{ev}_N \otimes 1_{M^*})(1_N \otimes 1_{N^*} \otimes f \otimes 1_{M^*})(1_N \otimes 1_{N^*} \otimes \mathrm{db}_M) \mathrm{db}_N = (1_N \otimes (\mathrm{ev}_N \otimes 1_{M^*})(1_{N^*} \otimes f \otimes 1_{M^*})(1_{N^*} \otimes \mathrm{db}_M)) \mathrm{db}_N = (1_N \otimes f^*) \mathrm{db}_N$. □

Corollary 3.3.10. *Let M, N have the left duals (M^*, ev_M) and (N^*, ev_N) and let $f : M \longrightarrow N$ be a morphism in $\mathcal{C}$. Then the following diagram commutes*

$$\begin{tikzcd} N^* \otimes M \arrow[r, "f^*\otimes 1"] \arrow[d, "1\otimes f"'] & M^* \otimes M \arrow[d, "\mathrm{ev}_M"] \\ N^* \otimes N \arrow[r, "\mathrm{ev}_N"'] & I. \end{tikzcd}$$

Proof. This statement follows immediately from the symmetry of the definition of a left dual. □

Example 3.3.11. Let $M \in R\text{-}\mathcal{M}od\text{-}R$ be an R-R-bimodule. Then $\mathrm{Hom}_R(M., R.)$ is an R-R-bimodule by $(rfs)(x) = rf(sx)$. Furthermore we have the morphism $\mathrm{ev} : \mathrm{Hom}_R(M., R.) \otimes_R M \longrightarrow R$ defined by $\mathrm{ev}(f \otimes_R m) = f(m)$.

(Dual Basis Lemma:) A module $M \in \mathcal{M}od\text{-}R$ is called *finitely generated and projective* if there are elements $m_1, \ldots, m_n \in M$ und $m^1, \ldots, m^n \in \mathrm{Hom}_R(M., R.)$ such that

$$\forall m \in M : \sum_{i=1}^{n} m_i m^i(m) = m.$$

Actually this is a consequence of the dual basis lemma. But this definition is equivalent to the usual definition.

Let $M \in R\text{-}\mathcal{M}od\text{-}R$. M as a right R-module is finitely generated and projective iff M has a left dual. The left dual is isomorphic to $\mathrm{Hom}_R(M., R.)$.

If M_R is finitely generated projective then we use $\mathrm{db} : R \to M \otimes_R \mathrm{Hom}_R(M., R.)$ with $\mathrm{db}(1) = \sum_{i=1}^n m_i \otimes_R m^i$. In fact we have $(1 \otimes_R \mathrm{ev})(\mathrm{db} \otimes_R 1)(m) = (1 \otimes_R \mathrm{ev})(\sum m_i \otimes_R m^i \otimes_R m) = \sum m_i m^i(m) = m$. We have furthermore $(\mathrm{ev} \otimes_R 1)(1 \otimes_R \mathrm{db})(f)(m) = (\mathrm{ev} \otimes_R 1)(\sum_{i=1}^n f \otimes_R m_i \otimes_R m^i)(m) = \sum f(m_i)m^i(m) = f(\sum m_i m^i(m))$ $= f(m)$ for all $m \in M$ hence $(\mathrm{ev} \otimes_R 1)(1 \otimes_R \mathrm{db})(f) = f$.

Conversely if M has a left dual M^* then $\mathrm{ev} : M^* \otimes_R M \to R$ defines a homomorphism $\iota : M^* \to \mathrm{Hom}_R(M., R.)$ in $R\text{-}\mathcal{M}od\text{-}R$ by $\iota(m^*)(m) = \mathrm{ev}(m^* \otimes_R m)$. We define $\sum_{i=1}^n m_i \otimes m^i := \mathrm{db}(1) \in M \otimes M^*$, then $m = (1 \otimes \mathrm{ev})(\mathrm{db} \otimes 1)(m) = (1 \otimes \mathrm{ev})(\sum m_i \otimes m^i \otimes m) = \sum m_i \iota(m^i)(m)$ so that $m_1, \ldots, m_n \in M$ and $\iota(m^1), \ldots, \iota(m^n) \in \mathrm{Hom}_R(M., R.)$ form a dual basis for M, i. e. M is finitely generated and projective as an R-module. Thus M^* and $\mathrm{Hom}_R(M., R.)$ are isomorphic by the map ι.

Analogously $\mathrm{Hom}_R(.M, .R)$ is a right dual for M iff M is finitely generated and projective as a left R-module.

Problem 3.3.26. Find an example of an object M in a monoidal category $\mathcal{C}$ that has a left dual but no right dual.

Definition 3.3.12. Given objects M, N in $\mathcal{C}$. An object $[M, N]$ is called a *left inner Hom* of M and N if there is a natural isomorphism $\mathrm{Mor}_{\mathcal{C}}(\text{-} \otimes M, N) \cong \mathrm{Mor}_{\mathcal{C}}(\text{-}, [M, N])$, i. e. if it represents the functor $\mathrm{Mor}_{\mathcal{C}}(\text{-} \otimes M, N)$.

If there is an isomorphism $\mathrm{Mor}_{\mathcal{C}}(P \otimes M, N) \cong \mathrm{Mor}_{\mathcal{C}}(P, [M, N])$ natural in the three variable M, N, P then the category $\mathcal{C}$ is called *monoidal and left closed.*

If there is an isomorphism $\mathrm{Mor}_{\mathcal{C}}(M \otimes P, N) \cong \mathrm{Mor}_{\mathcal{C}}(P, [M, N])$ natural in the three variable M, N, P then the category $\mathcal{C}$ is called *monoidal and right closed.*

If M has a left dual M^* in $\mathcal{C}$ then there are inner Homs $[M, \text{-}]$ defined by $[M, N] := N \otimes M^*$. In particular left rigid monoidal categories are left closed.

Example 3.3.13.
1. The category of finite dimensional vector spaces is a monoidal category where each object has a (left and right) dual. Hence it is (left and right) closed and rigid.
2. Let **Ban** be the category of (complex) Banach spaces where the morphisms satisfy $\| f(m) \| \leq \| m \|$ i. e. the maps are bounded by 1 or contracting. **Ban** is a monoidal category by the complete tensor product $M \widehat{\otimes} N$. In **Ban** exists an inner Hom functor $[M, N]$ that consists of the set of bounded linear maps from M to N made into a Banach space by an appropriate topology. Thus **Ban** is a monoidal closed category.
3. Let H be a Hopf algebra. The category $H\text{-}\mathcal{M}od$ of left H-modules is a monoidal category (see Example 3.2.5 2.). Then $\mathrm{Hom}_{\mathbb{K}}(M, N)$ is an object in $H\text{-}\mathcal{M}od$ by the multiplication
$$(hf)(m) := \sum h_{(1)} f(mS(h_{(2)})$$
as in Proposition 3.1.13.

$\mathrm{Hom}_{\mathbb{K}}(M, N)$ is an inner Hom functor in the monoidal category H-$\mathcal{M}od$. The isomorphism $\phi : \mathrm{Hom}_{\mathbb{K}}(P, \mathrm{Hom}_{\mathbb{K}}(M, N)) \cong \mathrm{Hom}_{\mathbb{K}}(P \otimes M, N)$ can be restricted to an isomorphism

$$\mathrm{Hom}_H(P, \mathrm{Hom}_{\mathbb{K}}(M, N)) \cong \mathrm{Hom}_H(P \otimes M, N),$$

because $\phi(f)(h(p \otimes m)) = \phi(f)(\sum h_{(1)}p \otimes h_{(2)}m) = \sum f(h_{(1)}p)(h_{(2)}m) = \sum(h_{(1)}(f(p)))(h_{(2)}m) = \sum h_{(1)}(f(p)(S(h_{(2)})h_{(3)}m)) = h(f(p)(m)) = h(\phi(f)(p\otimes m))$ and conversely $(h(f(p)))(m) = \sum h_{(1)}(f(p)(S(h_{(2)})m)) = \sum h_{(1)}(\phi(f)(p\otimes S(h_{(2)})m)) = \sum \phi(f)(h_{(1)}p\otimes h_{(2)}S(h_{(3)})m) = \phi(f)(hp\otimes m) = f(hp)(m)$. Thus H-$\mathcal{M}od$ is left closed.

If $M \in H$-$\mathcal{M}od$ is a finite dimensional vector space then the dual vector space $M^* := \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$ again is an H-module by $(hf)(m) := f(S(h)m)$. Furthermore M^* is a left dual for M with the morphisms

$$\mathrm{db} : \mathbb{K} \ni 1 \mapsto \sum_i m_i \otimes m^i \in M \otimes M^*$$

and

$$\mathrm{ev} : M^* \otimes M \ni f \otimes m \mapsto f(m) \in \mathbb{K}$$

where m_i and m^i are a dual basis of the vector space M. Clearly we have $(1 \otimes \mathrm{ev})(\mathrm{db} \otimes 1) = 1_M$ and $(\mathrm{ev} \otimes 1)(1 \otimes \mathrm{db}) = 1_{M^*}$ since M is a finite dimensional vector space. We have to show that db and ev are H-module homomorphisms. We have

$$\begin{aligned}&(h\,\mathrm{db}(1))(m) = (h(\textstyle\sum m_i \otimes m^i))(m) = (\sum h_{(1)}m_i \otimes h_{(2)}m^i)(m) = \\ &\textstyle\sum(h_{(1)}m_i)((h_{(2)}m^i)(m)) = \sum(h_{(1)}m_i)(m^i(S(h_{(2)})m)) = \\ &\textstyle\sum h_{(1)}S(h_{(2)})m = \varepsilon(h)m = \varepsilon(h)(\sum m_i \otimes m^i)(m) = \varepsilon(h)\,\mathrm{db}(1)(m) = \\ &\mathrm{db}(\varepsilon(h)1)(m) = \mathrm{db}(h1)(m),\end{aligned}$$

hence $h\,\mathrm{db}(1) = \mathrm{db}(h1)$. Furthermore we have

$$\begin{aligned}&h\,\mathrm{ev}(f \otimes m) = hf(m) = \textstyle\sum h_{(1)}f(S(h_{(2)})h_{(3)}m) = \sum(h_{(1)}f)(h_{(2)}m) = \\ &\textstyle\sum \mathrm{ev}(h_{(1)}f \otimes h_{(2)}m) = \mathrm{ev}(h(f \otimes m)).\end{aligned}$$

(4) Let H be a Hopf algebra. Then the category of left H-comodules (see Example 3.2.5 3.) is a monoidal category. Let $M \in H$-$\mathcal{C}omod$ be a finite dimensional vector space. Let m_i be a basis for M and let the comultiplication of the comodule be $\delta(m_i) = \sum h_{ij} \otimes m_j$. Then we have $\Delta(h_{ik}) = \sum h_{ij} \otimes h_{jk}$. $M^* := \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$ becomes a left H-comodule $\delta(m^j) := \sum S(h_{ij}) \otimes m^i$. One verifies that M^* is a left dual for M.

Lemma 3.3.14. *Let $M \in \mathcal{C}$ be an object with left dual (M^*, ev). Then*
1. *$M \otimes M^*$ is an algebra with multiplication*

$$\nabla := 1_M \otimes \mathrm{ev} \otimes 1_{M^*} : M \otimes M^* \otimes M \otimes M^* \longrightarrow M \otimes M^*$$

and unit

$$u := \mathrm{db} : I \longrightarrow M \otimes M^*;$$

2. *$M^* \otimes M$ is a coalgebra with comultiplication*

$$\Delta := 1_{M^*} \otimes \mathrm{db} \otimes 1_M : M^* \otimes M \longrightarrow M^* \otimes M \otimes M^* \otimes M$$

and counit

$$\varepsilon := \mathrm{ev} : M^* \otimes M \longrightarrow I.$$

Proof. 1. The associativity is given by $(\nabla \otimes 1)\nabla = (1_M \otimes \mathrm{ev} \otimes 1_{M^*} \otimes 1_M \otimes 1_{M^*})(1_M \otimes \mathrm{ev} \otimes 1_{M^*}) = 1_M \otimes \mathrm{ev} \otimes \mathrm{ev} \otimes 1_{M^*} = (1_M \otimes 1_{M^*} \otimes 1_M \otimes \mathrm{ev} \otimes 1_{M^*})(1_M \otimes \mathrm{ev} \otimes 1_{M^*}) = (1 \otimes \nabla)\nabla$. The axiom for the left unit is $\nabla(u \otimes 1) = (1_M \otimes \mathrm{ev} \otimes 1_{M^*})(\mathrm{db} \otimes 1_M \otimes 1_{M^*}) = 1_M \otimes 1_{M^*}$.

2. is dual to the statement for algebras. $\square$

Lemma 3.3.15. 1. *Let A be an algebra in $\mathcal{C}$ and left $M \in \mathcal{C}$ be a left rigid object with left dual (M^*, ev). There is a bijection between the set of morphisms $f : A \otimes M \longrightarrow M$ making M a left A-module and the set of algebra morphisms $\widetilde{f} : A \longrightarrow M \otimes M^*$.*
2. *Let C be a coalgebra in $\mathcal{C}$ and left $M \in \mathcal{C}$ be a left rigid object with left dual (M^*, ev). There is a bijection between the set of morphisms $f : M \longrightarrow M \otimes C$ making M a right C-comodule and the set of coalgebra morphisms $\widetilde{f} : M^* \otimes M \longrightarrow C$.*

Proof. 1. By Lemma 3.3.14 the object $M \otimes M^*$ is an algebra. Given $f : A \otimes M \longrightarrow M$ such that M becomes an A-module. By Lemma 3.3.3 we associate $\widetilde{f} := (f \otimes 1)(1 \otimes \mathrm{db}) : A \longrightarrow A \otimes M \otimes M^* \longrightarrow M \otimes M^*$. The compatibility of $\widetilde{f}$ with the multiplication is given by the commutative diagram

$$\begin{tikzcd}
A \otimes A \arrow[rrrr, "\nabla"] \arrow[dddd, "\widetilde{f}\otimes\widetilde{f}"'] \arrow[dr, "1\otimes1\otimes\mathrm{db}"] \arrow[ddr, "1\otimes\widetilde{f}"'] & & & & A \arrow[dl, "1\otimes\mathrm{db}"'] \arrow[dddd, "\widetilde{f}"] \\
& A \otimes A \otimes M \otimes M^* \arrow[rr, "\nabla\otimes1\otimes1"] \arrow[d, "1\otimes f\otimes1"] & & A \otimes M \otimes M^* \arrow[d, "f\otimes1"'] & \\
& A \otimes M \otimes M^* \arrow[rr, "f\otimes1"] \arrow[d, "1\otimes\mathrm{db}\otimes1\otimes1"] \arrow[drr, "1"] \arrow[ddl, "\widetilde{f}\otimes1\otimes1"'] & & M \otimes M^* \arrow[ddr, "1"] & \\
& A \otimes M \otimes M^* \otimes M \otimes M^* \arrow[rr, "1\otimes1\otimes\mathrm{ev}\otimes1"'] \arrow[dl, "f\otimes1\otimes1"] & & A \otimes M \otimes M^* \arrow[dr, "f\otimes1"'] & \\
M \otimes M^* \otimes M \otimes M^* \arrow[rrrr, "1\otimes\mathrm{ev}\otimes1"'] & & & & M \otimes M^*
\end{tikzcd}$$

The unit axiom is given by

$$\begin{tikzcd}
I \arrow[r, "\mathrm{db}"] \arrow[d, "u"'] & M \otimes M^* \arrow[d, "u\otimes1"'] \arrow[dr, "1\otimes1"] & \\
A \arrow[r, "1\otimes\mathrm{db}"] & A \otimes M \otimes M^* \arrow[r, "f\otimes1"] & M \otimes M^*
\end{tikzcd}$$

Conversely let $g : A \longrightarrow M \otimes M^*$ be an algebra homomorphism and consider $\widetilde{g} := (1 \otimes \mathrm{ev})(g \otimes 1) : A \otimes M \longrightarrow M \otimes M^* \otimes M \longrightarrow M$. Then M becomes a left A-module since

$$
\begin{tikzcd}
A \otimes A \otimes M \arrow[rrrr, "\nabla\otimes 1"] \arrow[dr, "1\otimes g\otimes 1"] \arrow[drr, "g\otimes g\otimes 1"] \arrow[ddd, "1\otimes\widetilde{g}"'] & & & & A \otimes M \arrow[dl, "g\otimes 1"'] \arrow[ddd, "\widetilde{g}"] \\
& A \otimes M \otimes M^* \otimes M \arrow[r, "g\otimes 1\otimes 1\otimes 1"'] \arrow[ddl, "1\otimes 1\otimes \mathrm{ev}"] & M \otimes M^* \otimes M \otimes M^* \otimes M \arrow[r, "1\otimes \mathrm{ev}\otimes 1\otimes 1"] \arrow[d, "1\otimes 1\otimes 1\otimes \mathrm{ev}"] & M \otimes M^* \otimes M \arrow[ddr, "1\otimes \mathrm{ev}"] & \\
& & M \otimes M^* \otimes M \arrow[drr, "1\otimes \mathrm{ev}"] & & \\
A \otimes M \arrow[urr, "g\otimes 1"] \arrow[rrrr, "\widetilde{g}"] & & & & M
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
M \arrow[d, "u\otimes 1"'] \arrow[dr, "\mathrm{db}\otimes 1"] \arrow[drr, "1"] & & \\
A \otimes M \arrow[r, "g\otimes 1"'] & M \otimes M^* \otimes M \arrow[r, "1\otimes \mathrm{ev}"'] & M
\end{tikzcd}
$$

commute.

2. (M^*, ev) is a left dual for M in the category $\mathcal{C}$ if and only if (M^*, db) is the right dual for M in the dual category $\mathcal{C}^{op}$. So if we dualize the result of part 1. we have to change sides, hence 2. $\square$

4. Finite reconstruction

The endomorphism ring of a vector space enjoys the following universal property. It is a vector space itself and allows a homomorphism $\rho : \mathrm{End}(V) \otimes V \longrightarrow V$. It is universal with respect to this property, i. e. if Z is a vector space and $f : Z \otimes V \longrightarrow V$ is a homomorphism, then there is a unique homomorphism $g : Z \longrightarrow \mathrm{End}(V)$ such that

$$
\begin{tikzcd}
Z \otimes V \arrow[d, "g\otimes 1"'] \arrow[dr, "f"] & \\
\mathrm{End}(V) \otimes V \arrow[r, "\rho"'] & V
\end{tikzcd}
$$

commutes.

The algebra structure of $\mathrm{End}(V)$ comes for free from this universal property.

If we replace the vector space V by a diagram of vector spaces $\omega : \mathcal{D} \longrightarrow \mathcal{V}ec$ we get a similar universal object $\mathrm{End}(\omega)$. Again the universal property induces a unique algebra structure on $\mathrm{End}(\omega)$.

Problem 3.4.27. (1) Let V be a vector space. Show that there is a universal vector space E and homomorphism $\rho : E \otimes V \longrightarrow V$ (such that for each vector space Z and each homomorphism $f : Z \otimes V \longrightarrow V$ there is a unique homomorphism $g : Z \longrightarrow E$ such that

$$\begin{tikzcd} Z \otimes V \arrow[d, "g\otimes 1"'] \arrow[dr, "f"] & \\ E \otimes V \arrow[r, "\rho"'] & V \end{tikzcd}$$

commutes). We call E and $\rho : E \otimes V \longrightarrow V$ a *vector space acting universally on* V.

(2) Let E and $\rho : E \otimes V \longrightarrow V$ be a vector space acting universally on V. Show that E uniquely has the structure of an algebra such that V becomes a left E-module.

(3) Let $\omega : \mathcal{D} \longrightarrow \mathcal{V}ec$ be a diagram of vector spaces. Show that there is a universal vector space E and natural transformation $\rho : E \otimes \omega \longrightarrow \omega$ (such that for each vector space Z and each natural transformation $f : Z \otimes \omega \longrightarrow \omega$ there is a unique homomorphism $g : Z \longrightarrow E$ such that

$$\begin{tikzcd} Z \otimes \omega \arrow[d, "g\otimes 1"'] \arrow[dr, "f"] & \\ E \otimes \omega \arrow[r, "\rho"'] & \omega \end{tikzcd}$$

commutes). We call E and $\rho : E \otimes \omega \longrightarrow \omega$ a *vector space acting universally on* ω.

(4) Let E and $\rho : E \otimes \omega \longrightarrow \omega$ be a vector space acting universally on ω. Show that E uniquely has the structure of an algebra such that ω becomes a diagram of left E-modules.

Similar considerations can be carried out for coactions $V \longrightarrow V \otimes C$ or $\omega \longrightarrow \omega \otimes C$ and a coalgebra structure on C. There is one restriction, however. We can only use finite dimensional vector spaces V or diagrams of finite dimensional vector spaces. This will be done further down.

We want to find a universal natural transformation $\delta : \omega \longrightarrow \omega \otimes \mathrm{coend}(\omega)$. For this purpose we consider the isomorphisms

$$\mathrm{Mor}_{\mathcal{C}}(\omega(X), \omega(X) \otimes M) \cong \mathrm{Mor}_{\mathcal{C}}(\omega(X)^* \otimes \omega(X), M)$$

that are given by $f \mapsto (\mathrm{ev} \otimes 1)(1 \otimes f)$ and as inverse $g \mapsto (1 \otimes g)(\mathrm{db} \otimes 1)$. We first develop techniques to describe the properties of a natural transformation $\phi : \omega \longrightarrow \omega \otimes M$ as properties of the associated family $g(X) : \omega(X)^* \otimes \omega(X) \longrightarrow M$. We will see that $g : \omega^* \otimes \omega \longrightarrow M$ will be a *cone*. Then we will show that ϕ is a universal natural transformation if and only if its associated cone is universal. In the literature this is called a coend.

Throughout this section assume the following. Let $\mathcal{D}$ be an arbitrary diagram scheme. Let $\mathcal{C}$ be a cocomplete monoidal category such that the tensor product preserves colimits in both arguments. Let $\mathcal{C}_0$ be the full subcategory of those objects in $\mathcal{C}$ that have a left dual. Let $\omega : \mathcal{D} \longrightarrow \mathcal{C}$ be a diagram in $\mathcal{C}$ such that $\omega(X) \in \mathcal{C}_0$ for all $X \in \mathcal{D}$, i. e. ω is given by a functor $\omega_0 : \mathcal{D} \longrightarrow \mathcal{C}_0$. We call such a diagram a *finite diagram* in $\mathcal{C}$. Finally for an object $M \in \mathcal{C}$ let $\omega \otimes M : \mathcal{D} \longrightarrow \mathcal{C}$ be the functor with $(\omega \otimes M)(X) = \omega(X) \otimes M$.

Remark 3.4.1. Consider the following category $\widetilde{\mathcal{D}}$. For each morphism $f : X \longrightarrow Y$ there is an object $\widetilde{f} \in \widetilde{D}$. The object corresponding to the identity $1_X : X \longrightarrow X$ is denoted by $\widetilde{X} \in \widetilde{\mathcal{D}}$. For each morphism $f : X \longrightarrow Y$ in $\mathcal{D}$ there are two morphisms $f_1 : \widetilde{f} \longrightarrow \widetilde{X}$ and $f_2 : \widetilde{f} \longrightarrow \widetilde{Y}$ in $\widetilde{\mathcal{D}}$. Furthermore there are the identities $1_f : \widetilde{f} \longrightarrow \widetilde{f}$ in $\widetilde{\mathcal{D}}$.

Since there are no morphisms with $\widetilde{X}$ as domain other than $(1_X)_i : \widetilde{X} \longrightarrow \widetilde{X}$ and $1_f : \widetilde{f} \longrightarrow \widetilde{f}$ we only have to define the following compositions $(1_X)_i \circ f_j := f_j$. Then $\widetilde{\mathcal{D}}$ becomes a category and we have $1_{\widetilde{X}} = (1_X)_1 = (1_X)_2$.

We define a diagram $\omega^* \otimes \omega : \widetilde{\mathcal{D}} \longrightarrow \mathcal{C}$ as follows. If $f : X \longrightarrow Y$ is given then

$$(\omega^* \otimes \omega)(\widetilde{f}) := \omega(Y)^* \otimes \omega(X)$$

and

$$\begin{aligned} \omega(f_1) &:= \omega(f)^* \otimes \omega(1_X), \\ \omega(f_2) &:= \omega(1_Y)^* \otimes \omega(f). \end{aligned}$$

The colimit of $\omega^* \otimes \omega$ consists of an object $\mathrm{coend}(\omega) \in \mathcal{C}$ together with a family of morphisms $\iota(X, X) : \omega(X)^* \otimes \omega(X) \longrightarrow \mathrm{coend}(\omega)$ such that the diagrams

$$\begin{tikzcd}
& \omega(X)^* \otimes \omega(X) \arrow[dr, "\iota(X,X)"] & \\
(\omega^* \otimes \omega)(\widetilde{f}) = \omega(Y)^* \otimes \omega(X) \arrow[ur, "\omega(f)^* \otimes 1"] \arrow[dr, "1 \otimes \omega(f)"'] & & \mathrm{coend}(\omega) \\
& \omega(Y)^* \otimes \omega(Y) \arrow[ur, "\iota(Y,Y)"'] &
\end{tikzcd}$$

commute for all $f : X \longrightarrow Y$ in $\mathcal{D}$. Indeed, such a family $\iota(\widetilde{X}) := \iota(X, X)$ can be uniquely extended to a natural transformation by defining $\iota(\widetilde{f}) := \iota(X, X)(\omega(f)^* \otimes \omega(X)) = \iota(Y, Y)(\omega(Y)^* \otimes \omega(f))$. In addition the pair $(\mathrm{coend}(\omega), \iota)$ is universal with respect to this property.

In the literature such a universal object is called a *coend* of the bifunctor $\omega^* \otimes \omega : \mathcal{D}^{op} \times \mathcal{D} \longrightarrow \mathcal{C}$.

Corollary 3.4.2. *The following is a coequalizer*

$$\coprod_{f \in \mathrm{Mor}\mathcal{D}} \omega(\mathrm{Dom}(f))^* \otimes \omega(\mathrm{Codom}(f)) \overset{p}{\underset{q}{\rightrightarrows}} \coprod_{X \in \mathrm{Ob}\,\mathcal{D}} \omega(X)^* \otimes \omega(X) \longrightarrow \mathrm{coend}(\omega)$$

PROOF. This is just a reformulation of [Advanced Algebra] Remark 6.11, since the colimit may also be built from the commutative squares given above.

Observe that for the construction of the colimit not all objects of the diagram have to be used but only those of the form $\omega(X)^* \otimes \omega(X)$. $\square$

Theorem 3.4.3. *(***Tannaka-Krein***)*

Let $\omega : \mathcal{D} \longrightarrow \mathcal{C}_0 \subseteq \mathcal{C}$ be a finite diagram. Then there exists an object $\mathrm{coend}(\omega) \in \mathcal{C}$ and a natural transformation $\delta : \omega \longrightarrow \omega \otimes \mathrm{coend}(\omega)$ such that for each object $M \in \mathcal{C}$ and each natural transformation $\varphi : \omega \longrightarrow \omega \otimes M$ there exists a unique morphism $\widetilde{\varphi} : \mathrm{coend}(\omega) \longrightarrow M$ such that the diagram

$$\begin{tikzcd} \omega \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\widetilde{\varphi}"] \\ & \omega \otimes M \end{tikzcd}$$

commutes.

PROOF. Let $\mathrm{coend}(\omega) \in \mathcal{C}$ together with morphisms $\iota(\widetilde{f}) : \omega(Y)^* \otimes \omega(X) \longrightarrow \mathrm{coend}(\omega)$ be the colimit of the diagram $\omega^* \otimes \omega : \widetilde{\mathcal{D}} \longrightarrow \mathcal{C}$. So we get commutative diagrams

$$\begin{tikzcd} & \omega(X)^* \otimes \omega(X) \arrow[dr, "\iota(X,X)"] & \\ \omega(Y)^* \otimes \omega(X) \arrow[ur, "\omega(f)^*\otimes 1"] \arrow[rr, "\iota(\widetilde{f})"] \arrow[dr, "1\otimes\omega(f)"'] & & \mathrm{coend}(\omega) \\ & \omega(Y)^* \otimes \omega(Y) \arrow[ur, "\iota(Y,Y)"'] & \end{tikzcd}$$

for each $f : X \longrightarrow Y$ in $\mathcal{C}$.

For $X \in \mathcal{C}$ we define a morphism $\delta(X) : \omega(X) \longrightarrow \omega(X) \otimes \mathrm{coend}(\omega)$ by $(1 \otimes \iota(X,X))(\mathrm{db} \otimes 1) : \omega(X) \longrightarrow \omega(X) \otimes \omega(X)^* \otimes \omega(X) \longrightarrow \omega(X) \otimes \mathrm{coend}(\omega)$. Then we get as in Corollary 3.3.5 $\quad \iota(X,X) = (1 \otimes \mathrm{ev})(1 \otimes \delta(X))$.

We show that δ is a natural transformation. For each $f : X \longrightarrow Y$ the square

$$\begin{tikzcd} I \arrow[r, "\mathrm{db}_X"] \arrow[d, "\mathrm{db}_Y"'] & \omega(X) \otimes \omega(X)^* \arrow[d, "\omega(f)\otimes 1"] \\ \omega(Y) \otimes \omega(Y)^* \arrow[r, "1\otimes\omega(f)^*"'] & \omega(Y) \otimes \omega(X)^*. \end{tikzcd}$$

commutes by Corollary 3.3.9. Thus the following diagram commutes

$$\begin{tikzcd}
\omega(X) \arrow[rr, "\mathrm{db}\otimes 1"] \arrow[dd, "\omega(f)"'] \arrow[dr, "\mathrm{db}\otimes 1"] & & \omega(X)\otimes\omega(X)^*\otimes\omega(X) \arrow[rr, "1\otimes\iota(X,X)"] \arrow[dr, "\omega(f)\otimes 1\otimes 1"] & & \omega(X)\otimes\mathrm{coend}(\omega) \arrow[dd, "\omega(f)\otimes 1"] \\
& \omega(Y)\otimes\omega(Y)^*\otimes\omega(X) \arrow[rr, "1\otimes\omega(f)^*\otimes 1"] \arrow[dr, "1\otimes 1\otimes\omega(f)"'] & & \omega(Y)\otimes\omega(X)^*\otimes\omega(X) \arrow[dr, "1\otimes\iota(X,X)"'] & \\
\omega(Y) \arrow[rr, "\mathrm{db}\otimes 1"] & & \omega(Y)\otimes\omega(Y)^*\otimes\omega(Y) \arrow[rr, "1\otimes\iota(Y,Y)"] & & \omega(Y)\otimes\mathrm{coend}(\omega).
\end{tikzcd}$$

Now let $M \in \mathcal{C}$ be an object and $\varphi : \omega \longrightarrow \omega\otimes M$ a natural transformation. Observe that

$$\begin{tikzcd}
\omega(Y)^*\otimes\omega(X) \arrow[r, "\omega(f)^*\otimes 1"] \arrow[d, "1\otimes\omega(f)"'] & \omega(X)^*\otimes\omega(X) \arrow[d, "\mathrm{ev}"] \\
\omega(Y)^*\otimes\omega(Y) \arrow[r, "\mathrm{ev}"'] & I
\end{tikzcd}$$

commutes by Corollary 3.3.10. Thus also the diagram

$$\begin{tikzcd}
\omega(X)^*\otimes\omega(X) \arrow[rr, "1\otimes\varphi(X)"] & & \omega(X)^*\otimes\omega(X)\otimes M \arrow[dr, "\mathrm{ev}\otimes 1"] & \\
\omega(Y)^*\otimes\omega(X) \arrow[u, "\omega(f)^*\otimes 1"] \arrow[r, "1\otimes\varphi(X)"] \arrow[d, "1\otimes\omega(f)"] & \omega(Y)^*\otimes\omega(X)\otimes M \arrow[ur, "\omega(f)^*\otimes 1\otimes 1"] \arrow[dr, "1\otimes\omega(f)\otimes 1"'] & & M \\
\omega(Y)^*\otimes\omega(Y) \arrow[rr, "1\otimes\varphi(Y)"] & & \omega(Y)^*\otimes\omega(Y)\otimes M \arrow[ur, "\mathrm{ev}\otimes 1"'] &
\end{tikzcd}$$

commutes. We define $\widetilde{\varphi} : \mathrm{coend}(\omega) \longrightarrow M$ from the colimit property as universal factorization

$$\begin{tikzcd}
& \omega(X)^*\otimes\omega(X) \arrow[dr, "\iota(X,X)"'] \arrow[drr, "(\mathrm{ev}\otimes 1)(1\otimes\varphi(X))"] & & \\
\omega(Y)^*\otimes\omega(X) \arrow[ur, "\omega(f)^*\otimes 1"] \arrow[dr, "1\otimes\omega(f)"'] & & \mathrm{coend}(\omega) \arrow[r, "\widetilde{\varphi}"] & M \\
& \omega(Y)^*\otimes\omega(Y) \arrow[ur, "\iota(Y,Y)"] \arrow[urr, "(\mathrm{ev}\otimes 1)(1\otimes\varphi(Y))"'] & &
\end{tikzcd}$$

Hence the diagram

$$\begin{tikzcd}
\omega(X) \arrow[rr, "\delta(X)"] \arrow[dr, "\mathrm{db}\otimes 1"'] \arrow[ddd, "\varphi(X)"'] & & \omega(X)\otimes\mathrm{coend}(\omega) \arrow[ddd, "1\otimes\widetilde{\varphi}"] \\
& \omega(X)\otimes\omega(X)^*\otimes\omega(X) \arrow[ur, "1\otimes\iota(X,X)"'] \arrow[d, "1\otimes 1\otimes\varphi(X)"'] & \\
& \omega(X)\otimes\omega(X)^*\otimes\omega(X)\otimes M \arrow[dr, "1\otimes\mathrm{ev}\otimes 1"] & \\
\omega(X)\otimes M \arrow[ur, "\mathrm{db}\otimes 1\otimes 1"] \arrow[rr, "1\otimes 1"'] & & \omega(X)\otimes M
\end{tikzcd}$$

commutes. The exterior portion of this diagram yields

$$\begin{tikzcd}
\omega(X) \arrow[r, "\delta(X)"] \arrow[dr, "\varphi(X)"'] & \omega(X)\otimes\mathrm{coend}(\omega) \arrow[d, "1\otimes\widetilde{\varphi}"] \\
& \omega(X)\otimes M.
\end{tikzcd}$$

It remains to show that $\widetilde{\varphi} : \mathrm{coend}(\omega) \to M$ is uniquely determined. Let $\widetilde{\varphi}_0 : \mathrm{coend}(\omega) \to M$ be another morphism with $\varphi(X) = (1\otimes\widetilde{\varphi}_0)\delta(X)$ for all $X \in \mathcal{D}$. Then the following diagram commutes

$$\begin{tikzcd}
\omega(X)^*\otimes\omega(X) \arrow[rr, "\iota(X,X)"] \arrow[dr, "1\otimes\delta(X)"] \arrow[ddr, "1\otimes\varphi(X)"'] & & \mathrm{coend}(\omega) \arrow[dd, "\widetilde{\varphi}_0"] \\
& \omega(X)^*\otimes\omega(X)\otimes\mathrm{coend}(\omega) \arrow[ur, "\mathrm{ev}\otimes 1"] \arrow[d, "1\otimes 1\otimes\widetilde{\varphi}_0"] & \\
& \omega(X)^*\otimes\omega(X)\otimes M \arrow[r, "\mathrm{ev}\otimes 1"] & M,
\end{tikzcd}$$

hence we have $\widetilde{\varphi}_0 = \widetilde{\varphi}$. $\square$

Corollary 3.4.4. *The functor* $\mathrm{Nat}(\omega, \omega\otimes M)$ *is a representable functor in* M *represented by* $\mathrm{coend}(\omega)$.

PROOF. The universal problem implies the isomorphism

$$\mathrm{Nat}(\omega, \omega\otimes M) \cong \mathrm{Mor}_{\mathcal{C}}(\mathrm{coend}(\omega), M)$$

and the universal natural transformation $\delta : \omega \to \omega\otimes\mathrm{coend}(\omega)$ is mapped to the identity under this isomorphism. $\square$

It is also possible to construct an isomorphism

$$\mathrm{Nat}(\omega, \omega' \otimes M) \cong \mathrm{Mor}_{\mathcal{C}}(\mathrm{cohom}(\omega', \omega), M)$$

for different functors $\omega, \omega' : \mathcal{D} \longrightarrow \mathcal{C}$ and thus define *cohomomorphism objects*. Observe that only ω' has to take values in $\mathcal{C}_0$ since then we can build objects $\omega'(X)^* \otimes \omega(X)$.

5. The coalgebra coend

Proposition 3.5.1. *Let $\mathcal{C}$ be a monoidal category and $\omega : \mathcal{D} \longrightarrow \mathcal{C}$ be a diagram in $\mathcal{C}$. Assume that there is a universal object* $\mathrm{coend}(\omega)$ *and natural transformation $\delta : \omega \longrightarrow \omega \otimes \mathrm{coend}(\omega)$.*

Then there is exactly one coalgebra structure on $\mathrm{coend}(\omega)$ *such that the diagrams*

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[d, "\delta"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\Delta"] \\
\omega \otimes \mathrm{coend}(\omega) \arrow[r, "\delta\otimes 1"'] & \omega \otimes \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[dr, "\mathrm{id}_\omega"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\epsilon"] \\
 & \omega \otimes I
\end{tikzcd}$$

commute.

Proof. Because of the universal property of $\mathrm{coend}(\omega)$ there are structure morphisms $\Delta : \mathrm{coend}(\omega) \longrightarrow \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega)$ and $\epsilon : \mathrm{coend}(\omega) \longrightarrow I$. This implies the coalgebra property similar to the proof of Corollary 3.3.8. $\square$

Observe that by this construction all objects and all morphisms of the diagram $\omega : \mathcal{D} \longrightarrow \mathcal{C}_0 \subseteq \mathcal{C}$ are comodules or morphisms of comodules over the coalgebra $\mathrm{coend}(\omega)$. In fact $C := \mathrm{coend}(\omega)$ is the universal coalgebra over which the given diagram becomes a diagram of comodules.

Corollary 3.5.2. *Let $(\mathcal{D}, \omega)$ be a diagram $\mathcal{C}$ with objects in $\mathcal{C}_0$. Then all objects of the diagram are comodules over the coalgebra $C :=$* $\mathrm{coend}(\omega)$ *and all morphisms are morphisms of comodules. If D is another coalgebra and all objects of the diagram are D-comodules by $\varphi(X) : \omega(X) \longrightarrow \omega(X) \otimes D$ and all morphisms of the diagram are morphisms of D-comodules then there exists a unique morphism of coalgebras*

$\widetilde{\varphi} : \mathrm{coend}(\omega) \longrightarrow D$ *such that the diagram*

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\widetilde{\varphi}"] \\
& \omega \otimes D
\end{tikzcd}$$

commutes.

PROOF. The morphisms $\varphi(X) : \omega(X) \longrightarrow \omega(X) \otimes D$ define a natural transformation since all morphisms of the diagram are morphisms of comodules. So the existence and the uniqueness of a morphism $\widetilde{\varphi} : \mathrm{coend}(\omega) \longrightarrow D$ is clear. The only thing to show is that this is a morphism of coalgebras. This follows from the universal property of $C = \mathrm{coend}(\omega)$ and the diagram

$$\begin{tikzcd}
\omega \arrow[rr, "\delta"] \arrow[dr, "\delta"] \arrow[dd, "1"'] & & \omega \otimes C \arrow[dr, "1\otimes\Delta"] \arrow[dd, "1\otimes\widetilde{\varphi}"' near start] & \\
& \omega \otimes C \arrow[rr, "\delta\otimes 1"] \arrow[dd, "1\otimes\widetilde{\varphi}"' near start] & & \omega \otimes C \otimes C \arrow[dd, "1\otimes\widetilde{\varphi}\otimes\widetilde{\varphi}"] \\
\omega \arrow[rr, "\varphi"' near start] \arrow[dr, "\varphi"'] & & \omega \otimes D \arrow[dr, "1\otimes\Delta"'] & \\
& \omega \otimes D \arrow[rr, "\varphi\otimes 1"'] & & \omega \otimes D \otimes D
\end{tikzcd}$$

where the right side of the cube commutes by the universal property. Similarly we get that $\widetilde{\varphi}$ preserves the counit since the following diagram commutes

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[d, "1"'] \arrow[drr, "1"] & \omega \otimes C \arrow[d, "1\otimes\widetilde{\varphi}"'] \arrow[dr, "1\otimes\varepsilon"] & \\
\omega \arrow[r, "\varphi"] \arrow[drr, "1"'] & \omega \otimes D \arrow[dr, "1\otimes\varepsilon"] & \omega \otimes \mathbb{K} \arrow[d, "1"] \\
& & \omega \otimes \mathbb{K}
\end{tikzcd}$$

$\square$

6. The bialgebra coend

Let $\omega : \mathcal{D} \to \mathcal{C}$ and $\omega' : \mathcal{D}' \to \mathcal{C}$ be diagrams in $\mathcal{C}$. We call the diagram $(\mathcal{D}, \omega) \otimes (\mathcal{D}', \omega') := (\mathcal{D} \times \mathcal{D}', \omega \otimes \omega')$ with $(\omega \otimes \omega')(X, Y) := \omega(X) \otimes \omega'(Y)$ the *tensor product* of these two diagrams. The new diagram consists of all possible tensor products of all objects and all morphisms of the original diagrams.

From now on we assume that the category $\mathcal{C}$ is the category of vector spaces and we use the symmetry $\tau : V \otimes W \to W \otimes V$ in $\mathcal{V}ec$.

Proposition 3.6.1. *Let $(\mathcal{D}, \omega)$ and $(\mathcal{D}', \omega')$ be finite diagrams in $\mathcal{V}ec$. Then*

$$\mathrm{coend}(\omega \otimes \omega') \cong \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega').$$

PROOF. First observe the following. If two diagrams $\omega : \mathcal{D} \to \mathcal{V}ec$ and $\omega' : \mathcal{D}' \to \mathcal{V}ec$ are given then $\varinjlim_{\mathcal{D}} \varinjlim_{\mathcal{D}'} (\omega \otimes \omega') \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'} (\omega \otimes \omega') \cong \varinjlim_{\mathcal{D}} (\omega) \otimes \varinjlim_{\mathcal{D}'} (\omega')$ since the tensor product preserves colimits and colimits commute with colimits. For this consider the diagram

$$\begin{tikzcd}
\omega(X) \otimes \omega'(Y) \arrow[r] \arrow[d] & \omega(X) \otimes \varinjlim_{\mathcal{D}'} (\omega') \arrow[d] & \\
\varinjlim_{\mathcal{D}} (\omega) \otimes \omega'(Y) \arrow[r] & \varinjlim_{\mathcal{D}} (\omega) \otimes \varinjlim_{\mathcal{D}'} (\omega') & \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'} (\omega \otimes \omega').
\end{tikzcd}$$

The maps in the diagram are the injections for the corresponding colimits. In particular we have $\mathrm{coend}(\omega \otimes \omega') \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'} ((\omega \otimes \omega')^* \otimes (\omega \otimes \omega')) \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'} (\omega^* \otimes \omega \otimes \omega'^* \otimes \omega') \cong \varinjlim_{\mathcal{D}} (\omega^* \otimes \omega) \otimes \varinjlim_{\mathcal{D}'} (\omega'^* \otimes \omega') \cong \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega')$.

The (universal) morphism

$$(\iota(X) \otimes \iota'(Y))(1 \otimes \tau \otimes 1) : \omega(X)^* \otimes \omega'(Y)^* \otimes \omega(X) \otimes \omega'(Y) \to \varinjlim (\omega^* \otimes \omega) \otimes \varinjlim (\omega'^* \otimes \omega')$$

can be identified with the universal morphism

$$\iota(X, Y) : \omega(X)^* \otimes \omega'(Y)^* \otimes \omega(X) \otimes \omega'(Y) \to \varinjlim ((\omega \otimes \omega')^* \otimes (\omega \otimes \omega')).$$

Hence the induced morphisms

$$(1 \otimes \tau \otimes 1)(\delta \otimes \delta') : \omega(X) \otimes \omega'(Y) \to \omega(X) \otimes \omega'(Y) \otimes \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega')$$

and

$$\delta : \omega(X) \otimes \omega'(Y) \to \omega(X) \otimes \omega'(Y) \otimes \mathrm{coend}(\omega \otimes \omega')$$

can be identified. $\square$

Corollary 3.6.2. *For all finite diagrams $(\mathcal{D}, \omega)$ and $(\mathcal{D}', \omega')$ in $\mathcal{D}$ there is a universal natural transformation $\delta : \omega \otimes \omega' \to \omega \otimes \omega' \otimes \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega')$ so that for* *each object M and each natural transformation $\varphi : \omega \otimes \omega' \longrightarrow \omega \otimes \omega' \otimes M$ there exists a unique morphism $\widetilde{\varphi} : \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega') \longrightarrow M$ such that*

$$\begin{tikzcd}
\omega \otimes \omega' \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \omega \otimes \omega' \otimes \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega') \arrow[d, "1 \otimes 1 \otimes \widetilde{\varphi}"] \\
 & \omega \otimes \omega' \otimes M
\end{tikzcd}$$

commutes.

Definition 3.6.3. Let $(\mathcal{D}, \omega)$ be a diagram in $\mathcal{C} = \mathcal{V}ec$. Then ω is called *reconstructive*

- if there is an object $\mathrm{coend}(\omega)$ in $\mathcal{C}$ and a universal natural transformation $\delta : \omega \longrightarrow \omega \otimes \mathrm{coend}(\omega)$
- and if $(1 \otimes \tau \otimes 1)(\delta \otimes \delta) : \omega \otimes \omega \longrightarrow \omega \otimes \omega \otimes \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega)$ is a univesarl natural transformation of bifunctors.

Definition 3.6.4. Let $(\mathcal{D}, \omega)$ be a diagram in $\mathcal{V}ec$. Let $\mathcal{D}$ be a monoidal category and ω be a monoidal functor. Then $(\mathcal{D}, \omega)$ is called a *monoidal diagram.*

Let $(\mathcal{D}, \omega)$ be a monoidal diagram $\mathcal{V}ec$. Let $A \in \mathcal{V}ec$ be an algebra. A natural transformation $\varphi : \omega \longrightarrow \omega \otimes B$ is called monoidal *monoidal* if the diagrams

$$\begin{tikzcd}
\omega(X) \otimes \omega(Y) \arrow[r, "\varphi(X) \otimes \varphi(Y)"] \arrow[d, "\rho"'] & \omega(X) \otimes \omega(Y) \otimes B \otimes B \arrow[d, "\rho \otimes m"] \\
\omega(X \otimes Y) \arrow[r, "\varphi(X \otimes Y)"] & \omega(X \otimes Y) \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\cong"] \arrow[d] & \mathbb{K} \otimes \mathbb{K} \arrow[d] \\
\omega(I) \arrow[r, "\varphi(I)"] & \omega(I) \otimes B
\end{tikzcd}$$

commute.

We denote the set of monoidal natural transformations by $\mathrm{Nat}^{\otimes}(\omega, \omega \otimes B)$.

Problem 3.6.28. Show that $\mathrm{Nat}^{\otimes}(\omega, \omega \otimes B)$ is a functor in B.

Theorem 3.6.5. *Let $(\mathcal{D}, \omega)$ be a reconstructive, monoidal diagram in $\mathcal{V}ec$. Then $\mathrm{coend}(\omega)$ is a bialgebra and $\delta : \omega \longrightarrow \omega \otimes \mathrm{coend}(\omega)$ is a monoidal natural transformation.*

If B is a bialgebra and $\partial : \omega \longrightarrow \omega \otimes B$ is a monoidal natural transformation, then there is a unique homomorphism of bialgebras $f : coend(\omega) \longrightarrow B$ such that the *diagram*

$$\begin{tikzcd} \omega \arrow[r, "\delta"] \arrow[dr, "\partial"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes f"] \\ & \omega \otimes B \end{tikzcd}$$

commutes.

Proof. The multiplication of $\mathrm{coend}(\omega)$ arises from the following diagram

$$\begin{tikzcd} \omega(X)\otimes\omega(Y) \arrow[r, "\delta\otimes\delta"] \arrow[d] & \omega(X)\otimes\omega(Y)\otimes\mathrm{coend}(\omega)\otimes\mathrm{coend}(\omega) \arrow[d] \arrow[dr] & \\ \omega(X\otimes Y) \arrow[r, "\delta"] & \omega(X\otimes Y)\otimes\mathrm{coend}(\omega) \cong & \omega(X)\otimes\omega(Y)\otimes\mathrm{coend}(\omega) \end{tikzcd}$$

For the construction of the unit we consider the diagram $\mathcal{D}_0 = (\{I\}, \{\mathrm{id}\})$ together with $\omega_0 : \mathcal{D}_0 \to \mathcal{V}ec$, $\omega_0(I) = \mathbb{K}$, the monoidal unit object in the monoidal category of diagrams in $\mathcal{V}ec$. Then $(\mathbb{K} \to \mathbb{K}\otimes\mathbb{K}) = (\omega_0 \to \omega_0 \otimes \mathrm{coend}(\omega_0))$ is the universal map. The following diagram then induced the unit for $\mathrm{coend}(\omega)$

$$\begin{tikzcd} \mathbb{K} \arrow[r, "\cong"] \arrow[d] & \mathbb{K}\otimes\mathbb{K} \arrow[d] \arrow[dr] & \\ \omega(I) \arrow[r] & \omega(I)\otimes\mathrm{coend}(\omega) \cong & \mathbb{K}\otimes\mathrm{coend}(\omega) \end{tikzcd}$$

By using the universal property one checks the laws for bialgebras.

The above diagrams show in particular that the natural transformation $\delta : \omega \to \omega \otimes \mathrm{coend}(\omega)$ is monoidal. $\square$

7. The quantum monoid of a quantum space

Problem 3.7.29. If A is a finite dimensional algebra and $\delta : A \to M(A) \otimes A$ the universal cooperation of the Tambara bialgebra on A from the left then $\tau\delta : A \to A \otimes M(A)$ (with the same multiplication on $M(A)$) is a universal cooperation of $M(A)$ on A from the right. The comultiplication defined by this cooperation is $\tau\Delta : M(A) \to M(A) \otimes M(A)$. Thus we have to distinguish between the left and the right Tambara bialgebra on A and we have $M_r(A) = M_l(A)^{cop}$.

Now consider the special monoidal diagram scheme $\mathcal{D} := \mathcal{D}[X; m, u]$. To make things simpler we assume that $\mathcal{V}ec$ is strict monoidal. The category $\mathcal{D}[X; m, u]$ has the objects $X \otimes \ldots \otimes X = X^{\otimes n}$ for all $n \in \mathbb{N}$ (and $I := X^{\otimes 0}$) and the morphisms $m : X \otimes X \to X$, $u : I \to X$ and all morphisms formally constructed from m, u, id by taking tensor products and composition of morphisms.

Let A be an algebra with multiplication $m_A : A \otimes A \longrightarrow A$ and unit $u_A : \mathbb{K} \longrightarrow A$. Then $\omega_A : \mathcal{D} \longrightarrow \mathcal{C}$ defined by $\omega(X) = A$, $\omega(X^{\otimes n}) = A^{\otimes n}$, $\omega(m) = m_A$ and $\omega(u) = u_A$ is a strict monoidal functor. If A is finite dimensional then the diagram is finite. We get

Theorem 3.7.1. *Let A be a finite dimensional algebra. Then the algebra $M(A)$ coacting universally from the right on A (the right Tambara bialgebra) $M(A)$ and* $\mathrm{coend}(\omega_A)$ *are isomorphic as bialgebras.*

Proof. We have studied the Tambara bialgebra for left coaction $f : A \longrightarrow M(A) \otimes A$ but here we need the analogue for universal right coaction $f : A \longrightarrow A \otimes M(A)$ (see Problem (44).

Let B be an algebra and $f : A \longrightarrow A \otimes B$ be a homomorphism of algebras. For $\omega = \omega_A$ we define

$$\varphi(X^{\otimes n}) : \omega(X^{\otimes n}) = A^{\otimes n} \xrightarrow{f^{\otimes n}} A^{\otimes n} \otimes B^{\otimes n} \xrightarrow{1 \otimes m_B^n} A^{\otimes n} \otimes B = \omega(X^{\otimes n}) \otimes B,$$

where $m_B^n : B^{\otimes n} \longrightarrow B$ is the n-fold multiplication on B. The map φ is a natural transformation since the diagrams

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\varphi(I)"] \arrow[d, "u"'] & \mathbb{K} \otimes B \arrow[d, "1 \otimes u"] \\
A \arrow[r, "\varphi(X)"'] & A \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\varphi(X \otimes X)"] \arrow[dr, "f \otimes f"] \arrow[ddd, "m"'] & & A \otimes A \otimes B \arrow[ddd, "m \otimes 1"] \\
& A \otimes A \otimes B \otimes B \arrow[ur, "1 \otimes m"'] \arrow[d, "m \otimes m"] & \\
& A \otimes B \arrow[dr, "1 \otimes 1"] & \\
A \arrow[ur, "f"] \arrow[rr, "\varphi(X)"'] & & A \otimes B
\end{tikzcd}$$

commute. Furthermore the following commute

$$\begin{tikzcd}
A^{\otimes r} \otimes A^{\otimes s} \arrow[rr, "\varphi(X^{\otimes r}) \otimes \varphi(X^{\otimes s})"] \arrow[dr] \arrow[ddd] & & A^{\otimes r} \otimes A^{\otimes s} \otimes B \otimes B \arrow[ddd] \\
& A^{\otimes r} \otimes A^{\otimes s} \otimes B^{\otimes r} \otimes B^{\otimes s} \arrow[ur] \arrow[d] & \\
& A^{\otimes (r+s)} \otimes B^{\otimes (r+s)} \arrow[dr] & \\
A^{\otimes (r+s)} \arrow[ur] \arrow[rr, "\varphi(X^{\otimes (r+s)})"'] & & A^{\otimes (r+s)} \otimes B
\end{tikzcd}$$

so that $\varphi : \omega_A \to \omega_A \otimes B$ is a monoidal natural transformation.

Conversely let $\varphi : \omega_A \to \omega_A \otimes B$ be a natural transformation. Let $f := \varphi(X) : A \to A \otimes B$. Then the following commute

$$\begin{tikzcd}
A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "="'] & A \otimes A \otimes B \otimes B \arrow[d, "1 \otimes m"] \\
A \otimes A \arrow[r, "\varphi(X \otimes X)"] \arrow[d, "m"'] & A \otimes A \otimes B \arrow[d, "m \otimes 1"] \\
A \arrow[r, "f"] & A \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\cong"] \arrow[d, "="'] & \mathbb{K} \otimes \mathbb{K} \arrow[d] \\
\mathbb{K} \arrow[r] \arrow[d, "u"'] & \mathbb{K} \otimes B \arrow[d, "u \otimes 1"] \\
A \arrow[r, "f"] & A \otimes B.
\end{tikzcd}$$

Hence $f : A \to A \otimes B$ is a homomorphism of algebras.

Thus we have defined an isomorphism

$$\mathbb{K}\mathcal{A}lg\,(A, A \otimes B) \cong \mathrm{Nat}^{\otimes}(\omega_A, \omega_A \otimes B)$$

that is natural in B. If A is finite dimensional then the left hand side is represented by the Tambara bialgebra $M_r(A)$ and the right hand side by the bialgebra $\mathrm{coend}(\omega_A)$. Thus both bialgebras must be isomorphic. $\square$

Corollary 3.7.2. *There is a unique isomorphism of bialgebras $M_r(A) \cong \mathrm{coend}(\omega_A)$ such that the diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & A \otimes M_r(A) \arrow[d] \\
& A \otimes \mathrm{coend}(\omega_A)
\end{tikzcd}$$

commutes

Proof. This is a direct consequence of the universal property. $\square$

Thus the Tambara bialgebra that represents the universal quantum monoid acting on a finite quantum space may be reconstructed by the Tannaka-Krein reconstruction from representation theory. Similar reconstructions can be given for more complicated quantum spaces such as so called quadratic quantum spaces.

8. Reconstruction and $\mathcal{C}$-categories

Now we show that an arbitrary coalgebra C can be reconstructed by the methods introduced above from its (co-)representations or more precisely from the underlying functor $\omega : \mathcal{C}omod\text{-}C \to \mathcal{V}ec$. In this case one can not use the usual construction of $\operatorname{coend}(\omega)$ that is restricted to finite dimensional comodules.

The following Theorem is an example that shows that the restriction to finite dimensional comodules in general is too strong for Tannaka reconstruction. There may be universal coendomorphism bialgebras for more general diagrams. On the other hand the following Theorem also holds if one only considers finite dimensional corepresentations of C. However the proof then becomes somewhat more complicated.

Definition 3.8.1. Let $\mathcal{C}$ be a monoidal category. A category $\mathcal{D}$ together with a bifunctor $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ and natural isomorphisms $\beta : (A \otimes B) \otimes M \to A \otimes (B \otimes M)$, $\eta : I \otimes M \to M$ is called a *$\mathcal{C}$-category* if the following diagrams commute

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes M \arrow[r, "\alpha(A,B,C)\otimes 1"] \arrow[d, "\beta(A\otimes B,C,M)"] & (A \otimes (B \otimes C)) \otimes M \arrow[r, "\beta(A,B\otimes C,M)"] & A \otimes ((B \otimes C) \otimes M) \arrow[d, "1\otimes\beta(B,C,M)"'] \\
(A \otimes B) \otimes (C \otimes M) \arrow[rr, "\beta(A,B,C\otimes M)"] & & A \otimes (B \otimes (C \otimes M))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes M \arrow[rr, "\beta(A,I,M)"] \arrow[dr, "\rho(A)\otimes 1"'] & & A \otimes (I \otimes M) \arrow[dl, "1\otimes\eta(M)"] \\
& A \otimes M &
\end{tikzcd}$$

A $\mathcal{C}$-category is called *strict* if the morphisms β, η are the identities.

Let $(\mathcal{D}, \otimes)$ and $(\mathcal{D}', \otimes)$ be $\mathcal{C}$-categories. A functor $\mathcal{F} : \mathcal{D} \to \mathcal{D}'$ together with a natural transformation $\zeta(A, M) : A \otimes \mathcal{F}(M) \to \mathcal{F}(A \otimes M)$ is called a *weak $\mathcal{C}$-functor* if the following diagrams commute

$$\begin{tikzcd}
(A \otimes B) \otimes \mathcal{F}(M) \arrow[rr, "\zeta"] \arrow[d, "\beta"'] & & \mathcal{F}((A \otimes B) \otimes M) \arrow[d, "\mathcal{F}(\beta)"] \\
A \otimes (B \otimes \mathcal{F}(M)) \arrow[r, "1\otimes\zeta"] & A \otimes \mathcal{F}(B \otimes M) \arrow[r, "\zeta"] & \mathcal{F}(A \otimes (B \otimes M))
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes \mathcal{F}(M) \arrow[rr, "\zeta"] \arrow[dr, "\eta"'] & & \mathcal{F}(I \otimes M) \arrow[dl, "\mathcal{F}(\eta)"] \\
& \mathcal{F}(M) &
\end{tikzcd}$$

If, in addition, ζ is an isomorphism then we call $\mathcal{F}$ a $\mathcal{C}$-functor. The functor is called a *strict $\mathcal{C}$-functor* if ζ is the identity morphism.

A natural transformation $\varphi : \mathcal{F} \longrightarrow \mathcal{F}'$ between (weak) $\mathcal{C}$-functors is called a *$\mathcal{C}$-transformation* if

$$\begin{tikzcd} A \otimes \mathcal{F}(M) \arrow[r, "\zeta"] \arrow[d, "1_A \otimes \varphi(M)"'] & \mathcal{F}(A \otimes M) \arrow[d, "\varphi(A \otimes M)"] \\ A \otimes \mathcal{F}'(M) \arrow[r, "\zeta'"'] & \mathcal{F}'(A \otimes M) \end{tikzcd}$$

commutes.

Example 3.8.2. Let C be a coalgebra and $\mathcal{C} := \mathcal{V}ec$. Then the category $\mathcal{C}omod\text{-}C$ of right C-comodules is a $\mathcal{C}$-category since $N \in \mathcal{C}omod\text{-}C$ and $V \in \mathcal{C} = \mathcal{V}ec$ implies that $V \otimes N$ is a comodules with the comodule structure of N.

The underlying functor $\omega : \mathcal{C}omod\text{-}C \longrightarrow \mathcal{V}ec$ is a strict $\mathcal{C}$-functor since we have $V \otimes \omega(N) = \omega(V \otimes N)$. Similarly $\omega \otimes M : \mathcal{C}omod\text{-}C \longrightarrow \mathcal{V}ec$ is a $\mathcal{C}$-functor since $V \otimes (\omega(N) \otimes M) \cong \omega(V \otimes N) \otimes M$.

Lemma 3.8.3. *Let C be a coalgebra. Let $\omega : \mathcal{C}omod\text{-}C \longrightarrow \mathcal{V}ec$ be the underlying functor. Let $\varphi : \omega \longrightarrow \omega \otimes M$ be a natural transformation. Then φ is a $\mathcal{C}$-transformation with $\mathcal{C} = \mathcal{V}ec$.*

Proof. It suffices to show $1_V \otimes \varphi(N) = \varphi(V \otimes N)$ for an arbitrary comodule N. We show that the diagram

$$\begin{tikzcd} V \otimes N \arrow[r, "\varphi(V \otimes N)"] \arrow[d, "1"'] & V \otimes N \otimes M \arrow[d, "1"] \\ V \otimes N \arrow[r, "1_V \otimes \varphi(N)"'] & V \otimes N \otimes M \end{tikzcd}$$

commutes. Let (v_i) be a basis of V. For an arbitrary vector space W let $p_i : V \otimes W \longrightarrow W$ be the projections defined by $p_i(t) = p_i(\sum_j v_j \otimes w_j) = w_i$ where $\sum_j v_j \otimes w_j$ is the unique representation of an arbitrary tensor in $V \otimes W$. So we get

$$t = \sum_i v_i \otimes p_i(t)$$

for all $t \in V \otimes W$. Now we consider $V \otimes N$ as a comodule by the comodule structure of N. Then the $p_i : V \otimes N \longrightarrow N$ are homomorphisms of comodules. Hence all diagrams of the form

$$\begin{tikzcd} V \otimes N \arrow[r, "\varphi(V \otimes N)"] \arrow[d, "p_i"'] & V \otimes N \otimes M \arrow[d, "p_i \otimes M"] \\ N \arrow[r, "\varphi(N)"'] & N \otimes M. \end{tikzcd}$$

commute. Expressed in formulas this means $\varphi(N)p_i(t) = p_i\varphi(V \otimes N)(t)$ for all $t \in V \otimes N$. Hence we have

$$\begin{aligned}(1_V \otimes \varphi(N))(t) = (1_V \otimes \varphi(N))(\textstyle\sum v_i \otimes p_i(t)) = \sum v_i \otimes \varphi(N)p_i(t) \\ = \textstyle\sum_i v_i \otimes p_i\varphi(V \otimes N)(t) = \varphi(V \otimes N)(t)\end{aligned}$$

So we have $1_V \otimes \varphi(N) = \varphi(V \otimes N)$ as claimed. $\square$

We prove the following Theorem only for the category $\mathcal{C} = \mathcal{V}ec$ of vector spaces. The Theorem holds in general and says that in an arbitrary symmetric monoidal category $\mathcal{C}$ the coalgebra C represents the functor $\mathcal{C}\text{-}\operatorname{Nat}(\omega, \omega \otimes M) \cong \operatorname{Mor}_{\mathcal{C}}(C, M)$ of natural $\mathcal{C}$-transformations.

Theorem 3.8.4. (Reconstruction of coalgebras) *Let C be a coalgebra. Let $\omega : \mathcal{C}omod\text{-}C \longrightarrow \mathcal{V}ec$ be the underlying functor. Then $C \cong \operatorname{coend}(\omega)$.*

Proof. Let M in $\mathcal{V}ec$ and let $\varphi : \omega \longrightarrow \omega \otimes M$ be a natural transformation. We define the homomorphism $\widetilde{\varphi} : C \longrightarrow M$ by $\widetilde{\varphi} = (\epsilon \otimes 1)\varphi(C)$ using the fact that C is a comodule.

Let N be a C-comodule. Then N is a subcomodule of $N \otimes C$ by $\delta : N \longrightarrow N \otimes C$ since the diagram

$$\begin{tikzcd}
N \arrow[r, "\delta"] \arrow[d, "\delta"'] & N \otimes C \arrow[d, "1\otimes\Delta"] \\
N \otimes C \arrow[r, "\delta\otimes 1"'] & N \otimes C \otimes C
\end{tikzcd}$$

commutes. Thus the following diagram commutes

$$\begin{tikzcd}
N \arrow[r, "\delta"] \arrow[d, "\varphi(N)"'] & N \otimes C \arrow[d, "\varphi(N\otimes C)=1_N\otimes\varphi(C)"'] \arrow[ddr, "1\otimes\widetilde{\varphi}"] & \\
N \otimes M \arrow[r, "\delta\otimes 1"] \arrow[drr, "1"'] & N \otimes C \otimes M \arrow[dr, "1\otimes\epsilon\otimes 1"] & \\
 & & N \otimes M
\end{tikzcd}$$

In particular we have shown that the diagram

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \omega \otimes C \arrow[d, "1\otimes\widetilde{\varphi}"] \\
 & \omega \otimes M
\end{tikzcd}$$

commutes.

To show the uniqueness of $\widetilde{\varphi}$ let $g : C \longrightarrow M$ be another homomorphism with $(1 \otimes g)\delta = \varphi$. For $c \in C$ we have $g(c) = g(\epsilon \otimes 1)\Delta(c) = (\epsilon \otimes 1)(1 \otimes g)\Delta(c) = (\epsilon \otimes 1)\varphi(C)(c) = \widetilde{\varphi}(c)$.

The coalgebra structure from Corollary 3.5.1 is the original coalgebra structure of C. This can be seen as follows. The comultiplication $\delta : \omega \longrightarrow \omega \otimes C$ is a natural transformation hence $(\delta \otimes 1_C)\delta : \omega \longrightarrow \omega \otimes C \otimes C$ is also a natural transformation. As in Corollary 3.5.1 this induced a unique homomorphism $\Delta : C \longrightarrow C \otimes C$ so that the diagram

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[d, "\delta"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\Delta"] \\
\omega \otimes \mathrm{coend}(\omega) \arrow[r, "\delta\otimes 1"'] & \omega \otimes \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega)
\end{tikzcd}$$

commutes. In a similar way the natural isomorphism $\omega \cong \omega \otimes \mathbb{K}$ induces a unique homomorphism $\epsilon : C \longrightarrow \mathbb{K}$ so that the diagram

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[dr, "\mathrm{id}_\omega"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\epsilon"] \\
 & \omega \otimes I
\end{tikzcd}$$

commutes. Because of the uniqueness these must be the structure homomorphisms of C. $\square$

We need a more general version of this Theorem in the next chapter. So let C be a coalgebra. Let $\omega : \mathcal{C}omod\text{-}C \longrightarrow \mathcal{V}ec$ be the underlying functor and $\delta : \omega \longrightarrow \omega \otimes C$ the universal natural transformation for $C \cong \mathrm{coend}(\omega)$.

We use the permutation map τ on the tensor product that gives the natural isomorphism

$$\tau : N_1 \otimes T_1 \otimes N_2 \otimes T_2 \otimes \ldots \otimes N_n \otimes T_n \cong N_1 \otimes N_2 \otimes \ldots \otimes N_n \otimes T_1 \otimes T_2 \ldots \otimes \otimes T_n$$

which is uniquely determined by the coherence theorems and is constructed by suitable applications of the flip $\tau : N \otimes T \cong T \otimes N$.

Let $\omega^n : \mathcal{C}omod\text{-}C \times \mathcal{C}omod\text{-}C \times \ldots \times \mathcal{C}omod\text{-}C \longrightarrow \mathcal{V}ec$ be the functor $\omega^n(N_1, N_2, \ldots, N_n) = \omega(N_1) \otimes \omega(N_2) \otimes \ldots \otimes \omega(N_n)$. For notational convenience we abbreviate $\{N\}^n := N_1 \otimes N_2 \otimes \ldots \otimes N_n$, similarly $\{C\}^n = C \otimes C \otimes \ldots \otimes C$ and $\{f\}^n := f_1 \otimes f_2 \otimes \ldots \otimes f_n$. So we get $\tau : \{N \otimes T\}^n \cong \{N\}^n \otimes \{T\}^n$.

Lemma 3.8.5. *Let $\varphi : \omega^n \longrightarrow \omega^n \otimes M$ be a natural transformation. Then φ is a $\mathcal{C}$-transformation in the sense that the diagrams*

$$\begin{tikzcd}
\{V \otimes N\}^n \arrow[r, "\varphi(V_1 \otimes N_1, \ldots, V_n \otimes N_n)"] \arrow[d, "\tau"'] & \{V \otimes N\}^n \otimes M \arrow[d, "\tau \otimes M"] \\
\{V\}^n \otimes \{N\}^n \arrow[r, "\{V\}^n \otimes \varphi(N_1, \ldots, N_n)"'] & \{V\}^n \otimes \{N\}^n \otimes M
\end{tikzcd}$$

commute for all vector spaces V_i and C-comodules N_i.

Proof. Choose bases $\{v_{ij}\}$ of the vector spaces V_i with corresponding projections $p_{ij} : V_i \otimes N_i \longrightarrow N_i$. Then we have $\tau(t_1 \otimes \ldots \otimes t_n) = \sum v_{1i_1} \otimes \ldots \otimes v_{ni_n} \otimes p_{1i_1}(t_1) \otimes \ldots \otimes p_{ni_n}(t_n)$ so $\tau = \sum v_{1i_1} \otimes \ldots \otimes v_{ni_n} \otimes \{p\}^n$.

The $p_{ij_i} : V_i \otimes N_i \longrightarrow N_i$ are homomorphisms of C-comodules. Hence the diagrams

$$\begin{tikzcd}
\{V \otimes N\}^n \arrow[r, "\varphi(V_1 \otimes N_1, \ldots, V_n \otimes N_n)"] \arrow[d, "\{p\}^n"'] & \{V \otimes N\}^n \otimes M \arrow[d, "\{p\}^n \otimes M"] \\
\{N\}^n \arrow[r, "\varphi(N_1, \ldots, N_n)"'] & \{N\}^n \otimes M
\end{tikzcd}$$

commute for all choices of $\{p\}^n = p_{1i_1} \otimes \ldots \otimes p_{ni_n}$.

So we get for all $t_i \in V_i \otimes N_i$

$$\begin{aligned}
&(\{V\}^n \otimes \varphi(N_1, \ldots, N_n))\tau(t_1 \otimes \ldots \otimes t_n) = \\
&= (\{V\}^n \otimes \varphi(N_1, \ldots, N_n))(\textstyle\sum v_{1i_1} \otimes \ldots \otimes v_{ni_n} \otimes p_{1i_1}(t_1) \otimes \ldots \otimes p_{ni_n}(t_n)) \\
&= \textstyle\sum v_{1i_1} \otimes \ldots \otimes v_{ni_n} \otimes \varphi(N_1, \ldots, N_n)\{p\}^n(t_1 \otimes \ldots \otimes t_n) \\
&= \textstyle\sum v_{1i_1} \otimes \ldots \otimes v_{ni_n} \otimes (\{p\}^n \otimes M)\varphi(V_1 \otimes N_1, \ldots, V_n \otimes N_n)(t_1 \otimes \ldots \otimes t_n) \\
&= (\tau \otimes M)\varphi(V_1 \otimes N_1, \ldots, V_n \otimes N_n)(t_1 \otimes \ldots \otimes t_n).
\end{aligned}$$

□

Theorem 3.8.6. *With the notation given above we have*

$$\mathrm{coend}(\omega^n) \cong C \otimes C \otimes \ldots \otimes C$$

with the universal natural transformation

$$\begin{gathered}
\delta^{(n)}(N_1, N_2, \ldots, N_n) := \tau(\delta(N_1) \otimes \delta(N_2) \otimes \ldots \otimes \delta(N_n)) : \\
\omega(N_1) \otimes \omega(N_2) \otimes \ldots \otimes \omega(N_n) \longrightarrow \omega(N_1) \otimes C \otimes \omega(N_2) \otimes C \otimes \ldots \otimes \omega(N_n) \otimes C \\
\cong \omega(N_1) \otimes \omega(N_2) \otimes \ldots \otimes \omega(N_n) \otimes C \otimes C \otimes \ldots \otimes C.
\end{gathered}$$

Proof. We proceed as in the proof of the previous Theorem.

Let M in $\mathcal{V}ec$ and let $\varphi : \omega^n \longrightarrow \omega^n \otimes M$ be a natural transformation. We define the homomorphism $\widetilde{\varphi} : C^n = \omega(C) \otimes \omega(C) \otimes \ldots \otimes \omega(C) = C \otimes C \otimes \ldots \otimes C \longrightarrow M$ by $\widetilde{\varphi} = (\varepsilon^n \otimes 1_M)\varphi(C, \ldots, C)$ using the fact that C is a comodule.

As in the preceding proof we get that $\delta : N_i \to N_i \otimes C$ are homomorphisms of C-comodules. Thus the following diagram commutes

$$\begin{tikzcd}
N_1 \otimes \ldots \otimes N_n \arrow[r, "\delta\otimes\ldots\otimes\delta"] \arrow[d, "\varphi(N_1\otimes\ldots\otimes N_n)"'] & N_1 \otimes C \otimes \ldots \otimes N_n \otimes C \arrow[r, "\tau"] \arrow[d, "\varphi(N_1\otimes C,\ldots,N_n\otimes C)"'] & N_1 \otimes \ldots \otimes N_n \otimes C \otimes \ldots \otimes C \arrow[d, "N_1\otimes\ldots\otimes N_n\otimes\varphi(C,\ldots,C)"'] \\
N_1 \otimes \ldots \otimes N_n \otimes M \arrow[r, "\delta\otimes\ldots\otimes\delta\otimes M"] \arrow[drr, "1"] & N_1 \otimes C \otimes \ldots \otimes N_n \otimes C \otimes M \arrow[r, "\tau\otimes M"] \arrow[dr, "\{1\otimes\varepsilon\}^n\otimes 1"] & N_1 \otimes \ldots \otimes N_n \otimes C \otimes \ldots \otimes C \otimes M \arrow[d, "1\otimes\{\epsilon\}^n\otimes 1"'] \\
 & & N_1 \otimes \ldots \otimes N_n \otimes M
\end{tikzcd}$$

Hence we get the commutative diagram

$$\begin{tikzcd}
\omega^n \arrow[r, "\delta^{(n)}"] \arrow[dr, "\varphi"'] & \omega^n \otimes \{C\}^n \arrow[d, "1\otimes\widetilde{\varphi}"] \\
 & \omega^n \otimes M
\end{tikzcd}$$

To show the uniqueness of $\widetilde{\varphi}$ let $g : C^n \to M$ be another homomorphism with $(1_{\omega^n} \otimes g)\delta^{(n)} = \varphi$. We have $g = g(\varepsilon^n \otimes 1_{C^n})\tau\Delta^n = g(\varepsilon^n \otimes 1_{C^n})\delta^{(n)}(C, \ldots, C) = (\varepsilon^n \otimes 1_M)(1_{C^n} \otimes g)\delta^{(n)}(C, \ldots, C) = (\varepsilon^n \otimes 1_M)\varphi(C, \ldots, C) = \widetilde{\varphi}$. $\square$

Now we prove the finite dimensional case of reconstruction of coalgebras.

Proposition 3.8.7. (Reconstruction) *Let C be a coalgebra. Let $\mathcal{C}omod_0$-C be the category of finite dimensional C-comodules and $\omega : \mathcal{C}omod_0\text{-}C \to \mathcal{V}ec$ be the underlying functor. Then we have $C \cong \mathrm{coend}(\omega)$.*

Proof. Let M be in $\mathcal{V}ec$ and let $\varphi : \omega \to \omega \otimes M$ be a natural transformation. We define the homomorphism $\widetilde{\varphi} : C \to M$ as follows. Let $c \in C$. Let N be a finite dimensional C-subcomodule of C containing c. Then we define $g(c) := (\epsilon|_N \otimes 1)\varphi(N)(c)$. If N' is another finite dimensional subcomodule of C with $c \in N'$ and with $N \subseteq N'$ then the following commutes

$$\begin{tikzcd}
N \arrow[r, "\varphi(N)"] \arrow[d] & N \otimes M \arrow[d] \arrow[dr] & \\
 & & C \otimes M \arrow[r, "\epsilon\otimes 1"] & M \\
N' \arrow[r, "\varphi(N')"] & N' \otimes M \arrow[ur] &
\end{tikzcd}$$

Thus the definition of $\widetilde{\varphi}(c)$ is independent of the choice of N. Furthermore $\widetilde{\varphi} : N \to M$ is obviously a linear map. For any two elements $c, c' \in C$ there is a finite dimensional subcomodule $N \subseteq C$ with $c, c' \in N$ e. g. the sum of the finite dimensional subcomodules containing c and c' separately. Thus $\widetilde{\varphi}$ can be extended to all of C.

The rest of the proof is essentially the same as the proof of the first reconstruction theorem. $\square$

The representations allow to reconstruct further structure of the coalgebra. We prove a reconstruction theorem about bialgebras. Recall that the category of B-comodules over a bialgebra B is a monoidal category, furthermore that the underlying functor $\omega : \mathcal{C}omod\text{-}B \to \mathcal{V}ec$ is a monoidal functor. From this information we can reconstruct the full bialgebra structure of B. We have

Theorem 3.8.8. *Let B be a coalgebra. Let $\mathcal{C}omod\text{-}B$ be a monoidal category such that the underlying functor $\omega : \mathcal{C}omod\text{-}B \to \mathcal{V}ec$ is a monoidal functor. Then there is a unique bialgebra structure on B that induces the given monoidal structure on the corepresentations.*

PROOF. First we prove the uniqueness of the multiplication $\nabla : B \otimes B \to B$ and of the unit $\eta : \mathbb{K} \to B$. The natural transformation $\delta : \omega \to \omega \otimes B$ becomes a monoidal natural transformation with $\nabla : B \otimes B \to B$ and $\eta : \mathbb{K} \to B$ We show that ∇ and η are uniquely determined by ω and δ.

Let $\nabla' : B \otimes B \to B$ and $\eta' : B \to \mathbb{K}$ be morphisms that make δ a monoidal natural transformation. The diagrams

$$\begin{tikzcd}
\omega(X) \otimes \omega(Y) \arrow[r, "\delta(X)\otimes\delta(Y)"] \arrow[d, "\rho"'] & \omega(X) \otimes \omega(Y) \otimes B \otimes B \arrow[d, "\rho\otimes\nabla'"] \\
\omega(X \otimes Y) \arrow[r, "\delta(X\otimes Y)"] & \omega(X \otimes Y) \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\cong"] \arrow[d] & \mathbb{K} \otimes \mathbb{K} \arrow[d, "1 \otimes \eta'"] \\
\omega(\mathbb{K}) \arrow[r, "\delta(\mathbb{K})"] & \omega(\mathbb{K}) \otimes B
\end{tikzcd}$$

commute. In particular the following diagrams commute

$$\begin{tikzcd}
\omega(B) \otimes \omega(B) \arrow[r, "\delta(B)\otimes\delta(B)"] \arrow[d, "\rho"'] & \omega(B) \otimes \omega(B) \otimes B \otimes B \arrow[d, "\rho\otimes\nabla'"] \\
\omega(B \otimes B) \arrow[r, "\delta(B\otimes B)"] & \omega(B \otimes B) \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\cong"] \arrow[d] & \mathbb{K} \otimes \mathbb{K} \arrow[d, "1 \otimes \eta'"] \\
\omega(\mathbb{K}) \arrow[r, "\delta(\mathbb{K})"] & \omega(\mathbb{K}) \otimes B
\end{tikzcd}$$

Hence we get $\sum b_{(1)} \otimes c_{(1)} \otimes b_{(2)} c_{(2)} = \sum b_{(1)} \otimes c_{(1)} \otimes \nabla'(b_{(2)} \otimes c_{(2)})$ and $1 \otimes 1 = 1 \otimes \eta'(1)$. This implies $bc = \sum \epsilon(b_{(1)})\epsilon(c_{(1)}) b_{(2)} c_{(2)} = \sum \epsilon(b_{(1)})\epsilon(c_{(1)}) \nabla'(b_{(2)} \otimes c_{(2)}) = \nabla'(b \otimes c)$ and $1 = \eta'(1)$.

Now we show the existence of a bialgebra structure. Let B be a coalgebra only and let $\omega : \mathcal{C}omod\text{-}B \longrightarrow \mathcal{V}ec$ be a monoidal functor with $\xi : \omega(M) \otimes \omega(N) \longrightarrow \omega(M \otimes N)$ and $\xi_0 : \mathbb{K} \longrightarrow \omega(\mathbb{K})$. First we observe that the new tensor product between the comodules M and N coincides with the tensor product of the underlying vector spaces (up to an isomorphism ξ). Because of the coherence theorems for monoidal categories (that also hold in our situation) we may identify along the maps ξ and ξ_0.

We define $\eta := (\mathbb{K} \stackrel{\delta(\mathbb{K})}{\longrightarrow} \mathbb{K} \otimes B \cong B)$ and $\nabla := (B \otimes B \stackrel{\delta(B \otimes B)}{\longrightarrow} B \otimes B \otimes B \stackrel{\epsilon \otimes \epsilon \otimes 1_B}{\longrightarrow} \mathbb{K} \otimes \mathbb{K} \otimes B \cong B)$.

Since the structural morphism for the comodule $\delta : M \longrightarrow M \otimes B$ is a homomorphism of of B comodules where the comodule structure on $M \otimes B$ is only given by the diagonal of B that is the $\mathcal{C}$-structure on $\omega : \mathcal{C}omod\text{-}B \longrightarrow \mathcal{V}ec$ we get that also $\delta(M) \otimes \delta(N) : M \otimes N \longrightarrow M \otimes N \otimes B$ is a comodule homomorphism. Hence the first square in the following diagram commutes

$$\begin{tikzcd}
M \otimes N \arrow[r, "\delta(M)\otimes\delta(N)"] \arrow[d, "\delta(M\otimes N)"'] & M \otimes B \otimes N \otimes B \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "\delta(M\otimes B\otimes N\otimes B)"'] & M \otimes N \otimes B \otimes B \arrow[d, "1\otimes 1\otimes\delta(B\otimes B)"'] \\
M \otimes N \otimes B \arrow[r, "\delta(M)\otimes\delta(N)\otimes 1_B"] & M \otimes B \otimes N \otimes B \otimes B \arrow[r, "1\otimes\tau\otimes 1\otimes 1"] & M \otimes N \otimes B \otimes B \otimes B
\end{tikzcd}$$

The second square commutes by a similar reasoning since the comodule structure on $M \otimes B$ resp. $N \otimes B$ is given by the diagonal on B hence $M \otimes N$ can be factored out of the natural ($\mathcal{C}$-)transformation. Now we attach

$$1_M \otimes 1_N \otimes \epsilon \otimes \epsilon \otimes 1_B : M \otimes N \otimes B \otimes B \otimes B \longrightarrow M \otimes N \otimes B$$

to the commutative rectangle and obtain $\delta(M \otimes N) = (1_M \otimes 1_N \otimes \nabla)(1 \otimes \tau \otimes 1)(\delta(M) \otimes \delta(N)$. Thus the comodule structure on $M \otimes N$ is induced by the multiplication $\nabla : B \otimes B \longrightarrow B$ defined above.

So the following diagrams commute

$$\begin{tikzcd}
B \otimes B \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "1_B\otimes 1_B"] & B \otimes B \otimes B \otimes B \arrow[rr, "1\otimes\tau\otimes 1"] & & B \otimes B \otimes B \otimes B \arrow[d, "1\otimes\nabla"] \\
B \otimes B \arrow[rrr, "\delta(B\otimes B)"] \arrow[dr, "\delta(B\otimes B)"] \arrow[dd, "\nabla"'] & & & B \otimes B \otimes B \arrow[dl, "\delta(B\otimes B)\otimes 1"'] \arrow[dd, "\nabla\otimes 1"] \\
& B \otimes B \otimes B \arrow[r, "1\otimes 1\otimes\Delta"] \arrow[dl, "\epsilon\otimes\epsilon\otimes 1"] & B \otimes B \otimes B \otimes B \arrow[dr, "\epsilon\otimes\epsilon\otimes 1\otimes 1"] & \\
B \arrow[rrr, "\Delta"] & & & B \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
B \otimes B \arrow[rr, "\nabla"] \arrow[dr, "\delta(B\otimes B)"] \arrow[dd, "1"'] & & B \arrow[ddl, "\epsilon"] \\
& B \otimes B \otimes B \arrow[ur, "\epsilon\otimes\epsilon\otimes 1"'] \arrow[dl, "1\otimes 1\otimes\epsilon"] & \\
B \otimes B \arrow[r, "\epsilon\otimes\epsilon"] & \mathbb{K} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{K} \arrow[rrr, "\eta"] \arrow[drr, "\delta(\mathbb{K})"] \arrow[ddr, "\delta(\mathbb{K})"] \arrow[ddd, "\eta"'] & & & B \arrow[dd, "\Delta"] \\
& & \mathbb{K} \otimes B \arrow[ur, "\cong"'] \arrow[d, "1\otimes\Delta"] & \\
& \mathbb{K} \otimes B \arrow[r, "\delta(\mathbb{K})\otimes 1"] \arrow[dl, "\cong"] \arrow[dr, "\eta\otimes 1"] & \mathbb{K} \otimes B \otimes B \arrow[r, "\cong"] \arrow[d, "\cong"] & B \otimes B \arrow[dl, "1_B"] \\
B \arrow[rr, "\eta\otimes 1"] & & B \otimes B &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \arrow[rr, "1"] \arrow[dr, "\delta(\mathbb{K})"] \arrow[ddr, "\eta"'] & & \mathbb{K} \\
& \mathbb{K} \otimes B \arrow[ur, "1\otimes\epsilon"'] \arrow[d, "\cong"] & \\
& B \arrow[uur, "\epsilon"'] &
\end{tikzcd}$$

Hence η and ∇ are coalgebra homomorphisms.

To show the associativity of ∇ we identify along the maps $\alpha : (M \otimes N) \otimes P \cong M \otimes (N \otimes P)$ and furthermore simplify the relevant diagram by fixing that σ represents a suitable permutation of the tensor factors. Then the following commute

$$\begin{tikzcd}
B \otimes B \otimes B \arrow[r, "\sigma(\delta(B)\otimes\delta(B)\otimes\delta(B))"] \arrow[d, "1"] & B \otimes B \otimes B \otimes B \otimes B \otimes B \arrow[r, "\epsilon\otimes\epsilon\otimes\epsilon\otimes 1"] \arrow[d, shift right, "1\otimes(\nabla\otimes 1)"'] \arrow[d, shift left, "1\otimes(1\otimes\nabla)"] & B \otimes B \otimes B \arrow[d, shift right, "\nabla\otimes 1"'] \arrow[d, shift left, "1\otimes\nabla"] \\
B \otimes B \otimes B \arrow[r, shift left, "\sigma(\delta(B\otimes B)\otimes\delta(B))"] \arrow[r, shift right, "\sigma(\delta(B)\otimes\delta(B\otimes B))"'] \arrow[d, "1"] & B \otimes B \otimes B \otimes B \otimes B \otimes B \arrow[r, "\epsilon\otimes\epsilon\otimes\epsilon\otimes 1"] \arrow[d, "1\otimes\nabla"] & B \otimes B \arrow[d, "\nabla"] \\
B \otimes B \otimes B \arrow[r, "\delta(B\otimes B\otimes B)"] & B \otimes B \otimes B \otimes B \arrow[r, "\epsilon\otimes\epsilon\otimes\epsilon\otimes 1"] & B
\end{tikzcd}$$

The upper row is the identity hence we get the associative law.

For the proof that η has the properties of a unit we must explicitly consider the coherence morphisms λ and ρ By reasons of symmetry we will only show one half of of the unit axiom. This axiom follows from the commutativity of the following diagram

$$\begin{tikzcd}
B \arrow[r, "\delta(B)"] \arrow[d, "\rho^{-1}"'] & B \otimes B \arrow[r, "\rho^{-1}"] & B \otimes B \otimes \mathbb{K} \arrow[rr, "\rho"] \arrow[d, "1\otimes 1\otimes\delta(\mathbb{K})"] & & B \otimes B \arrow[r, "\epsilon\otimes 1"] \arrow[d, "1\otimes 1\otimes\eta"'] & B \arrow[d, "1\otimes\eta"'] \\
B \otimes \mathbb{K} \arrow[rr, "\delta(B)\otimes\delta(\mathbb{K})"] \arrow[urr, "\delta(B)\otimes 1"] \arrow[d, "="'] & & B \otimes B \otimes \mathbb{K} \otimes B \arrow[r, "1\otimes\tau\otimes 1"] & B \otimes \mathbb{K} \otimes B \otimes B \arrow[r, "\rho\otimes 1\otimes 1"] \arrow[d, "1\otimes 1\otimes\nabla"] & B \otimes B \otimes B \arrow[r, "\epsilon\otimes 1\otimes 1"] \arrow[dd, "1\otimes\nabla"'] & B \otimes B \arrow[dd, "\nabla"'] \\
B \otimes \mathbb{K} \arrow[rrr, "\delta(B\otimes\mathbb{K})"] \arrow[d, "\rho"'] & & & B \otimes \mathbb{K} \otimes B \arrow[dr, "\rho\otimes 1"] & & \\
B \arrow[rrrr, "\delta(B)"] & & & & B \otimes B \arrow[r, "\epsilon\otimes 1"] & B
\end{tikzcd}$$

□

CHAPTER 4

The Infinitesimal Theory

1. Integrals and Fourier Transforms

Assume for this chapter that $\mathbb{K}$ is a field.

Lemma 4.1.1. *Let C be a finite dimensional coalgebra. Every right C-comodule M is a left C^*-module by $c^*m = \sum m_{(M)}\langle c^*, m_{(1)}\rangle$ and conversely by $\delta(m) = \sum_i c_i^* m \otimes c_i$ where $\sum c_i^* \otimes c_i$ is the dual basis.*

Proof. We check that M becomes a left C^*-module

$$\begin{aligned}(c^*c'^*)m &= \sum m_{(M)}\langle c^*c'^*, m_{(1)}\rangle = \sum m_{(M)}\langle c^*, m_{(1)}\rangle\langle c'^*, m_{(2)}\rangle \\ &= c^* \sum m_{(M)}\langle c'^*, m_{(1)}\rangle = c^*(c'^* m).\end{aligned}$$

It is easy to check that the two constructions are inverses of each other. In particular assume that M is a right C-comodule. Choose m_i such that $\delta(m) = \sum m_i \otimes c_i$. Then $c_j^* m = \sum m_i \langle c_j^*, c_i\rangle = m_j$ and $\sum c_i^* m \otimes c_i = \sum m_i \otimes c_i = \delta(m)$. $\square$

Definition 4.1.2. 1. Let A be an algebra with augmentation $\varepsilon : A \to \mathbb{K}$, an algebra homomorphism. Let M be a left A-module. Then ${}^AM = \{m \in M | am = \varepsilon(a)m\}$ is called the *space of left invariants* of M.

This defines a functor ^{A}- : A-$\mathcal{M}od \to \mathcal{V}ec$.

2. Let C be a coalgebra with a grouplike element $1 \in C$. Let M be a right C-comodule. Then $M^{coC} := \{m \in M | \delta(m) = m \otimes 1\}$ is called the *space of right coinvariants* of M.

This defines a functor -^{coC} : $\mathcal{C}omod$-$C \to \mathcal{V}ec$.

Lemma 4.1.3. *Let C be a finite dimensional coalgebra with a grouplike element $1 \in C$. Then $A := C^*$ is an augmented algebra with augmentation $\varepsilon : C^* \ni a \mapsto \langle a, 1\rangle \in \mathbb{K}$. Let M be a right C-comodule. Then M is a left C^*-module and we have*

$${}^{C^*}M = M^{coC}.$$

Proof. Since $1 \in C$ is grouplike we have $\varepsilon_A(ab) = \langle ab, 1\rangle = \langle a, 1\rangle\langle b, 1\rangle = \varepsilon_A(a)\varepsilon_A(b)$ and $\varepsilon_A(1_A) = \langle 1_A, 1_C\rangle = \varepsilon_C(1_C) = 1$.

We have $m \in M^{coH}$ iff $\delta(m) = \sum m_{(M)} \otimes m_{(1)} = m \otimes 1$ iff $\sum m_{(M)}\langle a, m_{(1)}\rangle = m\langle a, 1\rangle$ for all $a \in A = C^*$ and by identifying $C^* \otimes C = \mathrm{Hom}(C^*, C^*)$ iff $am = \varepsilon_A(a)m$ iff $m \in {}^AM$. $\square$

Remark 4.1.4. The theory of Fourier transforms contains the following statements. Let H be the (Schwartz) space of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$, such that f and all derivatives rapidly decrease at infinity. (f decreases rapidly at infinity if $|x|^m f(x)$ is bounded for all m.) This space is an algebra (without unit) under the multiplication of values. There is a second multiplication on H, the convolution

$$(f * g)(x) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} f(t)g(x-t)dt.$$

The Fourier transform is a homomorphism $\hat{-} : H \longrightarrow H$ defined by

$$\hat{f}(x) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} f(t)e^{-itx}dt.$$

It satisfies the identity $(f * g)\hat{} = \hat{f}\hat{g}$ hence it is an algebra homomorphism. We want to find an analogue of this theory for finite quantum groups.

A similar example is the following. Let G be a locally compact topological group. Let μ be the (left) *Haar measure* on G and $\int f := \int_G f(x)d\mu(x)$ be the *Haar integral.*

The Haar measure is left invariant in the sense that $\mu(E) = \mu(gE)$ for all $g \in G$ and all compact subsets E of G. The Haar measure exists and is unique up to a positive factor. The Haar integral is translation invariant i.e. for all $y \in G$ we have $\int f(yx)d\mu(x) = \int f(x)d\mu(x)$.

If μ is a left-invariant Haar measure then there is a continuous homomorphism $\mod : G \longrightarrow (\mathbb{R}^+, \cdot)$ such that $\int f(xy^{-1})d\mu(x) = \mod(y) \int(f(x)d\mu(x)$. The homomorphism μ does not depend on f and is called the *modulus* of G. The group G is called *unimodular* if the homomorphism $\mod$ is the identity.

If G is a compact, or discrete, or Abelian group, or a connected semisimple or nilpotent Lie group, then G is unimodular.

Let G be a quantum group (or a quantum monoid) with function algebra H an arbitrary Hopf algebra. We also use the algebra of linear functionals $H^* = \mathrm{Hom}(H, \mathbb{K})$ (called the bialgebra of G in the French literature). The operation $H^* \otimes H \ni a \otimes f \mapsto \langle a, f \rangle \in \mathbb{K}$ is nondegenerate on both sides. We denote the elements of H by $f, g, h \in H$, the elements of H^* by $a, b, c \in H^*$, the (non existing) elements of the quantum group G by $x, y, z \in G$.

Remark 4.1.5. In 2.4.8 we have seen that the dual vector space H^* of a finite dimensional Hopf algebra H is again a Hopf algebra. The Hopf algebra structures are connected by the evaluation bilinear form

$$\mathrm{ev} : H^* \otimes H \ni a \otimes f \mapsto \langle a, f \rangle \in \mathbb{K}$$

as follows:

$$\begin{array}{cc} \langle a \otimes b, \sum f_{(1)} \otimes f_{(2)} \rangle = \langle ab, f \rangle, & \langle \sum a_{(1)} \otimes a_{(2)}, f \otimes g \rangle = \langle a, fg \rangle, \\ \langle a, 1 \rangle = \varepsilon(a), & \langle 1, f \rangle = \varepsilon(f), \\ \multicolumn{2}{c}{\langle a, S(f) \rangle = \langle S(a), f \rangle.} \end{array}$$

Definition 4.1.6. 1. The linear functionals $a \in H^*$ are called *generalized integrals on H* ([Riesz-Nagy] S.123).

2. An element $\int \in H^*$ is called a *left (invariant) integral on* H if

$$a\int = \langle a, 1_H\rangle \int$$

or $a\int = \varepsilon_{H^*}(a)\int$ for all $a \in H^*$.

3. An element $\delta \in H$ is called a *left integral in* H if

$$f\delta = \varepsilon(f)\delta$$

for all $f \in H$.

4. The set of left integrals in H is denoted by $\mathrm{Int}_l(H)$, the set of right integrals by $\mathrm{Int}_r(H)$. The set of left (right) integrals on H is $\mathrm{Int}_l(H^*)$ ($\mathrm{Int}_r(H^*)$).

5. A Hopf algebra H is called *unimodular* if $\mathrm{Int}_l(H) = \mathrm{Int}_r(H)$.

Lemma 4.1.7. *The left integrals* $\mathrm{Int}_l(H^*)$ *form a two sided ideal of* H^*. *If the antipode* S *is bijective then* S *induces an isomorphism* $S : \mathrm{Int}_l(H^*) \longrightarrow \mathrm{Int}_r(H^*)$.

Proof. For $\int$ in $\mathrm{Int}_l(H^*)$ we have $a\int = \varepsilon(a)\int \in \mathrm{Int}_l(H^*)$ and $a\int b = \varepsilon(a)\int b$ hence $\int b \in \mathrm{Int}_l(H^*)$. If S is bijective then the induced map $S : H^* \longrightarrow H^*$ is also bijective and satisfies $S(\int)b = S(\int)S(S^{-1}(b)) = S(S^{-1}(b)\int) = S(\int)\varepsilon(b)$ hence $S(\int) \in \mathrm{Int}_r(H^*)$. $\square$

Remark 4.1.8. Maschke's Theorem has an extension to finite dimensional Hopf algebras: $\varepsilon(\int) \neq 0$ iff H^* is semisimple.

Corollary 4.1.9. *Let* H *be a finite dimensional Hopf algebra. Then* H^* *is a left* H^**-module by the usual multiplication, hence a right* H*-comodule. We have*

$$(H^*)^{coH} = \mathrm{Int}_l(H^*).$$

Proof. By definition we have $\mathrm{Int}_l(H^*) = {}^{H^*}H^*$. $\square$

Example 4.1.10. Let G be a finite group. Let $H := \mathrm{Map}(G, \mathbb{K})$ be the Hopf algebra defined by the following isomorphism

$$\mathbb{K}^G = \mathrm{Map}(G, \mathbb{K}) \cong \mathrm{Hom}(\mathbb{K}G, \mathbb{K}) = (\mathbb{K}G)^*.$$

This isomorphism between the vector space $\mathbb{K}^G$ of all set maps from the group G to the base ring $\mathbb{K}$ and the dual vector space $(\mathbb{K}G)^*$ of the group algebra $\mathbb{K}G$ defines the structure of a Hopf algebra on $\mathbb{K}^G$.

We regard $H := \mathbb{K}^G$ as the function algebra on the set G. In the sense of algebraic geometry this is not quite true. The algebra $\mathbb{K}^G$ represents a functor from $\mathbb{K}$-$c\mathcal{A}lg$ to $\mathcal{S}et$ that has G as value for all connected algebras A in particular for all field extensions of $\mathbb{K}$.

As before we use the map $\mathrm{ev} : \mathbb{K}G \otimes \mathbb{K}^G \longrightarrow \mathbb{K}$. The multiplication of $\mathbb{K}^G$ is given by pointwise multiplication of maps since $\langle x, ff'\rangle = \langle \sum x_{(1)} \otimes x_{(2)}, f \otimes f'\rangle = \langle x \otimes x, f \otimes f'\rangle = \langle x, f\rangle\langle x, f'\rangle$ for all $f, f' \in \mathbb{K}^G$ and all $x \in G$. The unit element $1_{\mathbb{K}^G}$ of $\mathbb{K}^G$ is the map $\varepsilon : \mathbb{K}G \longrightarrow \mathbb{K}$ restricted to G, hence $\varepsilon(x) = 1 = \langle x, 1_{\mathbb{K}^G}\rangle$ for all $x \in G$. The antipode of $f \in \mathbb{K}^G$ is given by $S(f)(x) = \langle x, S(f)\rangle = f(x^{-1})$.

The elements of the dual basis $(x^*|x \in G)$ with $\langle x, y^* \rangle = \delta_{x,y}$ considered as maps from G to $\mathbb{K}$ form a basis of $\mathbb{K}^G$. They satisfy the conditions

$$x^* y^* = \delta_{x,y} x^* \text{ and } \sum_{x \in G} x^* = 1_{\mathbb{K}^G}$$

since $\langle z, x^* y^* \rangle = \langle z, x^* \rangle \langle z, y^* \rangle = \delta_{z,x} \delta_{z,y} = \delta_{x,y} \langle z, x^* \rangle$ and $\langle z, \sum_{x \in G} x^* \rangle = 1 = \langle z, 1_{\mathbb{K}^G} \rangle$.

Hence the dual basis $(x^*|x \in G)$ is a decomposition of the unit into a set of minimal orthogonal idempotents and the algebra of $\mathbb{K}^G$ has the structure

$$\mathbb{K}^G = \oplus_{x \in G} \mathbb{K} x^* \cong \mathbb{K} \times \ldots \times \mathbb{K}.$$

In particular $\mathbb{K}^G$ is commutative and semisimple.

The diagonal of $\mathbb{K}^G$ is

$$\Delta(x^*) = \sum_{y \in G} y^* \otimes (y^{-1} x)^* = \sum_{y,z \in G, yz = x} y^* \otimes z^*$$

since

$$\begin{aligned} \langle z \otimes u, \Delta(x^*) \rangle &= \langle zu, x^* \rangle = \delta_{x,zu} = \delta_{z^{-1}x,u} = \textstyle\sum_{y \in G} \delta_{y,z} \delta_{y^{-1}x,u} \\ &= \textstyle\sum_{y \in G} \langle z, y^* \rangle \langle u, (y^{-1}x)^* \rangle = \langle z \otimes u, \sum_{y \in G} y^* \otimes (y^{-1}x)^* \rangle. \end{aligned}$$

Let $a \in \mathbb{K}G$. Then a defines a map $\widetilde{a} : G \to \mathbb{K} \in \mathbb{K}^G$ by $a = \sum_{x \in G} \widetilde{a}(x) x$. For arbitrary $f \in \mathbb{K}^G$ and $a \in \mathbb{K}G$ this gives

$$\langle a, f \rangle = f(\sum_{x \in G} \widetilde{a}(x) x) = \sum_{x \in G} \widetilde{a}(x) f(x).$$

The counit of $\mathbb{K}^G$ is given by $\varepsilon(x^*) = \delta_{x,e}$ where $e \in G$ is the unit element.

The antipode is, as above, $S(x^*) = (x^{-1})^*$.

We consider $H = \mathbb{K}^G$ as the function algebra on the finite group G and $\mathbb{K}G$ as the dual space of $H = \mathbb{K}^G$ hence as the set of distributions on H.

Then

$$\textstyle\int := \sum_{x \in G} x \in H^* = \mathbb{K}G \tag{7}$$

is a (two sided) integral on H since $\sum_{x \in G} yx = \sum_{x \in G} x = \varepsilon(y) \sum_{x \in G} x = \sum_{x \in G} yx$.

We write

$$\int f(x) dx := \langle \textstyle\int, f \rangle = \sum_{x \in G} f(x).$$

We have seen that there is a decomposition of the unit $1 \in \mathbb{K}^G$ into a set of primitive orthogonal idempotents $\{x^*|x \in G\}$ such that every element $f \in \mathbb{K}^G$ has a unique representation $f = \sum f(x) x^*$. Since $\int y^* = \sum_{x \in G} \langle x, y^* \rangle$ we get $\int f y^* = \sum_{x \in G} \langle x, f y^* \rangle = \sum f(x) y^*(x) = f(y)$ hence

$$f = \sum (\int f(x) y^*(x) dx) y^*.$$

Problem 4.1.30. Describe the group valued functor $\mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}^G, -)$ in terms of sets and their group structure.

Definition and Remark 4.1.11. Let $\mathbb{K}$ be an algebraicly closed field and let G be a finite abelian group (replacing $\mathbb{R}$ above). Assume that the characteristic of $\mathbb{K}$ does not divide the order of G. Let $H = \mathbb{K}^G$. We identify $\mathbb{K}^G = \mathrm{Hom}(\mathbb{K}G, \mathbb{K})$ along the linear expansion of maps as in Example 2.1.10.

Let us consider the set $\hat{G} := \{\chi : G \longrightarrow \mathbb{K}^* | \chi \text{ group homomorphism}\}$. Since $\mathbb{K}^*$ is an abelian group, the set $\hat{G}$ is an abelian group by pointwise multiplication.

The group $\hat{G}$ is called the *character group* of G.

Obviously the character group is a multiplicative subset of $\mathbb{K}^G = \mathrm{Hom}(\mathbb{K}G, \mathbb{K})$. Actually it is a subgroup of $\mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}G, \mathbb{K}) \subseteq \mathrm{Hom}(\mathbb{K}G, \mathbb{K})$ since the elements $\chi \in \hat{G}$ expand to algebra homomorphisms: $\chi(ab) = \chi(\sum \alpha_x x \sum \beta_y y) = \sum \alpha_x \beta_y \chi(xy) = \chi(a)\chi(b)$ and $\chi(1) = \chi(e) = 1$. Conversely an algebra homomorphism $f \in \mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}G, \mathbb{K})$ restricts to a character $f : G \longrightarrow \mathbb{K}^*$. Thus $\hat{G} = \mathbb{K}\text{-}c\mathcal{A}lg(\mathbb{K}G, \mathbb{K})$, the set of rational points of the affine algebraic group represented by $\mathbb{K}G$.

There is a more general observation behind this remark.

Lemma 4.1.12. *Let H be a finite dimensional Hopf algebra. Then the set $Gr(H^*)$ of grouplike elements of H^* is equal to $\mathbb{K}\text{-}\mathcal{A}lg(H, \mathbb{K})$.*

Proof. In fact $f : H \longrightarrow \mathbb{K}$ is an algebra homomorphism iff $\langle f \otimes f, a \otimes b\rangle = \langle f, a\rangle\langle f, b\rangle = \langle f, ab\rangle = \langle \Delta(f), a \otimes b\rangle$ and $1 = \langle f, 1\rangle = \varepsilon(f)$. □

Hence there is a Hopf algebra homomorphism $\varphi : \mathbb{K}\hat{G} \longrightarrow \mathbb{K}^G$ by 2.1.5.

Proposition 4.1.13. *The Hopf algebra homomorphism $\varphi : \mathbb{K}\hat{G} \longrightarrow \mathbb{K}^G$ is bijective.*

Proof. We give the proof by several lemmas.

Lemma 4.1.14. *Any set of grouplike elements in a Hopf algebra H is linearly independent.*

Proof. Assume there is a linearly dependent set $\{x_0, x_1, \ldots, x_n\}$ of grouplike elements in H. Choose such a set with n minimal. Obviously $n \geq 1$ since all elements are non zero. Thus $x_0 = \sum_{i=1}^n \alpha_i x_i$ and $\{x_1, \ldots, x_n\}$ linearly independent. We get

$$\sum_{i,j} \alpha_i \alpha_j x_i \otimes x_j = x_0 \otimes x_0 = \Delta(x_0) = \sum_i \alpha_i x_i \otimes x_i.$$

Since all $\alpha_i \neq 0$ and the $x_i \otimes x_j$ are linearly independent we get $n = 1$ and $\alpha_1 = 1$ so that $x_0 = x_1$, a contradiction. □

Corollary 4.1.15. *(Dedekind's Lemma) Any set of characters in $\mathbb{K}^G$ is linearly independent.*

Thus $\varphi : \mathbb{K}\hat{G} \longrightarrow \mathbb{K}^G$ is injective. Now we prove that the map $\varphi : \mathbb{K}\hat{G} \longrightarrow \mathbb{K}^G$ is surjective.

Lemma 4.1.16. *(Pontryagin duality) The evaluation $\hat{G} \times G \longrightarrow \mathbb{K}^*$ is a non-degenerate bilinear map of abelian groups.*

Proof. First we observe that $\mathrm{Hom}(C_n, \mathbb{K}^*) \cong C_n$ for a cyclic group of order n since $\mathbb{K}$ has a primitive n-th root of unity ($\mathrm{char}(\mathbb{K}) \not| \ |G|$).

Since the direct product and the direct sum coincide in $\mathcal{A}b$ we can use the fundamental theorem for finite abelian groups $G \cong C_{n_1} \times \ldots \times C_{n_t}$ to get $\mathrm{Hom}(G, \mathbb{K}^*) \cong G$ for any abelian group G with $\mathrm{char}(\mathbb{K}) \not| \ |G|$. Thus $\hat{G} \cong G$ and $\hat{\hat{G}} = G$. In particular $\chi(x) = 1$ for all $x \in G$ iff $\chi = 1$. By the symmetry of the situation we get that the bilinear form $\langle ., . \rangle : \hat{G} \times G \longrightarrow \mathbb{K}^*$ is non-degenerate. $\square$

Thus $|\hat{G}| = |G|$ hence $\dim(\mathbb{K}\hat{G}) = \dim(\mathbb{K}^G)$. This proves Proposition 2.1.13. $\square$

Definition 4.1.17. Let H be a Hopf algebra. A $\mathbb{K}$-module M that is a right H-module by $\rho : M \otimes H \longrightarrow M$ and a right H-comodule by $\delta : M \longrightarrow M \otimes H$ is called a *Hopf module* if the diagram

$$\begin{tikzcd}
M \otimes H \arrow[r, "\rho"] \arrow[d, "\delta\otimes\Delta"'] & H \arrow[r, "\delta"] & M \otimes H \\
M \otimes H \otimes H \otimes H \arrow[rr, "1\otimes\tau\otimes 1"] & & M \otimes H \otimes H \otimes H \arrow[u, "\rho\otimes\nabla"']
\end{tikzcd}$$

commutes, i.e. if $\delta(mh) = \sum m_{(M)}h_{(1)} \otimes m_{(1)}h_{(2)}$ holds for all $h \in H$ and all $m \in M$.

Observe that H is an Hopf module over itself. Furthermore each module of the form $V \otimes H$ is a Hopf module by the induced structure. More generally there is a functor $\mathcal{V}ec \ni V \mapsto V \otimes H \in$ **Hopf-Mod**-H.

Proposition 4.1.18. *The two functors -^{coH} :* **Hopf-Mod**-$H \longrightarrow \mathcal{V}ec$ *and* $- \otimes H$: $\mathcal{V}ec \ni V \mapsto V \otimes H \in$ **Hopf-Mod**-H *are inverse equivalences of each other.*

Proof. Define natural isomorphisms

$$\alpha : M^{coH} \otimes H \ni m \otimes h \mapsto mh \in M$$

with inverse map

$$\alpha^{-1} : M \ni m \mapsto \sum m_{(M)}S(m_{(1)}) \otimes m_{(2)} \in M^{coH} \otimes H$$

and

$$\beta : V \ni v \mapsto v \otimes 1 \in (V \otimes H)^{coH}$$

with inverse map

$$(V \otimes H)^{coH} \ni v \otimes h \mapsto v\varepsilon(h) \in V.$$

Obviously these homomorphisms are natural transformations in M and V. Furthermore α is a homomorphism of H-modules. α^{-1} is well-defined since

$$\begin{aligned}\delta(\sum m_{(M)}S(m(1))) &= \sum m_{(M)}S(m_{(3)}) \otimes m_{(1)}S(m_{(2)}) \\ &\quad \text{(since } M \text{ is a Hopf module)} \\ &= \sum m_{(M)}S(m_{(2)}) \otimes \eta\varepsilon(m_{(1)}) \\ &= \sum m_{(M)}S(m_{(1)}) \otimes 1\end{aligned}$$

hence $\sum m_{(M)}S(m_{(1)}) \in M^{coH}$. Furthermore α^{-1} is a homomorphism of comodules since

$$\begin{aligned}\delta\alpha^{-1}(m) &= \delta(\sum m_{(M)}S(m_{(1)}) \otimes m_{(2)}) = \sum m_{(M)}S(m_{(1)}) \otimes m_{(2)} \otimes m_{(3)} \\ &= \sum \alpha^{-1}(m_{(M)}) \otimes m_{(1)} = (\alpha^{-1} \otimes 1)\delta(m).\end{aligned}$$

Finally α and α^{-1} are inverse to each other by

$$\alpha\alpha^{-1}(m) = \alpha(\sum m_{(M)}S(m_{(1)}) \otimes m_{(2)}) = \sum m_{(M)}S(m_{(1)})m_{(2)} = m$$

and

$$\begin{aligned}\alpha^{-1}\alpha(m \otimes h) &= \alpha^{-1}(mh) = \sum m_{(M)}h_{(1)}S(m_{(1)}h_{(2)}) \otimes m_{(2)}h_{(3)} \\ &= \sum mh_{(1)}S(h_{(2)}) \otimes h_{(3)} \text{ (by } \delta(m) = m \otimes 1 \text{)} = m \otimes h.\end{aligned}$$

Thus α and α^{-1} are mutually inverse homomorphisms of Hopf modules.

The image of β is in $(V \otimes H)^{coH}$ by $\delta(v \otimes 1) = v \otimes \Delta(1) = (v \otimes 1) \otimes 1$. Both β and β^{-1} are $\mathbb{K}$-linear maps. Furthermore we have

$$\beta^{-1}\beta(v) = \beta^{-1}(v \otimes 1) = v\varepsilon(1) = v$$

and

$$\begin{aligned}\beta\beta^{-1}(\sum v_i \otimes h_i) &= \beta(\sum v_i\varepsilon(h_i)) = \sum v_i\varepsilon(h_i) \otimes 1 = \sum v_i \otimes \varepsilon(h_i)1 \\ &= \sum v_i \otimes \varepsilon(h_{i(1)})h_{i(2)} \text{ (since } \textstyle\sum v_i \otimes h_i \in (V \otimes H)^{coH} \text{) } = \sum v_i \otimes h_i.\end{aligned}$$

Thus β and β^{-1} are mutually inverse homomorphisms. $\square$

Since $H^* = \mathrm{Hom}(H, \mathbb{K})$ and $S : H \longrightarrow H$ is an algebra antihomomorphism, the dual H^* is an H-module in four different ways:

(8)
$$\begin{aligned}\langle (f \rightharpoonup a), g\rangle &:= \langle a, gf\rangle, & \langle (a \leftharpoonup f), g\rangle &:= \langle a, fg\rangle, \\ \langle (f \rightharpoondown a), g\rangle &:= \langle a, S(f)g\rangle, & \langle (a \leftharpoondown f), g\rangle &:= \langle a, gS(f)\rangle.\end{aligned}$$

If H is finite dimensional then H^* is a Hopf algebra. The equality $\langle (f \rightharpoonup a), g\rangle = \langle a, gf\rangle = \sum\langle a_{(1)}, g\rangle\langle a_{(2)}, f\rangle$ implies

(9)
$$(f \rightharpoonup a) = \sum a_{(1)}\langle a_{(2)}, f\rangle.$$

Analogously we have

(10)
$$(a \leftharpoonup f) = \sum\langle a_{(1)}, f\rangle a_{(2)}.$$

Proposition 4.1.19. *Let H be a finite dimensional Hopf algebra. Then H^* is a right Hopf module over H.*

Proof. H^* is a left H^*-module by left multiplication hence by 2.1.1 a right H-comodule by $\delta(a) = \sum_i b_i^* a \otimes b_i$. Let $f, g \in H$ and $a, b \in H^*$. The (left) multiplication of H^* satisfies

$$ab = \sum b_{(H^*)} \langle a, b_{(1)} \rangle.$$

We use the right H-module structure

$$(a \leftharpoondown f) = \sum a_{(1)} \langle S(f), a_{(2)} \rangle.$$

on $H^* = \mathrm{Hom}(H, \mathbb{K})$.

Now we check the Hopf module property. Let $a, b \in H^*$ and $f, g \in H$. We apply $H^* \otimes H$ to its dual $H \otimes H^*$ and get

$$\begin{aligned}
\delta(a \leftharpoondown f)(g \otimes b) &= \sum \langle (a \leftharpoondown f)_{(H^*)}, g \rangle \langle b, (a \leftharpoondown f)_{(1)} \rangle = \langle b(a \leftharpoondown f), g \rangle \\
&= \sum \langle b, g_{(1)} \rangle \langle (a \leftharpoondown f), g_{(2)} \rangle = \sum \langle b, g_{(1)} \rangle \langle a, g_{(2)} S(f) \rangle \\
&= \sum \langle b, g_{(1)} \varepsilon(f_{(2)}) \rangle \langle a, g_{(2)} S(f_{(1)}) \rangle = \sum \langle (f_{(3)} \rightharpoonup b), g_{(1)} S(f_{(2)}) \rangle \langle a, g_{(2)} S(f_{(1)}) \rangle \\
&= \sum \langle (f_{(2)} \rightharpoonup b) a, g S(f_{(1)}) \rangle = \sum \langle ((f_{(2)} \rightharpoonup b) a) \leftharpoondown f_{(1)}, g \rangle \\
&= \sum \langle (a_{(H^*)} \langle (f_{(2)} \rightharpoonup b), a_{(1)} \rangle) \leftharpoondown f_{(1)}, g \rangle \\
&= \sum \langle (a_{(H^*)} \leftharpoondown f_{(1)}) \langle (f_{(2)} \rightharpoonup b), a_{(1)} \rangle, g \rangle = \sum \langle (a_{(H^*)} \leftharpoondown f_{(1)}) \langle b, a_{(1)} f_{(2)} \rangle, g \rangle
\end{aligned}$$

hence $\delta(a \leftharpoondown f) = \sum (a_{(H^*)} \leftharpoondown f_{(1)}) \otimes a_{(1)} f_{(2)}$. $\square$

Theorem 4.1.20. *Let H be a finite dimensional Hopf algebra. Then the antipode S is bijective, the space of left integrals $\mathrm{Int}_l(H^*)$ has dimension 1, and the homomorphism*

$$H \ni f \mapsto (f \rightharpoonup \textstyle\int) = \sum \int_{(1)} \langle \int_{(2)}, f \rangle \ni H^*$$

is bijective for any $0 \neq \int \in \mathrm{Int}_l(H^*)$.

Proof. By Proposition 2.1.19 H^* is a right Hopf module over H. By Proposition 2.1.18 there is an isomorphism $\alpha : (H^*)^{coH} \otimes H \ni a \otimes f \mapsto (a \leftharpoondown f) = (S(f) \rightharpoonup a) \in H^*$. Since $(H^*)^{coH} \cong \mathrm{Int}_l(H^*)$ by 2.1.9 we get

$$\mathrm{Int}_l(H^*) \otimes H \cong H^*$$

as right H-Hopf modules by the given map. This shows $\dim(\mathrm{Int}_l(H^*)) = 1$. So we get an isomorphism $H \ni f \mapsto (\int \leftharpoondown f) \in H^*$ that is a composition of S and $f \mapsto (f \rightharpoonup \int)$. Since H is finite dimensional both of these maps are bijective. $\square$

If G is a finite group then every generalized integral $a \in \mathbb{K}G$ can be written with a uniquely determined $g \in H = \mathbb{K}^G$ as

$$\langle a, f \rangle = \int f(x) S(g)(x) dx = \sum_{x \in G} f(x) g(x^{-1}) \tag{11}$$

for all $f \in H$.

If G is a finite Abelian group then each group element (rational integral) $y \in G \subseteq \mathbb{K}G$ can be written as

$$y = \sum_{x \in G} \sum_{\chi \in \hat{G}} \beta_\chi \langle x^{-1}, \chi \rangle x$$

since

$$\begin{aligned} \langle y, f \rangle &= \langle (\textstyle\int \leftharpoonup \sum_{\chi \in \hat{G}} \beta_\chi \chi), f \rangle = \langle \textstyle\int, f S(\sum_{\chi \in \hat{G}} \beta_\chi \chi) \rangle \\ &= \textstyle\sum_{x \in G} \langle x, f \rangle \sum_{\chi \in \hat{G}} \beta_\chi \langle x, S(\chi) \rangle = \langle \sum_{x \in G} \sum_{\chi \in \hat{G}} \beta_\chi \langle x^{-1}, \chi \rangle x, f \rangle. \end{aligned}$$

In particular the matrix $(\langle x^{-1}, \chi \rangle)$ is invertible.

Let H be finite dimensional. Since $\langle \int, fg \rangle = \langle (\int \leftharpoonup f), g \rangle$ as a functional on g is a generalized integral, there is a unique $\nu(f) \in H$ such that

$$\langle \textstyle\int, fg \rangle = \langle \textstyle\int, g\nu(f) \rangle \tag{12}$$

or

$$\int f(x)g(x)dx = \int g(x)\nu(f)(x)dx. \tag{13}$$

Although the functions $f, g \in H$ of the quantum group do not commute under multiplication, there is a simple commutation rule if the product is integrated.

Proposition and Definition 4.1.21. *The map $\nu : H \longrightarrow H$ is an algebra automorphism, called the* Nakayama automorphism.

PROOF. It is clear that ν is a linear map. We have $\int f\nu(gh) = \int ghf = \int hf\nu(g) = \int f\nu(g)\nu(h)$ hence $\nu(gh) = \nu(g)\nu(h)$ and $\int f\nu(1) = \int f$ hence $\nu(1) = 1$. Furthermore if $\nu(g) = 0$ then $0 = \langle \int, f\nu(g) \rangle = \langle \int, gf \rangle = \langle (f \rightharpoonup \int), g \rangle$ for all $f \in H$ hence $\langle a, g \rangle = 0$ for all $a \in H^*$ hence $g = 0$. So ν is injective hence bijective. $\square$

Corollary 4.1.22. *The map $H \ni f \mapsto (\int \leftharpoonup f) \in H^*$ is an isomorphism.*

PROOF. We have

$$(\textstyle\int \leftharpoonup f) = (\nu(f) \rightharpoonup \int)$$

since $\langle (\int \leftharpoonup f), g \rangle = \langle \int, fg \rangle = \langle \int, g\nu(f) \rangle = \langle (\nu(f) \rightharpoonup \int), g \rangle$. This implies the corollary. $\square$

If G is a finite group and $H = \mathbb{K}^G$ then H is commutative hence $\nu = \mathrm{id}$.

Definition 4.1.23. An element $\delta \in H$ is called a *Dirac δ-function* if δ is a left invariant integral in H with $\langle \int, \delta \rangle = 1$, i.e. if δ satisfies

$$f\delta = \varepsilon(f)\delta \qquad \text{and} \qquad \int \delta(x)dx = 1$$

for all $f \in H$. If H has a Dirac δ-function then we write

$$\int^* a(x)dx = \textstyle\int^* a := \langle a, \delta \rangle. \tag{14}$$

Proposition 4.1.24.

(1) *If H is finite dimensional then there exists a unique Dirac δ-function δ.*
(2) *If H is infinite dimensional then there exists no Dirac δ-function.*

PROOF. 1. Since $H \ni f \mapsto (f \rightharpoonup \int) \in H^*$ is an isomorphism there is a $\delta \in H$ such that $(\delta \rightharpoonup \int) = \varepsilon$. Then $(f\delta \rightharpoonup \int) = (f \rightharpoonup (\delta \rightharpoonup \int)) = (f \rightharpoonup \varepsilon) = \varepsilon(f)\varepsilon = \varepsilon(f)(\delta \rightharpoonup \int)$ which implies $f\delta = \varepsilon(f)\delta$. Furthermore we have $\langle \int, \delta \rangle = \langle \int, 1_H \delta \rangle = \langle (\delta \rightharpoonup \int), 1_H \rangle = \varepsilon(1_H) = 1_{\mathbb{K}}$.

2. is [Sweedler] exercise V.4. □

Lemma 4.1.25. *Let H be a finite dimensional Hopf algebra. Then $\int \in H^*$ is a left integral iff*

$$a(\sum \textstyle\int_{(1)} \otimes S(\int_{(2)})) = (\sum \int_{(1)} \otimes S(\int_{(2)}))a \tag{15}$$

iff

$$\sum S(a)\textstyle\int_{(1)} \otimes \int_{(2)} = \sum \int_{(1)} \otimes a\int_{(2)} \tag{16}$$

iff

$$\sum f_{(1)} \langle \textstyle\int, f_{(2)} \rangle = \langle \int, f \rangle 1_H. \tag{17}$$

PROOF. Let $\int$ be a left integral. Then

$$\sum a_{(1)} \textstyle\int_{(1)} \otimes S(\int_{(2)}) S(a_{(2)}) = \sum (a\int)_{(1)} \otimes S((a\int)_{(2)}) = \varepsilon(a)(\sum \int_{(1)} \otimes S(\int_{(2)}))$$

for all $a \in H$. Hence

$$\begin{aligned}(\textstyle\sum \int_{(1)} \otimes S(\int_{(2)}))a &= \textstyle\sum \varepsilon(a_{(1)})(\int_{(1)} \otimes S(\int_{(2)}))a_{(2)} \\ &= \textstyle\sum a_{(1)}\int_{(1)} \otimes S(\int_{(2)}) S(a_{(2)}) a_{(3)} \\ &= \textstyle\sum a_{(1)}\int_{(1)} \otimes S(\int_{(2)}) \varepsilon(a_{(2)}) = a(\sum \int_{(1)} \otimes S(\int_{(2)})).\end{aligned}$$

Conversely $a(\sum \int_{(1)} \varepsilon(S(\int_{(2)}))) = (\sum \int_{(1)} \varepsilon(S(\int_{(2)})a)) = \varepsilon(a)(\sum \int_{(1)} \varepsilon(S(\int_{(2)})))$, hence $\int = \sum \int_{(1)} \varepsilon(S(\int_{(2)}))$ is a left integral.

Since S is bijective the following holds

$$\begin{aligned}&\textstyle\sum S(a) \int_{(1)} \otimes \int_{(2)} = \sum S(a) \int_{(1)} \otimes S^{-1}(S(\int_{(2)})) \\ &= \textstyle\sum \int_{(1)} \otimes S^{-1}(S(\int_{(2)})S(a)) = \sum \int_{(1)} \otimes a \int_{(2)}.\end{aligned}$$

The converse follows easily.

If $\int \in \mathrm{Int}_l(H)$ is a left integral then $\sum \langle a, f_{(1)} \rangle \langle \int, f_{(2)} \rangle = \langle a \int, f \rangle = \langle a, 1_H \rangle \langle \int, f \rangle$.

Conversely if $\lambda \in H^*$ with (17) is given then $\langle a\lambda, f \rangle = \sum \langle a, f_{(1)} \rangle \langle \lambda, f_{(2)} \rangle = \langle a, 1_H \rangle \langle \lambda, f \rangle$ hence $a\lambda = \varepsilon(a)\lambda$. □

If G is a finite group then

$$\delta(x) = \begin{cases} 0 \text{ if } x \neq e; \\ 1 \text{ if } x = e. \end{cases} \tag{18}$$

In fact since δ is left invariant we get $f(x)\delta(x) = f(e)\delta(x)$ for all $x \in G$ and $f \in \mathbb{K}^G$. Since $G \subset H^* = \mathbb{K}G$ is a basis, we get $\delta(x) = 0$ if $x \neq e$. Furthermore $\int \delta(x)dx = \sum_{x\in G} \delta(x) = 1$ implies $\delta(e) = 1$. So we have $\delta = e^*$.

If G is a finite Abelian group we get $\delta = \alpha \sum_{\chi\in\hat{G}} \chi$ for some $\alpha \in \mathbb{K}$. The evaluation gives $1 = \langle \int, \delta\rangle = \alpha \sum_{x\in G,\chi\in\hat{G}} \langle \chi, x\rangle$. Now let $\lambda \in \widehat{G}$. Then $\sum_{\chi\in\hat{G}} \langle \chi, x\rangle = \sum_{\chi\in\hat{G}} \langle \lambda\chi, x\rangle = \langle \lambda, x\rangle \sum_{\chi\in\hat{G}} \langle \chi, x\rangle$. Since for each $x \in G \setminus \{e\}$ there is a λ such that $\langle \lambda, x\rangle \neq 1$ and we get

$$\sum_{\chi\in\hat{G}} \langle \chi, x\rangle = |G|\delta_{e,x}.$$

Hence $\sum_{x\in G,\chi\in\hat{G}} \langle \chi, x\rangle = |G| = \alpha^{-1}$ and

$$\delta = |G|^{-1} \sum_{\chi\in\hat{G}} \chi. \tag{19}$$

Let H be finite dimensional for the rest of this section. In Corollary 1.22 we have seen that the map $H \ni f \mapsto (\int \leftharpoonup f) \in H^*$ is an isomorphism. This map will be called the *Fourier transform.*

Theorem 4.1.26. *The Fourier transform $H \ni f \mapsto \widetilde{f} \in H^*$ is bijective with*

$$\widetilde{f} = (\textstyle\int \leftharpoonup f) = \sum \langle \int_{(1)}, f\rangle \int_{(2)} \tag{20}$$

The inverse Fourier transform is defined by

$$\widetilde{a} = \sum S^{-1}(\delta_{(1)})\langle a, \delta_{(2)}\rangle. \tag{21}$$

Since these maps are inverses of each other the following formulas hold

$$\begin{array}{ll} \langle \widetilde{f}, g\rangle = \displaystyle\int f(x)g(x)dx & \langle a, \widetilde{b}\rangle = \displaystyle\int^* S^{-1}(a)(x)b(x)dx \\ f = \sum S^{-1}(\delta_{(1)})\langle \widetilde{f}, \delta_{(2)}\rangle & a = \sum \langle \int_{(1)}, \widetilde{a}\rangle \int_{(2)} . \end{array} \tag{22}$$

Proof. We use the isomorphisms $H \longrightarrow H^*$ defined by $\widehat{f} := \widetilde{f} = (\int \leftharpoonup f) = \sum \langle \int_{(1)}, f\rangle \int_{(2)}$ and $H^* \longrightarrow H$ defined by $\widehat{a} := (a \rightharpoonup \delta) = \sum \delta_{(1)}\langle a, \delta_{(2)}\rangle$. Because of

$$\langle a, \widehat{b}\rangle = \langle a, (b \rightharpoonup \delta)\rangle = \langle ab, \delta\rangle \tag{23}$$

and

$$\langle \widetilde{f}, g\rangle = \langle (\textstyle\int \leftharpoonup f), g\rangle = \langle \int, fg\rangle \tag{24}$$

we get for all $a \in H^*$ and $f \in H$

$$\begin{aligned} \langle a, \widehat{\widehat{f}}\rangle &= \langle a\widehat{f}, \delta\rangle = \textstyle\sum \langle a, \delta_{(1)}\rangle\langle \widehat{f}, \delta_{(2)}\rangle = \sum \langle a, \delta_{(1)}\rangle\langle \int, f\delta_{(2)}\rangle \qquad (\text{ by Lemma 1.25 }) \\ &= \textstyle\sum \langle a, S(f)\delta_{(1)}\rangle\langle \int, \delta_{(2)}\rangle = \langle a, S(f)\rangle\langle \int, \delta\rangle = \langle a, S(f)\rangle. \end{aligned}$$

This gives $\widehat{\widehat{f}} = S(f)$. So the inverse map of $H \to H^*$ with $\widehat{f} = (\int \leftharpoonup f) = \widetilde{f}$ is $H^* \to H$ with $S^{-1}(\widehat{a}) = \sum S^{-1}(\delta_{(1)})\langle a, \delta_{(2)}\rangle = \widetilde{a}$. Then the given inversion formulas are clear.

We note for later use $\langle a, \widetilde{b}\rangle = \langle a, S^{-1}(\widehat{b})\rangle = \langle S^{-1}(a), \widehat{b}\rangle = \langle S^{-1}(a)b, \delta\rangle$. $\square$

If G is a finite group and $H = \mathbb{K}^G$ then

$$\widetilde{f} = \sum_{x \in G} f(x)x.$$

Since $\Delta(\delta) = \sum_{x \in G} {x^{-1}}^* \otimes x^*$ where the $x^* \in \mathbb{K}^G$ are the dual basis to the $x \in G$, we get

$$\widetilde{a} = \sum_{x \in G} \langle a, x^*\rangle x^*.$$

If G is a finite Abelian group then the groups G and $\widehat{G}$ are isomorphic so the Fourier transform induces a linear automorphism $\widetilde{-} : \mathbb{K}^G \to \mathbb{K}^G$ and we have

$$\widetilde{a} = |G|^{-1} \sum_{\chi \in \hat{G}} \langle a, \chi\rangle \chi^{-1}$$

By substituting the formulas for the integral and the Dirac δ-function (7) and (19) we get

$$\begin{array}{ll} \widetilde{f} = \sum_{x \in G} f(x)x, & \widetilde{a} = |G|^{-1} \sum_{\chi \in \hat{G}} a(\chi)\chi^{-1}, \\ f = |G|^{-1} \sum_{\chi \in \hat{G}} \widetilde{f}(\chi)\chi^{-1}, & a = \sum_{x \in G} \widetilde{a}(x)x. \end{array} \tag{25}$$

This implies

$$\widetilde{f}(\chi) = \sum_{x \in G} f(x)\chi(x) = \int f(x)\chi(x)dx \tag{26}$$

with inverse transform

$$\widetilde{a}(x) = |G|^{-1} \sum_{\chi \in \hat{G}} \chi(a)\chi^{-1}(x). \tag{27}$$

Corollary 4.1.27. *The Fourier transforms of the left invariant integrals in H and H^* are*

$$\widetilde{\delta} = \varepsilon\nu^{-1} \in H^* \qquad \text{and} \qquad \widetilde{\int} = 1 \in H. \tag{28}$$

Proof. We have $\langle \widetilde{\delta}, f\rangle = \langle \int, \delta f\rangle = \langle \int, \nu^{-1}(f)\delta\rangle = \varepsilon\nu^{-1}(f)\langle \int, \delta\rangle = \varepsilon\nu^{-1}(f)$ hence $\widetilde{\delta} = \varepsilon\nu^{-1}$. From $\widetilde{1} = (\int \leftharpoonup 1) = \int$ we get $\widetilde{\int} = 1$. $\square$

Proposition 4.1.28. *Define a* convolution multiplication *on H^* by*

$$\langle a * b, f\rangle := \sum \langle a, S^{-1}(\delta_{(1)})f\rangle\langle b, \delta_{(2)}\rangle.$$

Then the following transformation rule holds for $f, g \in H$:

$$\widetilde{fg} = \widetilde{f} * \widetilde{g}. \tag{29}$$

In particular H^* with the convolution multiplication is an associative algebra with unit $\widetilde{1_H} = \int$, i.e.

$$\int * a = a * \int = a. \tag{30}$$

Proof. Given $f, g, h \in H^*$. Then

$$\begin{aligned}\langle \widetilde{fg}, h\rangle &= \langle \textstyle\int, fgh\rangle = \langle \textstyle\int, fS^{-1}(1_H)gh\rangle\langle \textstyle\int, \delta\rangle \\ &= \textstyle\sum\langle \textstyle\int, fS^{-1}(\delta_{(1)})gh\rangle\langle \textstyle\int, \delta_{(2)}\rangle = \textstyle\sum\langle \textstyle\int, fS^{-1}(\delta_{(1)})h\rangle\langle \textstyle\int, g\delta_{(2)}\rangle \\ &= \textstyle\sum\langle \widetilde{f}, S^{-1}(\delta_{(1)})h\rangle\langle \widetilde{g}, \delta_{(2)}\rangle = \langle \widetilde{f} * \widetilde{g}, h\rangle.\end{aligned}$$

From (28) we get $\widetilde{1}_H = \int$. So we have $\widetilde{f} = \widetilde{1f} = \widetilde{1} * \widetilde{f} = \int * \widetilde{f}$. □

If G is a finite Abelian group and $a, b \in H^* = \mathbb{K}^{\hat{G}}$. Then

$$(a * b)(\mu) = |G|^{-1} \sum_{\chi,\lambda\in\hat{G},\chi\lambda=\mu} a(\lambda)b(\chi).$$

In fact we have

$$\begin{aligned}(a * b)(\mu) &= \langle a * b, \mu\rangle = \textstyle\sum\langle a, S^{-1}(\delta_{(1)})\mu\rangle\langle b, \delta_{(2)}\rangle \\ &= |G|^{-1}\textstyle\sum_{\chi\in\hat{G}}\langle a, \chi^{-1}\mu\rangle\langle b, \chi\rangle = |G|^{-1}\textstyle\sum_{\chi,\lambda\in\hat{G},\chi\lambda=\mu} a(\lambda)b(\chi).\end{aligned}$$

One of the most important formulas for Fourier transforms is the Plancherel formula on the invariance of the inner product under Fourier transforms. We have

Theorem 4.1.29. *(The Plancherel formula)*

$$\langle a, f\rangle = \langle \widetilde{f}, \nu(\widetilde{a})\rangle. \tag{31}$$

Proof. First we have from (22)

$$\begin{aligned}\langle a, f\rangle &= \textstyle\sum\langle \textstyle\int_{(1)}, \widetilde{a}\rangle\langle \textstyle\int_{(2)}, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle = \textstyle\sum\langle \textstyle\int, \widetilde{a}S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \textstyle\sum\langle \textstyle\int, S^{-1}(\delta_{(1)})\nu(\widetilde{a})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle = \textstyle\sum\langle \textstyle\int, S^{-1}(S(\nu(\widetilde{a}))\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \textstyle\sum\langle \textstyle\int, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \nu(\widetilde{a})\delta_{(2)}\rangle = \textstyle\sum\langle \textstyle\int, S^{-1}(\delta)_{(2)}\rangle\langle \widetilde{f}, \nu(\widetilde{a})S(S^{-1}(\delta)_{(1)})\rangle \\ &= \langle \textstyle\int, S^{-1}(\delta)\rangle\langle \widetilde{f}, \nu(\widetilde{a})\rangle.\end{aligned}$$

Apply this to $\langle \int, \delta\rangle$. Then we get

$$1 = \langle \textstyle\int, \delta\rangle = \langle \textstyle\int, S^{-1}(\delta)\rangle\langle \widetilde{\delta}, \nu(\widetilde{\textstyle\int})\rangle = \langle \textstyle\int, S^{-1}(\delta)\rangle\varepsilon\nu^{-1}\nu(1) = \langle \textstyle\int, S^{-1}(\delta)\rangle.$$

Hence we get $\langle a, f\rangle = \langle \widetilde{f}, \nu(\widetilde{a})\rangle$. □

Corollary 4.1.30. *If H is unimodular then $\nu = S^2$.*

PROOF. H unimodular means that δ is left and right invariant. Thus we get

$$\begin{aligned}\langle a, f\rangle &= \sum\langle \textstyle\int_{(1)}, \widetilde{a}\rangle\langle \textstyle\int_{(2)}, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \sum\langle \textstyle\int, \widetilde{a}S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle = \sum\langle \textstyle\int, S^{-1}(\delta_{(1)}S(\widetilde{a}))\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \sum\langle \textstyle\int, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}S^2(\widetilde{a})\rangle \quad (\text{ since } \delta \text{ is right invariant}) \\ &= \langle \textstyle\int, S^{-1}(\delta)\rangle\langle \widetilde{f}, S^2(\widetilde{a})\rangle = \langle \widetilde{f}, S^2(\widetilde{a})\rangle.\end{aligned}$$

Hence $S^2 = \nu$. □

We also get a special representation of the inner product $H^* \otimes H \longrightarrow \mathbb{K}$ by both integrals:

Corollary 4.1.31.

$$\langle a, f\rangle = \int \widetilde{a}(x)f(x)dx = \int^* S^{-1}(a)(x)\widetilde{f}(x)dx. \tag{32}$$

PROOF. We have the rules for the Fourier transform. From (24) we get $\langle a, f\rangle = \langle \int, \widetilde{a}f\rangle = \int \widetilde{a}(x)f(x)dx$ and from (23) $\langle a, f\rangle = \langle S^{-1}(a)\widetilde{f}, \delta\rangle = \int^* S^{-1}(a)(x)\widetilde{f}(x)dx$. □

The Fourier transform leads to an interesting integral transform on H by double application.

Proposition 4.1.32. *The double transform* $\breve{f} := (\delta \leftharpoonup (\int \leftharpoonup f))$ *defines an automorphism* $H \longrightarrow H$ *with*

$$\breve{f}(y) = \int f(x)\delta(xy)dx.$$

PROOF. We have

$$\begin{aligned}\langle y, \breve{f}\rangle &= \langle y, (\delta \leftharpoonup (\textstyle\int \leftharpoonup f))\rangle = \langle (\textstyle\int \leftharpoonup f)y, \delta\rangle \\ &= \sum\langle (\textstyle\int \leftharpoonup f), \delta_{(1)}\rangle\langle y, \delta_{(2)}\rangle = \sum\langle \textstyle\int, f\delta_{(1)}\rangle\langle y, \delta_{(2)}\rangle \\ &= \sum\langle \textstyle\int_{(1)}, f\rangle\langle \textstyle\int_{(2)}, \delta_{(1)}\rangle\langle y, \delta_{(2)}\rangle = \sum\langle \textstyle\int_{(1)}, f\rangle\langle \textstyle\int_{(2)} y, \delta\rangle \\ &= \sum\langle \textstyle\int_{(1)}, f\rangle\langle \textstyle\int_{(2)}, (y \rightharpoonup \delta)\rangle = \langle \textstyle\int, f(y \rightharpoonup \delta)\rangle \\ &= \textstyle\int f(x)\delta(xy)dx\end{aligned}$$

since $\langle x, (y \rightharpoonup \delta)\rangle = \langle xy, \delta\rangle$. □

2. Derivations

Definition 4.2.1. Let A be a $\mathbb{K}$-algebra and ${}_AM_A$ be an A-A-bimodule (with identical $\mathbb{K}$-action on both sides). A linear map $D : A \longrightarrow M$ is called a *derivation* if

$$D(ab) = aD(b) + D(a)b.$$

The set of derivations $\mathrm{Der}_{\mathbb{K}}(A, {}_AM_A)$ is a $\mathbb{K}$-module and a functor in ${}_AM_A$.

By induction one sees that D satisfies

$$D(a_1 \dots a_n) = \sum_{i=1}^{n} a_1 \dots a_{i-1} D(a_i) a_{i+1} \dots a_n.$$

Let A be a commutative $\mathbb{K}$-algebra and ${}_AM$ be an A-module. Consider M as an A-A-bimodule by $ma := am$. We denote the set of derivations from A to M by $\mathrm{Der}_{\mathbb{K}}(A, M)_c$.

Proposition 4.2.2. 1. *Let A be a $\mathbb{K}$-algebra. Then the functor* $\mathrm{Der}_{\mathbb{K}}(A, \text{-})$: A-$\mathcal{M}od$-$A \longrightarrow \mathcal{V}ec$ *is representable by the* module of differentials Ω_A.

2. *Let A be a commutative $\mathbb{K}$-algebra. Then the functor* $\mathrm{Der}_{\mathbb{K}}(A, \text{-})_c$: A-$\mathcal{M}od \longrightarrow \mathcal{V}ec$ *is representable by the* module of commutative differentials Ω_A^c.

Proof. 1. Represent A as a quotient of a free $\mathbb{K}$-algebra $A := \mathbb{K}\langle X_i | i \in J\rangle / I$ where $B = \mathbb{K}\langle X_i | i \in J\rangle$ is the free algebra with generators X_i. We first prove the theorem for free algebras.

a) A representing module for $\mathrm{Der}_{\mathbb{K}}(B, \text{-})$ is $(\Omega_B, d : B \longrightarrow \Omega_B)$ with

$$\Omega_B := B \otimes F(dX_i | i \in J) \otimes B$$

where $F(dX_i | i \in J)$ is the free $\mathbb{K}$-module on the set of formal symbols $\{dX_i | i \in J\}$ as a basis.

We have to show that for every derivation $D : B \longrightarrow M$ there exists a unique homomorphisms $\varphi : \Omega_B \longrightarrow M$ of B-B-bimodules such that the diagram

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "D"'] & \Omega_B \arrow[d, "\varphi"] \\ & M \end{tikzcd}$$

commutes. The module Ω_B is a B-B-bimodule in the canonical way. The products $X_1 \dots X_n$ of the generators X_i of B form a basis for B. For any product $X_1 \dots X_n$ we define $d(X_1 \dots X_n) := \sum_{i=1}^n X_1 \dots X_{i-1} \otimes dX_i \otimes X_{i+1} \dots X_n$ in particular $d(X_i) = 1 \otimes dX_i \otimes 1$. To see that d is a derivation it suffices to show this on the basis elements:

$$\begin{aligned} & d(X_1 \dots X_k X_{k+1} \dots X_n) \\ & \qquad = \textstyle\sum_{j=1}^k X_1 \dots X_{j-1} \otimes dX_j \otimes X_{j+1} \dots X_k X_{k+1} \dots X_n \\ & \qquad\ + \textstyle\sum_{j=k+1}^n X_1 \dots X_k X_{k+1} \dots X_{j-1} \otimes dX_j \otimes X_{j+1} \dots X_n \\ & \qquad = d(X_1 \dots X_k) X_{k+1} \dots X_n + X_1 \dots X_k d(X_{k+1} \dots X_n) \end{aligned}$$

Now let $D : B \longrightarrow M$ be a derivation. Define φ by $\varphi(1 \otimes dX_i \otimes 1) := D(X_i)$. This map obviously extends to a homomorphism of B-B-bimodules. Furthermore we have

$$\begin{aligned} \varphi d(X_1 \dots X_n) &= \varphi(\textstyle\sum_j X_1 \dots X_{j-1} \otimes dX_j \otimes X_{j+1} \dots X_n) \\ &= \textstyle\sum_j X_1 \dots X_{j-1} \varphi(1 \otimes dX_j \otimes 1) X_{j+1} \dots X_n = D(X_1 \dots X_n) \end{aligned}$$

hence $\varphi d = D$.

To show the uniqueness of φ let $\psi : \Omega_B \longrightarrow M$ be a bimodule homomorphism such that $\psi d = D$. Then $\psi(1 \otimes dX_i \otimes 1) = \psi d(X_i) = D(X_i) = \varphi(1 \otimes dX_i \otimes 1)$. Since ψ and φ are B-B-bimodules homomorphisms this extends to $\psi = \varphi$.

b) Now let $A := \mathbb{K}\langle X_i | i \in J\rangle / I$ be an arbitrary algebra with $B = \mathbb{K}\langle X_i | i \in J\rangle$ free. Define

$$\Omega_A := \Omega_B/(I\Omega_B + \Omega_B I + Bd_B(I) + d_B(I)B).$$

We first show that $I\Omega_B + \Omega_B I + Bd_B(I) + d_B(I)B$ is a B-B-subbimodule. Since Ω_B and I are B-B-bimodules the terms $I\Omega_B$ and $\Omega_B I$ are bimodules. Furthermore we have $bd_B(i)b' = bd_B(ib') - bid_B(b') \in Bd_B(I) + I\Omega_B$ hence $I\Omega_B + \Omega_B I + Bd_B(I) + d_B(I)B$ is a bimodule.

Now $I\Omega_B$ and $\Omega_B I$ are subbimodules of $I\Omega_B + \Omega_B I + Bd_B(I) + d_B(I)B$. Hence $A = B/I$ acts on both sides on Ω_A so that Ω_A becomes an A-A-bimodule.

Let $\nu : \Omega_B \longrightarrow \Omega_A$ and also $\nu : B \longrightarrow A$ be the residue homomorphisms. Since $\nu d_B(i) \in \nu d_B(I) = 0 \subseteq \Omega_A$ we get a unique factorization map $d_A : A \longrightarrow \Omega_A$ such that

$$\begin{tikzcd} B \arrow[r, "d_B"] \arrow[d, "\nu"'] & \Omega_B \arrow[d, "\nu"] \\ A \arrow[r, "d_A"'] & \Omega_A \end{tikzcd}$$

commutes. Since $d_A(\overline{b}) = \overline{d_B(b)}$ it is clear that d_A is a derivation.

Let $D : A \longrightarrow M$ be a derivation. The A-A-bimodule M is also a B-B-bimodule by $bm = \overline{b}m$. Furthermore $D\nu : B \longrightarrow A \longrightarrow M$ is again a derivation. Let $\varphi_B : \Omega_B \longrightarrow M$ be the unique factorization map for the B-derivation $D\nu$. Consider the following diagram

$$\begin{tikzcd} B \arrow[r, "d_B"] \arrow[d] & \Omega_B \arrow[d] \arrow[dd, bend left, "\varphi"] \\ A \arrow[r, "d_A"] \arrow[dr, "D"'] & \Omega_A \arrow[d, "\psi"] \\ & M \end{tikzcd}$$

We want to construct ψ such that the diagram commutes. Let $i\omega \in I\Omega_B$. Then $\varphi(i\omega) = \overline{i}\varphi(\omega) = 0$ and similarly $\varphi(\omega i) = 0$. Let $bd_B(i) \in Bd_B(I)$ then $\varphi(bd_B(i)) = \overline{b}\varphi d_B(i) = \overline{b}D(\overline{i}) = 0$ and similarly $\varphi(d_B(i)b) = 0$. Hence φ vanishes on $I\Omega_B + \Omega_B I + Bd_B(I) + d_B(I)B$ and thus factorizes through a unique map $\psi : \Omega_A \longrightarrow M$. Obviously ψ is a homomorphism of A-A-bimodules. Furthermore we have $D\nu = \varphi d_B = \psi\nu d_B = \psi d_A \nu$ and, since ν is surjective, $D = \psi d_A$. It is clear that ψ is uniquely determined by this condition.

2. If A is commutative then we can write $A = \mathbb{K}[X_i | i \in J]/I$ and $\Omega^c_B = B \otimes F(dX_i)$. With $\Omega^c_A = \Omega^c_B/(I\Omega^c_B + Bd_B(I))$ the proof is analogous to the proof in the noncommutative situation. □

Remark 4.2.3. 1. Ω_A is generated by $d(A)$ as a bimodule, hence all elements are of the form $\sum_i a_i d(a'_i)a''_i$. These elements are called *differentials*.

2. If $A = \mathbb{K}\langle X_i \rangle/I$, then Ω_A is generated as a bimodule by the elements $\{\overline{d(X_i)}\}$.

3. Let $f \in B = \mathbb{K}\langle X_i \rangle$. Let B^{op} be the algebra opposite to B (with opposite multiplication). Then $\Omega_B = B \otimes F(dX_i) \otimes B$ is the free $B \otimes B^{op}$ left module over the free generating set $\{d(X_i)\}$. Hence $d(f)$ has a unique representation

$$d(f) = \sum_i \frac{\partial f}{\partial X_i} d(X_i)$$

with uniquely defined coefficients

$$\frac{\partial f}{\partial X_i} \in B \otimes B^{op}.$$

In the commutative situation we have unique coefficients

$$\frac{\partial f}{\partial X_i} \in \mathbb{K}[X_i].$$

4. We give the following examples for part 3:

$$\begin{aligned} \frac{\partial X_i}{\partial X_j} &= \delta_{ij}, \\ \frac{\partial X_1 X_2}{\partial X_1} &= 1 \otimes X_2, \\ \frac{\partial X_1 X_2}{\partial X_2} &= X_1 \otimes 1, \\ \frac{\partial X_1 X_2 X_3}{\partial X_2} &= X_1 \otimes X_3, \\ \frac{\partial X_1 X_3 X_2}{\partial X_2} &= X_1 X_3 \otimes 1. \end{aligned}$$

This is obtained by direct calculation or by the *product rule*

$$\frac{\partial fg}{\partial X_i} = (1 \otimes g)\frac{\partial f}{\partial X_i} + (f \otimes 1)\frac{\partial g}{\partial X_i}.$$

The product rule follows from

$$d(fg) = d(f)g + fd(g) = \sum((1 \otimes g)\frac{\partial f}{\partial X_i} + (f \otimes 1)\frac{\partial g}{\partial X_i})d(X_i).$$

Let $A = \mathbb{K}\langle X_i \rangle / I$. If $f \in I$ then $\overline{d(f)} = d_A(\overline{f}) = 0$ hence

$$\sum \frac{\partial f}{\partial X_i} d_A(\overline{X_i}) = 0.$$

These are the defining relations for the A-A-bimodule Ω_A with the generators $d_A(\overline{X_i})$.

For motivation of the quantum group case we consider an affine algebraic group G with representing commutative Hopf algebra A. Recall that $\mathrm{Hom}(A,R)$ is an algebra with the convolution multiplication for every $R \in \mathbb{K}\text{-}c\mathcal{A}lg$ and that $G(R) = \mathbb{K}\text{-}c\mathcal{A}lg(A,R) \subseteq \mathrm{Hom}(A,R)$ is a subgroup of the group of units of the algebra $\mathrm{Hom}(A,R)$.

Definition and Remark 4.2.4. A linear map $T : A \longrightarrow A$ is called *left translation invariant*, if the following diagram functorial in $R \in \mathbb{K}\text{-}c\mathcal{A}lg$ commutes:

$$\begin{tikzcd}
G(R) \times \mathrm{Hom}(A,R) \arrow[r, "*"] \arrow[d, "1 \otimes \mathrm{Hom}(T,R)"'] & \mathrm{Hom}(A,R) \arrow[d, "\mathrm{Hom}(T,R)"] \\
G(R) \times \mathrm{Hom}(A,R) \arrow[r, "*"'] & \mathrm{Hom}(A,R)
\end{tikzcd}$$

i. e. if we have

$$\forall g \in G(R), \forall x \in \mathrm{Hom}(A,R) : g * (x \circ T) = (g * x) \circ T.$$

This condition is equivalent to

$$\Delta_A \circ T = (1_A \otimes T) \circ \Delta_A. \tag{33}$$

In fact if (33) holds then $g * (x \circ T) = \nabla_R (g \otimes x)(1_A \otimes T)\Delta_A = \nabla_R(g \otimes x)\Delta_A T = (g * x) \circ T$.

Conversely if the diagram commutes, then take $R = A$, $g = 1_A$ and we get $\nabla_A(1_A \otimes x)(1_A \otimes T)\Delta_A = 1_A * (x \circ T) = (1_A * x) \circ T = \nabla_A(1_A \otimes x)\Delta_A T$ for all $x \in \mathrm{Hom}(A,A)$. To get (33) it suffices to show that the terms $\nabla_A(1_A \otimes x)$ can be cancelled in this equation. Let $\sum_{i=1}^n a_i \otimes b_i \in A \otimes A$ be given such that $\nabla_A(1_A \otimes x)(\sum a_i \otimes b_i) = 0$ for all $x \in \mathrm{Hom}(A,A)$ and choose such an element with a shortest representation (n minimal). Then $\sum a_i x(b_i) = 0$ for all x. Since the b_i are linearly independent in such a shortest representation, there are x_i with $x_j(b_i) = \delta_{ij}$. Hence $a_j = \sum a_i x_j(b_i) = 0$ and thus $\sum a_i \otimes b_i = 0$. From this follows (33).

Definition 4.2.5. Let H be an arbitrary Hopf algebra. An element $T \in \mathrm{Hom}(H,H)$ is called *left translation invariant* if it satisfies

$$\Delta_H T = (1_H \otimes T)\Delta_H.$$

Proposition 4.2.6. *Let H be an arbitrary Hopf algebra. Then $\Phi : H^* \longrightarrow \mathrm{End}(H)$ with $\Phi(f) := \mathrm{id} * u_H f$ is an algebra monomorphism satisfying*

$$\Phi(f * g) = \Phi(f) \circ \Phi(g).$$

The image of Φ is precisely the set of left translation invariant elements $T \in \mathrm{End}(H)$.

PROOF. For $f \in \mathrm{Hom}(H, \mathbb{K})$ we have $u_H f \in \mathrm{End}(H)$ hence $\mathrm{id} * u_H f \in \mathrm{End}(H)$. Thus Φ is a well defined homomorphism. Observe that

$$\Phi(f)(a) = (\mathrm{id}_H * u_H f)(a) = \sum a_{(1)} f(a_{(2)}).$$

Φ is injective since it has a retraction $\mathrm{End}(H) \ni g \mapsto \varepsilon_H \circ g \in \mathrm{Hom}(H, \mathbb{K})$. In fact we have $(\varepsilon\Phi(f))(a) = \varepsilon(\sum a_{(1)} f(a_{(2)})) = \sum \varepsilon(a_{(1)}) f(a_{(2)}) = f(\sum \varepsilon(a_{(1)}) a_{(2)}) = f(a)$ hence $\varepsilon\Phi(f) = f$.

The map Φ preserves the algebra unit since $\Phi(1_{H^*}) = \Phi(\varepsilon_H) = \mathrm{id}_H * u_H \varepsilon_H = \mathrm{id}_H$.

The map Φ is compatible with the multiplication: $\Phi(f * g)(a) = \sum a_{(1)}(f * g)(a_{(2)}) = \sum a_{(1)} f(a_{(2)}) g(a_{(3)}) = \sum (\mathrm{id} * u_H f)(a_{(1)}) g(a_{(2)}) = \Phi(f)(\sum a_{(1)} g(a_{(2)})) = \Phi(f)\Phi(g)(a)$ so that $\Phi(f * g) = \Phi(f) \circ \Phi(g)$.

For each $f \in H^*$ the element $\Phi(f)$ is left translation invariant since $\Delta\Phi(f)(a) = \Delta(\sum a_{(1)} f(a_{(2)})) = \sum a_{(1)} \otimes a_{(2)} f(a_{(3)}) = (1 \otimes \Phi(f))\Delta(a)$.

Let $T \in \mathrm{End}(H)$ be left translation invariant then $S * T = \nabla_H(S \otimes 1)(1 \otimes T)\Delta_H = \nabla_H(S \otimes 1)\Delta_H T = u_H \varepsilon_H T$. Thus $\Phi(\varepsilon T) = \mathrm{id} * u_H \varepsilon_H T = \mathrm{id} * S * T = T$, so that T is in the image of Φ. $\square$

Proposition 4.2.7. *Let $d \in \mathrm{Hom}(H, \mathbb{K})$ and $\Phi(d) = D \in \mathrm{Hom}(H, H)$ be given. The following are equivalent:*

(1) *$d : H \longrightarrow {}_\varepsilon\mathbb{K}_\varepsilon$ is a derivation.*
(2) *$D : H \longrightarrow {}_H H_H$ is a (left translation invariant) derivation.*

In particular Φ induces an isomorphism between the set of derivations $d : H \longrightarrow {}_\varepsilon\mathbb{K}_\varepsilon$ and the set of left translation invariant derivations $D : H \longrightarrow {}_H H_H$.

PROOF. Assume that 1. holds so that d satisfies $d(ab) = \varepsilon(a)d(b) + d(a)\varepsilon(b)$. Then we get $D(ab) = \Phi(d)(ab) = \sum a_{(1)} b_{(1)} d(a_{(2)} b_{(2)}) = \sum a_{(1)} b_{(1)} \varepsilon(a_{(2)}) d(b_{(2)}) + \sum a_{(1)} b_{(1)} d(a_{(2)}) \varepsilon(b_{(2)}) = aD(b) + D(a)b$. Conversely assume that $D(ab) = aD(b) + D(a)b$. Then $d(ab) = \varepsilon D(ab) = \varepsilon(a)\varepsilon D(b) + \varepsilon D(a)\varepsilon(b) = \varepsilon(a)d(b) + d(a)\varepsilon(b)$. $\square$

3. The Lie Algebra of Primitive Elements

Lemma 4.3.1. *Let H be a Hopf algebra and H^o be its Sweedler dual. If $d \in \mathrm{Der}_{\mathbb{K}}(H, {}_\varepsilon\mathbb{K}_\varepsilon) \subseteq \mathrm{Hom}(H, \mathbb{K})$ is a derivation then d is a primitive element of H^o. Furthermore every primitive element $d \in H^o$ is a derivation in $\mathrm{Der}_{\mathbb{K}}(H, {}_\varepsilon\mathbb{K}_\varepsilon)$.*

PROOF. Let $d : H \to \mathbb{K}$ be a derivation and let $a, b \in H$. Then $(b \rightharpoonup d)(a) = d(ab) = \varepsilon(a)d(b) + d(a)\varepsilon(b) = (d(b)\varepsilon + \varepsilon(b)d)(a)$ hence $(b \rightharpoonup d) = d(b)\varepsilon + \varepsilon(b)d$. Consequently we have $Hd = (H \rightharpoonup d) \subseteq \mathbb{K}\varepsilon + \mathbb{K}d$ so that $\dim Hd \leq 2 < \infty$. This shows $d \in H^o$. Furthermore we have $\langle \Delta d, a \otimes b \rangle = \langle d, ab \rangle = d(ab) = d(a)\varepsilon(b) + \varepsilon(a)d(b) = \langle d \otimes \varepsilon, a \otimes b \rangle + \langle \varepsilon \otimes d, a \otimes b \rangle = \langle 1_{H^o} \otimes d + d \otimes 1_{H^o}, a \otimes b \rangle$ hence $\Delta(d) = d \otimes 1_{H^o} + 1_{H^o} \otimes d$ so that d is a primitive element in H^o.

Conversely let $d \in H^o$ be primitive. then $d(ab) = \langle \Delta(d), a \otimes b \rangle = d(a)\varepsilon(b) + \varepsilon(a)d(b)$. $\square$

Proposition and Definition 4.3.2. *Let H be a Hopf algebra. The set of primitive elements of H will be denoted by $\mathcal{L}ie(H)$ and is a Lie algebra. If* $\operatorname{char}(\mathbb{K}) = p > 0$ *then $\mathcal{L}ie(H)$ is a restricted Lie algebra or a p-Lie algebra.*

PROOF. Let $a, b \in H$ be primitive elements. Then $\Delta([a,b]) = \Delta(ab - ba) = (a \otimes 1 + 1 \otimes a)(b \otimes 1 + 1 \otimes b) - (b \otimes 1 + 1 \otimes b)(a \otimes 1 + 1 \otimes a) = (ab - ba) \otimes 1 + 1 \otimes (ab - ba)$ hence $\mathcal{L}ie(H) \subseteq H^L$ is a Lie algebra. If the characteristic of $\mathbb{K}$ is $p > 0$ then we have $(a \otimes 1 + 1 \otimes a)^p = a^p \otimes 1 + 1 \otimes a^p$. Thus $\mathcal{L}ie(H)$ is a restricted Lie subalgebra of H^L with the structure maps $[a, b] = ab - ba$ and $a^{[p]} = a^p$. $\square$

Corollary 4.3.3. *Let H be a Hopf algebra. Then the set of left translation invariant derivations $D : H \longrightarrow H$ is a Lie algebra under $[D, D'] = DD' - D'D$. If* $\operatorname{char} = p$ *then these derivations are a restricted Lie algebra with $D^{[p]} = D^p$.*

PROOF. The map $\Psi : H^o \longrightarrow H^* \xrightarrow{\Phi} \operatorname{End}(H)$ is a homomorphism of algebras by 4.2.6. Hence $\Psi(d * d' - d' * d) = \Phi(d * d' - d' * d) = \Phi(d)\Phi(d') - \Phi(d')\Phi(d)$. If d is a primitive element in H^o then by 4.2.7 and 4.3.1 the image $D := \Psi(d)$ in $\operatorname{End}(H)$ is a left translation invariant derivation and all left translation invariant derivations are of this form. Since $[d, d'] = d * d' - d' * d$ is again primitive we get that $[D, D'] = DD' - D'D$ is a left translation invariant derivation so that the set of left translation invariant derivations $\operatorname{Der}^H_{\mathbb{K}}(H, H)$ is a Lie algebra resp. a restricted Lie algebra. $\square$

Definition 4.3.4. Let H be a Hopf algebra. An element $c \in H$ is called *cocommutative* if $\tau\Delta(c) = \Delta(c)$, i. e. if $\sum c_{(1)} \otimes c_{(2)} = \sum c_{(2)} \otimes c_{(1)}$. Let $C(H) := \{c \in H | c \text{ is cocommutative }\}$.

Let $G(H)$ denote the set of grouplike elements of H.

Lemma 4.3.5. *Let H be a Hopf algebra. Then the set of cocommutative elements $C(H)$ is a subalgebra of H and the grouplike elements $G(H)$ form a linearly independent subset of $C(H)$. Furthermore $G(H)$ is a multiplicative subgroup of the group of units $U(C(H))$.*

PROOF. It is clear that $C(H)$ is a linear subspace of H. If $a, b \in C(H)$ then $\Delta(ab) = \Delta(a)\Delta(b) = (\tau\Delta)(a)(\tau\Delta)(b) = \tau(\Delta(a)\Delta(b)) = \tau\Delta(ab)$ and $\Delta(1) = 1 \otimes 1 = \tau\Delta(1)$. Thus $C(H)$ is a subalgebra of H.

The grouplike elements obviously are cocommutative and form a multiplicative group, hence a subgroup of $U(C(H))$. They are linearly independent by Lemma 2.1.14. $\square$

Proposition 4.3.6. *Let H be a Hopf algebra with $S^2 = \mathrm{id}_H$. Then there is a left module structure*

$$C(H) \otimes \mathcal{L}ie(H) \ni c \otimes a \mapsto c \cdot a \in \mathcal{L}ie(H)$$

with $c \cdot a := \nabla_H(\nabla_H \otimes 1)(1 \otimes \tau)(1 \otimes S \otimes 1)(\Delta \otimes 1)(c \otimes a) = \sum c_{(1)}aS(c_{(2)})$ *such that*

$$c \cdot [a, b] = \sum [c_{(1)} \cdot a, c_{(2)} \cdot b].$$

In particular $G(H)$ *acts by Lie automorphisms on* $\mathcal{L}ie(H)$.

PROOF. The given action is actually the action $H \otimes H \to H$ with $h \cdot a = \sum h_{(1)}aS(h_{(2)})$, the so-called *adjoint action*.

We first show that the given map has image in $\mathcal{L}ie(H)$. For $c \in C(H)$ and $a \in \mathcal{L}ie(H)$ we have $\Delta(c \cdot a) = \Delta(\sum c_{(1)}aS(c_{(2)})) = \sum \Delta(c_{(1)})(a \otimes 1 + 1 \otimes a)\Delta(S(c_{(2)})) = \sum \Delta(c_{(1)})(a \otimes 1)\Delta(S(c_{(2)})) + \sum \Delta(c_{(2)})(1 \otimes a)\Delta(S(c_{(1)})) = \sum c_{(1)}aS(c_{(4)}) \otimes c_{(2)}S(c_{(3)}) + \sum c_{(3)}S(c_{(2)}) \otimes c_{(4)}aS(c_{(1)}) = c \cdot a \otimes 1 + 1 \otimes c \cdot a$ since c is cocommutative, $S^2 = \mathrm{id}_H$ and a is primitive.

We show now that $\mathcal{L}ie(H)$ is a $C(H)$-module. $(cd) \cdot a = \sum c_{(1)}d_{(1)}aS(c_{(2)}d_{(2)}) = \sum c_{(1)}d_{(1)}aS(d_{(2)})S(c_{(2)}) = c \cdot (d \cdot a)$. Furthermore we have $1 \cdot a = 1aS(1) = a$.

To show the given formula let $a, b \in \mathcal{L}ie(H)$ and $c \in C(H)$. Then $c \cdot [a, b] = \sum c_{(1)}(ab - ba)S(c_{(2)}) = \sum c_{(1)}aS(c_{(2)})c_{(3)}bS(c_{(4)}) - \sum c_{(1)}bS(c_{(2)})c_{(3)}aS(c_{(4)}) = \sum (c_{(1)} \cdot a)(c_{(2)} \cdot b) - \sum (c_{(1)} \cdot b)(c_{(2)} \cdot a) = \sum [c_{(1)} \cdot a, c_{(2)} \cdot b]$ again since $c \in C(H)$ is cocommutative.

Now let $g \in G(H)$. Then $g \cdot a = gaS(g) = gag^{-1}$ since $S(g) = g^{-1}$ for any grouplike element. Furthermore $g \cdot [a, b] = [g \cdot a, g \cdot b]$ hence g defines a Lie algebra automorphism of $\mathcal{L}ie(H)$. □

Problem 4.3.31. Show that the *adjoint action* $H \otimes H \ni h \otimes a \mapsto \sum h_{(1)}aS(h_{(2)}) \in H$ makes H an H-module algebra.

Definition and Remark 4.3.7. The algebra $\mathbb{K}(\delta) = \mathbb{K}[\delta]/(\delta^2)$ is called the algebra of *dual numbers*. Observe that $\mathbb{K}(\delta) = \mathbb{K} \oplus \mathbb{K}\delta$ as a $\mathbb{K}$-module.

We consider δ as a "small quantity" whose square vanishes.

The maps $p : \mathbb{K}(\delta) \to K$ with $p(\delta) = 0$ and $j : \mathbb{K} \to \mathbb{K}(\delta)$ are algebra homomorphism satisfying $pj = \mathrm{id}$.

Let $\mathbb{K}(\delta, \delta') := \mathbb{K}[\delta, \delta']/(\delta^2, \delta'^2)$. Then $\mathbb{K}(\delta, \delta') = \mathbb{K} \oplus \mathbb{K}\delta \oplus \mathbb{K}\delta' \oplus \mathbb{K}\delta\delta'$. The map $\mathbb{K}(\delta) \ni \delta \mapsto \delta\delta' \in \mathbb{K}(\delta, \delta')$ is an injective algebra homomorphism. Furthermore for every $\alpha \in \mathbb{K}$ we have an algebra homomorphism $\varphi_\alpha : \mathbb{K}(\delta) \ni \delta \mapsto \alpha\delta \in \mathbb{K}(\delta)$.

These algebra homomorphisms induce algebra homomorphisms $H \otimes \mathbb{K}(\delta) \to H \otimes \mathbb{K}(\delta)$ resp. $H \otimes \mathbb{K}(\delta) \to H \otimes \mathbb{K}(\delta, \delta')$ for every Hopf algebra H.

Proposition 4.3.8. *The map*

$$e^{\delta -} : \mathcal{L}ie(H) \to H \otimes \mathbb{K}(\delta) \subseteq H \otimes \mathbb{K}(\delta, \delta')$$

with $e^{\delta a} := 1 + a \otimes \delta = 1 + \delta a$ *is called the* exponential map *and satisfies*

$$\begin{aligned} e^{\delta(a+b)} &= e^{\delta a} e^{\delta b}, \\ e^{\delta \alpha a} &= \varphi_\alpha(e^{\delta a}), \\ e^{\delta\delta'[a,b]} &= e^{\delta a} e^{\delta' b} (e^{\delta a})^{-1} (e^{\delta' b})^{-1}. \end{aligned}$$

Furthermore all elements $e^{\delta a} \in H \otimes \mathbb{K}(\delta)$ *are grouplike in the* $\mathbb{K}(\delta)$*-Hopf algebra* $H \otimes \mathbb{K}(\delta)$.

PROOF. 1. $e^{\delta(a+b)} = (1+\delta(a+b)) = (1+\delta a)(1+\delta b) = e^{\delta a}e^{\delta b}$.

2. $e^{\delta\alpha a} = 1+\delta\alpha a = \varphi_\alpha(1+\delta a) = \varphi_\alpha(e^{\delta a})$.

3. Since $(1+\delta a)(1-\delta a) = 1$ we have $(e^{\delta a}) = 1-\delta a$. So we get $e^{\delta\delta'[a,b]} = 1+\delta[a,b] = 1+\delta(a-a)+\delta'(b-b)+\delta\delta'(ab-ab-ba+ab) = (1+\delta a)(1+\delta' b)(1-\delta a)(1-\delta' b) = e^{\delta a}e^{\delta' b}(e^{\delta a})^{-1}(e^{\delta' b})^{-1}$.

4. $\Delta_{\mathbb{K}(\delta)}(e^{\delta a}) = \Delta(1+a\otimes\delta) = 1\otimes_{\mathbb{K}(\delta)} 1 + (a\otimes 1 + 1\otimes a)\otimes\delta = 1\otimes_{\mathbb{K}(\delta)} 1 + \delta a\otimes_{\mathbb{K}(\delta)} 1 + 1\otimes_{\mathbb{K}(\delta)} \delta a + \delta a\otimes_{\mathbb{K}(\delta)} \delta a = (1+\delta a)\otimes_{\mathbb{K}(\delta)}(1+\delta a) = e^{\delta a}\otimes_{\mathbb{K}(\delta)} e^{\delta a}$ and $\varepsilon_{\mathbb{K}(\delta)}(e^{\delta a}) = \varepsilon_{\mathbb{K}(\delta)}(1+\delta a) = 1+\delta\varepsilon(a) = 1$. □

Corollary 4.3.9. *$(\mathcal{L}ie(H), e^{\delta -})$ is the kernel of the group homomorphism $p : G_{\mathbb{K}(\delta)}(H\otimes\mathbb{K}(\delta)) \longrightarrow G(H)$.*

PROOF. $p = 1\otimes p : H\otimes\mathbb{K}(\delta) \longrightarrow H\otimes\mathbb{K} = H$ is a homomorphism of $\mathbb{K}$-algebras. We show that it preserves grouplike elements. Observe that grouplike elements in $H\otimes\mathbb{K}(\delta)$ are defined by the Hopf algebra structure over $\mathbb{K}(\delta)$. Let $g\in G_{\mathbb{K}(\delta)}(H\otimes\mathbb{K}(\delta))$. Then $(\Delta_H\otimes 1)(g) = g\otimes_{\mathbb{K}(\delta)} g$ and $(\varepsilon_H\otimes 1)(g) = 1\in\mathbb{K}(\delta)$.

Since $p : \mathbb{K}(\delta)\longrightarrow\mathbb{K}$ is an algebra homomorphism the following diagram commutes

$$\begin{array}{ccc} (H\otimes\mathbb{K}(\delta))\otimes_{\mathbb{K}(\delta)}(H\otimes\mathbb{K}(\delta)) & \xrightarrow{\cong} & H\otimes H\otimes\mathbb{K}(\delta) \\ \downarrow{\scriptstyle (1\otimes p)\otimes(1\otimes p)} & & \downarrow{\scriptstyle 1\otimes p} \\ (H\otimes\mathbb{K})\otimes(H\otimes\mathbb{K}) & \xrightarrow[\cong]{} & H\otimes H\otimes\mathbb{K}. \end{array}$$

We identify elements along the isomorphisms. Thus we get $(\Delta_H\otimes 1_{\mathbb{K}})(1_H\otimes p)(g) = (1_{H\otimes H}\otimes p)(\Delta_H\otimes 1_{\mathbb{K}(\delta)})(g) = ((1_H\otimes p)\otimes_{\mathbb{K}(\delta)}(1_H\otimes p))(g\otimes_{\mathbb{K}(\delta)} g) = (1_H\otimes p)(g)\otimes(1_H\otimes p)(g)$, so that $1_H\otimes p : G_{\mathbb{K}(\delta)}(H\otimes\mathbb{K}(\delta))\longrightarrow G(H)$. Now we have $(1_H\otimes p)(gg') = (1_H\otimes p)(g)(1_H\otimes p)(g')$ so that $1_H\otimes p$ is a group homomorphism.

Now let $g = g_0\otimes 1 + g_1\otimes\delta\in G_{\mathbb{K}(\delta)}(H\otimes\mathbb{K}(\delta))\subseteq H\otimes\mathbb{K}\oplus H\otimes\mathbb{K}\delta$. Then we have $(1_H\otimes p)(g) = 1$ iff $g_0 = 1$ iff $g = 1_H\otimes 1_{\mathbb{K}(\delta)} + g_1\otimes\delta$. Furthermore we have

$$\begin{aligned} &\Delta_{H\otimes\mathbb{K}(\delta)}(g) = g\otimes_{\mathbb{K}(\delta)} g \iff \\ &1_H\otimes 1_H\otimes 1_{\mathbb{K}(\delta)} + \Delta_H(g_1)\otimes\delta = (1_H\otimes 1_{\mathbb{K}(\delta)} + g_1\otimes\delta)\otimes_{\mathbb{K}(\delta)}(1_H\otimes 1_{\mathbb{K}(\delta)} + g_1\otimes\delta) \\ &\qquad = 1_H\otimes 1_H\otimes 1_{\mathbb{K}(\delta)} + (g_1\otimes 1_H + 1_H\otimes g_1)\otimes\delta \iff \\ &\Delta_H(g_1) = g_1\otimes 1_H + 1_H\otimes g_1. \end{aligned}$$

Similarly we have $\varepsilon_{\mathbb{K}(\delta)})(g) = 1$ iff $1\otimes 1 + \varepsilon(g_1)\otimes\delta = 1$ iff $\varepsilon(g_1) = 0$. □

4. Derivations and Lie Algebras of Affine Algebraic Groups

Lemma and Definition 7.4.1. *Let $\mathcal{G} : \mathbb{K}\text{-}c\mathcal{A}lg \longrightarrow \mathcal{S}et$ be a group valued functor. The kernel $Lie(\mathcal{G})(R)$ of the sequence*

$$0 \longrightarrow Lie(\mathcal{G})(R) \longrightarrow \mathcal{G}(R(\delta)) \underset{\mathcal{G}(j)}{\overset{\mathcal{G}(p)}{\rightleftarrows}} \mathcal{G}(R) \longrightarrow 0$$

is called the Lie algebra *of $\mathcal{G}$ and is a group valued functor in R.*

PROOF. For every algebra homomorphism $f : R \to S$ the following diagram of groups commutes

$$\begin{tikzcd}
0 \arrow[r] & Lie(\mathcal{G})(R) \arrow[r] \arrow[d] & \mathcal{G}(R(\delta)) \arrow[r, shift left, "\mathcal{G}(p)"] \arrow[d, "\mathcal{G}(f(\delta))"] & \mathcal{G}(R) \arrow[r] \arrow[l, shift left, "\mathcal{G}(j)"] \arrow[d, "\mathcal{G}(f)"] & 0 \\
0 \arrow[r] & Lie(\mathcal{G})(S) \arrow[r] & \mathcal{G}(S(\delta)) \arrow[r, shift left, "\mathcal{G}(p)"] & \mathcal{G}(S) \arrow[r] \arrow[l, shift left, "\mathcal{G}(j)"] & 0
\end{tikzcd}$$

□

Proposition 4.4.2. *Let $\mathcal{G} : \mathbb{K}\text{-}c\mathcal{A}lg \to \mathcal{S}et$ be a group valued functor with multiplication $*$. Then there are functorial operations*

$$\mathcal{G}(R) \times Lie(\mathcal{G})(R) \ni (g, x) \mapsto g \cdot x \in Lie(\mathcal{G})(R)$$

$$R \times Lie(\mathcal{G})(R) \ni (a, x) \mapsto ax \in Lie(\mathcal{G})(R)$$

such that

$$\begin{gathered}
g \cdot (x + y) = g \cdot x + g \cdot y, \\
h \cdot (g \cdot x) = (h * g) \cdot x, \\
a(x + y) = ax + ay, \\
(ab)x = a(bx), \\
g \cdot (ax) = a(g \cdot x).
\end{gathered}$$

PROOF. First observe that the composition $+$ on $Lie(\mathcal{G})(R)$ is induced by the multiplication $*$ of $\mathcal{G}(R(\delta))$ so it is not necessarily commutative.

We define $g \cdot x := \mathcal{G}(j)(g) * x * \mathcal{G}(j)(g)^{-1}$. Then $\mathcal{G}(p)(g \cdot x) = \mathcal{G}(p)\mathcal{G}(j)(g) * \mathcal{G}(p)(x) * \mathcal{G}(p)\mathcal{G}(j)(g)^{-1} = g * 1 * g^{-1} = 1$ hence $g \cdot x \in Lie(\mathcal{G})(R)$.

Now let $a \in R$. To define $a : Lie(\mathcal{G})(R) \to Lie(\mathcal{G})(R)$ we use $u_a : R(\delta) \to R(\delta)$ defined by $u_a(\delta) := a\delta$ and thus $u_a(b + c\delta) := b + ac\delta$. Obviously u_a is a homomorphism of R-algebras. Furthermore we have $pu_a = p$ and $u_a j = j$. Thus we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Lie(\mathcal{G})(R) \arrow[r] \arrow[d, "a"] & \mathcal{G}(R(\delta)) \arrow[r, shift left, "\mathcal{G}(p)"] \arrow[d, "\mathcal{G}(u_a)"] & \mathcal{G}(R) \arrow[r] \arrow[l, shift left, "\mathcal{G}(j)"] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & Lie(\mathcal{G})(R) \arrow[r] & \mathcal{G}(R(\delta)) \arrow[r, shift left, "\mathcal{G}(p)"] & \mathcal{G}(R) \arrow[r] \arrow[l, shift left, "\mathcal{G}(j)"] & 0
\end{tikzcd}$$

that defines a group homomorphism $a : Lie(\mathcal{G})(R) \to Lie(\mathcal{G})(R)$ on the kernel of the exact sequences. In particular we have then $a(x + y) = ax + ay$.

Furthermore we have $u_{ab} = u_a u_b$ hence $(ab)x = a(bx)$.

The next formula follows from $g \cdot (x + y) = \mathcal{G}(j)(g) * x * y * \mathcal{G}(j)(g)^{-1} = \mathcal{G}(j)(g) * x * \mathcal{G}(j)(g)^{-1} * \mathcal{G}(j)(g) * y * \mathcal{G}(j)(g)^{-1} = g \cdot x + g \cdot y$.

We also see $(h*g)\cdot x = \mathcal{G}(j)(h*g)*x*\mathcal{G}(j)(h*g)^{-1} = \mathcal{G}(j)(h)*\mathcal{G}(j)(g)*x*\mathcal{G}(j)(g)^{-1}*\mathcal{G}(j)(h)^{-1} = h\cdot(g\cdot x)$. Finally we have $g\cdot(ax) = \mathcal{G}(j)(g)*\mathcal{G}(u_a)(x)*\mathcal{G}(j)(g^{-1}) = \mathcal{G}(u_a)(\mathcal{G}(j)(g)*x*\mathcal{G}(j)(g^{-1})) = a(g\cdot x)$. $\square$

Proposition 4.4.3. *Let $\mathcal{G} = \mathbb{K}\text{-}c\mathcal{A}lg(H,\text{-})$ be an affine algebraic group. Then $Lie(\mathcal{G})(\mathbb{K}) \cong \mathcal{L}ie(H^o)$ as additive groups. The isomorphism is compatible with the operations given in 4.4.2 and 4.3.6.*

PROOF. We consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & Lie(\mathcal{G})(\mathbb{K}) \arrow[r] & \mathbb{K}\text{-}c\mathcal{A}lg(H,\mathbb{K}(\delta)) \arrow[r, shift left, "p"] \arrow[d, "\omega"] & \mathbb{K}\text{-}c\mathcal{A}lg(H,\mathbb{K}) \arrow[l, shift left] \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathcal{L}ie(H^o) \arrow[r, "e^{\delta -}"] \arrow[u, "e"] & G_{\mathbb{K}(\delta)}(H^o\otimes\mathbb{K}(\delta)) \arrow[r, "p"] & G(H^o) \arrow[r] & 0
\end{tikzcd}$$

We know by definition that the top sequence is exact. The bottom sequence is exact by Corollary 4.3.9.

Let $f \in \mathbb{K}\text{-}c\mathcal{A}lg(H,\mathbb{K})$. Since $\mathrm{Ker}(f)$ is an ideal of codimension 1 we get $f \in H^o$. The map f is an algebra homomorphism iff $\langle f, ab\rangle = \langle f\otimes f, a\otimes b\rangle$ and $\langle f,1\rangle = 1$ iff $\Delta_{H^o}(f) = f\otimes f$ and $\varepsilon_{H^o}(f) = 1$ iff $f \in G(H^o)$. Hence we get the right hand vertical isomorphism $\mathbb{K}\text{-}c\mathcal{A}lg(H,\mathbb{K}) \cong G(H^o)$.

Consider an element $f \in \mathbb{K}\text{-}c\mathcal{A}lg(H,\mathbb{K}(\delta)) \subseteq \mathrm{Hom}(H,\mathbb{K}(\delta))$. It can be written as $f = f_0 + f_1\delta$ with $f_0, f_1 \in \mathrm{Hom}(H,\mathbb{K})$. The linear map f is an algebra homomorphism iff $f_0 : H \longrightarrow \mathbb{K}$ is an algebra homomorphism and f_1 satisfies $f_1(1) = 0$ and $f_1(ab) = f_0(a)f_1(b) + f_1(a)f_0(b)$. In fact we have $f(1) = f_0(1) + f_1(1)\delta = 1$ iff $f_0(1) = 1$ and $f_1(1) = 0$ (by comparing coefficients). Furthermore we have $f(ab) = f(a)f(b)$ iff $f_0(ab) + f_1(ab)\delta = (f_0(a) + f_1(a)\delta)(f_0(b) + f_1(b)\delta) = f_0(a)f_0(b) + f_0(a)f_1(b)\delta + f_1(a)f_0(b)\delta$ iff $f_0(ab) = f_0(a)f_0(b)$ and $f_1(ab) = f_0(a)f_1(b) + f_1(a)f_0(b)$.

Since f_0 is an algebra homomorphism we have as above $f_0 \in H^o$. For f_1 we have $(b \rightharpoonup f_1)(a) = f_1(ab) = f_0(a)f_1(b) + f_1(a)f_0(b) = (f_1(b)f_0 + f_0(b)f_1)(a)$ hence $(b \rightharpoonup f_1) = f_1(b)f_0 + f_0(b)f_1 \in \mathbb{K}f_0 + \mathbb{K}f_1$, a two dimensional subspace. Thus $f_1 \in H^o$.

In the following computations we will identify $(H^o\otimes\mathbb{K}(\delta))\otimes_{\mathbb{K}(\delta)}(H^o\otimes\mathbb{K}(\delta))$ with $H^o\otimes H^o\otimes|K(\delta)$.

Let $f = f_0 + f_1\delta = f_0\otimes 1 + f_1\otimes\delta \in H^o\oplus H^o\delta = H^o\otimes\mathbb{K}(\delta)$. Then f is a homomorphism of algebras iff $f(ab) = f(a)f(b)$ and $f(1) = 1$ iff $f_0(ab) = f_0(a)f_0(b)$ and $f_1(ab) = f_0(a)f_1(b) + f_1(a)f_0(b)$ and $f_0(1) = 1$ and $f_1(1) = 0$ iff $\Delta_{H^o}(f_0) = f_0\otimes f_0$ and $\Delta_{H^o}(f_1) = f_0\otimes f_1 + f_1\otimes f_0$ and $\varepsilon_{H^o}(f_0) = 1$ and $\varepsilon_{H^o}(f_1) = 0$ iff $(\Delta_{H^o}\otimes\mathrm{id}_{\mathbb{K}(\delta)})(f_0\otimes 1 + f_1\otimes\delta) = f_0\otimes f_0\otimes 1 + f_0\otimes f_1\otimes\delta + f_1\otimes f_0\otimes\delta = (f_0\otimes 1 + f_1\otimes\delta)\otimes_{\mathbb{K}(\delta)}(f_0\otimes 1 + f_1\otimes\delta)$ and $(\varepsilon_{H^o}\otimes\mathrm{id}_{\mathbb{K}(\delta)})(f_0\otimes 1 + f_1\otimes\delta) = 1\otimes 1$ iff $(\Delta_{H^o}\otimes\mathrm{id}_{\mathbb{K}(\delta)})(f) = f\otimes_{\mathbb{K}(\delta)} f$ and $(\varepsilon_{H^o}\otimes\mathrm{id}_{\mathbb{K}(\delta)})(f) = 1$ iff $f \in G_{\mathbb{K}(\delta)}(H^o\otimes\mathbb{K}(\delta))$.

Hence we have a bijective map $\omega : \mathbb{K}\text{-}c\mathcal{A}lg(H,\mathbb{K}(\delta)) \ni f = f_0 + f_1\delta \mapsto f_0\otimes 1 + f_1\otimes\delta \in G_{\mathbb{K}(\delta)}(H^o\otimes\mathbb{K}(\delta))$. Since the group multiplication in $\mathbb{K}\text{-}c\mathcal{A}lg(H,\mathbb{K}(\delta)) \subseteq \mathrm{Hom}(H,\mathbb{K}(\delta))$ is the convolution $*$ and the group multiplication in $G_{\mathbb{K}(\delta)}(H^o\otimes\mathbb{K}(\delta)) \subseteq$

$H^o \otimes \mathbb{K}(\delta)$ is the ordinary algebra multiplication, where the multiplication of H^o again is the convolution, it is clear that ω is a group homomorphism. Furthermore the right hand square of the above diagram commutes. Thus we get an isomorphism $e : \mathcal{L}ie(H^o) \longrightarrow Lie(\mathcal{G})(\mathbb{K})$ on the kernels. This map is defined by $e(d) = 1 + d\delta \in \mathbb{K}\text{-}c\mathcal{A}lg(H, \mathbb{K}(\delta))$.

To show that this isomorphism is compatible with the actions of $\mathbb{K}$ resp. $G(H^o)$ let $\alpha \in \mathbb{K}$, $a \in H$, and $d \in \mathcal{L}ie(H^o)$. We have $e(\alpha d)(a) = \varepsilon(a) + \alpha d(a)\delta = u_\alpha(\varepsilon(a) + d(a)\delta) = (u_\alpha \circ (1 + d\delta))(a) = (u_\alpha \circ e(d))(a) = (\alpha e(d))(a)$ hence $e(\alpha d) = \alpha e(d)$.

Furthermore let $g \in G(H^o) = \mathbb{K}\text{-}c\mathcal{A}lg(H, \mathbb{K})$, $a \in H$, and $d \in \mathcal{L}ie(H^o)$. Then we have $e(g \cdot d)(a) = e(gdg^{-1})(a) = (1 + gdg^{-1}\delta)(a) = \varepsilon(a) + gdg^{-1}(a)\delta = \sum g(a_{(1)})\varepsilon(a_{(2)})gS(a_{(3)}) + \sum g(a_{(1)})d(a_{(2)})gS(a_{(3)})\delta = \sum g(a_{(1)})e(d)(a_{(2)})gS(a_{(3)}) = (j \circ g * e(d) * j \circ g^{-1})(a) = (g \cdot e(d))(a)$ hence $e(g \cdot d) = g \cdot e(d)$. □

Proposition 4.4.4. *Let H be a Hopf algebra and let $I := \mathrm{Ker}(\varepsilon)$. Then $\mathrm{Der}_\varepsilon(H, \text{-}) : \mathcal{V}ec \longrightarrow \mathcal{V}ec$ is representable by I/I^2 and $d : H \xrightarrow{1-\varepsilon} I \xrightarrow{\nu} I/I^2$, in particular*

$$\mathrm{Der}_\varepsilon(H, \text{-}) \cong \mathrm{Hom}(I/I^2, \text{-}) \qquad \textit{and} \qquad \mathcal{L}ie(H^o) \cong \mathrm{Hom}(I/I^2, \mathbb{K}).$$

PROOF. Because of $\varepsilon(\mathrm{id} - u\varepsilon)(a) = \varepsilon(a) - \varepsilon u \varepsilon(a) = 0$ we have $\mathrm{Im}(\mathrm{id} - \varepsilon) \subseteq I$. Let $i \in I$. Then we have $i = i - \varepsilon(i) = (\mathrm{id} - \varepsilon)(i)$ hence $\mathrm{Im}(\mathrm{id} - \varepsilon) = \mathrm{Ker}(\varepsilon)$. We have $I^2 \ni (\mathrm{id} - \varepsilon)(a)(\mathrm{id} - \varepsilon)(b) = ab - \varepsilon(a)b - a\varepsilon(b) + \varepsilon(a)\varepsilon(b) = (\mathrm{id} - \varepsilon)(ab) - \varepsilon(a)(\mathrm{id} - \varepsilon)(b) - (\mathrm{id} - \varepsilon)(b)$. Hence we have in I/I^2 the equation $(\mathrm{id} - \varepsilon)(ab) = \varepsilon(a)(\mathrm{id} - \varepsilon)(b) + (\mathrm{id} - \varepsilon)(a)\varepsilon(b)$ so that $\nu(\mathrm{id} - \varepsilon) : H \longrightarrow I \longrightarrow I/I^2$ is an ε-derivation.

Now let $D : H \longrightarrow M$ be an ε-derivation. Then $D(1) = D(1\,1) = 1D(1) + D(1)1$ hence $D(1) = 0$. It follows $D(a) = D(\mathrm{id} - \varepsilon)(a)$. From $\varepsilon(I) = 0$ we get $D(I^2) \subseteq \varepsilon(I)D(I) + D(I)\varepsilon(I) = 0$ hence there is a unique factorization

$$\begin{tikzcd}
H \arrow[r, "\mathrm{id}-\varepsilon"] \arrow[drr, "D"'] & I \arrow[r, "\nu"] \arrow[dr, "D"] & I/I^2 \arrow[d, "f"] \\
& & M.
\end{tikzcd}$$

□

Corollary 4.4.5. *Let H be a Hopf algebra that is finitely generated a s an algebra. Then $\mathcal{L}ie(H^o)$ is finite dimensional.*

PROOF. Let $H = \mathbb{K}\langle a_1, \ldots, a_n \rangle$. Since $H = \mathbb{K} \oplus I$ we can choose $a_1 = 1$ and $a_2, \ldots, a_n \in I$. Thus any element in $i \in I$ can be written as $\sum \alpha_J a_{j_1} \ldots a_{j_k}$ so that $I/I^2 = \mathbb{K}\overline{a_2} + \ldots + \overline{a_n}$. This gives the result. □

Proposition 4.4.6. *Let H be a commutative Hopf algebra and ${}_HM$ be an H-module. Then we have $\Omega_H \cong H \otimes I/I^2$ and $d : H \longrightarrow H \otimes I/I^2$ is given by $d(a) = \sum a_{(1)} \otimes \overline{(\mathrm{id} - \varepsilon)(a_{(2)})}$.*

Proof. Consider the algebra $B := H \oplus M$ with $(a,m)(a',m') = (aa', am' + a'm)$. Let $\mathcal{G} = \mathbb{K}\text{-}c\mathcal{A}lg(H,\text{-})$. Then we have $\mathcal{G}(B) \subseteq \mathrm{Hom}(H,B) \cong \mathrm{Hom}(H,H) \oplus \mathrm{Hom}(H,M)$. An element $(\varphi, D) \in \mathrm{Hom}(H,B)$ is in $\mathcal{G}(B)$ iff $(\varphi,D)(1) = (\varphi(1), D(1)) = (1,0)$, hence $\varphi(1) = 1$ and $D(1) = 0$, and $(\varphi(ab), D(ab)) = (\varphi,D)(ab) = (\varphi,D)(a)(\varphi,D)(b) = (\varphi(a), D(a))(\varphi(b), D(b)) = (\varphi(a)\varphi(b), \varphi(a)D(b) + D(a)\varphi(b)$, hence $\varphi(ab) = \varphi(a)\varphi(b)$ and $D(ab) = \varphi(a)D(b) + D(a)\varphi(b)$. So (φ, D) is in $\mathcal{G}(B)$ iff $\varphi \in \mathcal{G}(H)$ and D is a φ-derivation. The $*$-multiplication in $\mathrm{Hom}(H,B)$ is given by $(\varphi,D) * (\varphi', D') = (\varphi * \varphi', \varphi * D' + D * \varphi')$ by applying this to an element $a \in H$. Since $(\varphi, 0) \in \mathcal{G}(B)$ and $(u\varepsilon, D) \in \mathcal{G}(B)$ for every ε-derivation D, there is a bijection $\mathrm{Der}_\varepsilon(H,M) \cong \{(u\varepsilon, D_\varepsilon) \in \mathcal{G}_\varepsilon(B)\} \cong \{(1_H, D_1) \in \mathcal{G}_1(B)\} \cong \mathrm{Der}_{\mathbb{K}}(H,M)$ by $(u\varepsilon, D_\varepsilon) \mapsto (1,0) * (u\varepsilon, D_\varepsilon) = (1, 1 * D_\varepsilon) \in \mathcal{G}_1(B)$ with inverse map $(1, D_1) \mapsto (S,0) * (1, D_1) = (u\varepsilon, S * D_1) \in \mathcal{G}_\varepsilon(B)$. Hence we have isomorphisms $\mathrm{Der}_{\mathbb{K}}(H,M) \cong \mathrm{Der}_\varepsilon(H,M) \cong \mathrm{Hom}(I/I^2, M) \cong \mathrm{Hom}_H(H \otimes I/I^2, M)$.

The universal ε-derivation for vector spaces is $\overline{\mathrm{id} - \varepsilon} : A \longrightarrow I/I^2$. The universal ε-derivation for H-modules is $D_\varepsilon(a) = 1 \otimes \overline{(\mathrm{id} - \varepsilon)(a)} \in A \otimes I/I^2$. The universal 1-derivation for H-modules is $1 * D_\varepsilon$ with $(1 * D_\varepsilon)(a) = \sum a_{(1)} \otimes \overline{(\mathrm{id} - \varepsilon)(a_{(2)})} \in A \otimes I/I^2$. $\square$

Bibliography

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