Mathematisches Institut
der Universität München
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WS 2001/02
Set 3

Problem set for Advanced Algebra

(9) (Tensors in physics: # 9,10,11) Let V be a finite dimensional vector space over the field $\mathbb{K}$ and let V^* be its dual space. Let t be a tensor in $V \otimes \ldots \otimes V \otimes V^* \otimes \ldots \otimes V^* = V^{\otimes r} \otimes (V^*)^{\otimes s}$.

Show that for each basis $B = (b_1, \ldots, b_n)$ and dual basis $B^* = (b^1, \ldots, b^n)$ there is a uniquely determined scheme (a family or an $(r+s)$-dimensional matrix) of coefficients $(a(B)^{i_1,\ldots,i_r}_{j_1,\ldots,j_s})$ with $a(B)^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \in \mathbb{K}$ such that

(1) $$t = \sum_{i_1=1}^n \ldots \sum_{i_r=1}^n \sum_{j_1=1}^n \ldots \sum_{j_s=1}^n a(B)^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} b_{i_1} \otimes \ldots \otimes b_{i_r} \otimes b^{j_1} \otimes \ldots \otimes b^{j_s}. (1)$$

(10) Show that for each change of bases $L : B \longrightarrow C$ with $c_j = \sum \lambda^i_j b_i$ (with inverse matrix (μ^i_j)) the following *transformation formula* holds

(2) $$a(B)^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} = \sum_{k_1=1}^n \ldots \sum_{k_r=1}^n \sum_{l_1=1}^n \ldots \sum_{l_s=1}^n \lambda^{i_1}_{k_1} \ldots \lambda^{i_r}_{k_r} \mu^{l_1}_{j_1} \ldots \mu^{l_s}_{j_s} a(C)^{k_1,\ldots,k_r}_{l_1,\ldots,l_s}$$

(11) Show that every family of schemes of coefficients

$$(a(B) | B \text{ basis of } V)$$

with $a(B) = (a(B)^{i_1,\ldots,i_r}_{j_1,\ldots,j_s})$ and $a(B)^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \in \mathbb{K}$ satisfying the transformation formula (2) defines a unique tensor (independent of the choice of the basis) $t \in V^{\otimes r} \otimes (V^*)^{\otimes s}$ such that (1) holds.
Rule for physicists: A tensor is a collection of schemes of coefficients that transform according to the transformation formula for tensors.

(12) Show that $(A, \nabla : A \otimes A \longrightarrow A, \eta : \mathbb{K} \longrightarrow A)$ is a $\mathbb{K}$-algebra if and only if A with the multiplication $A \times A \xrightarrow{\otimes} A \otimes A \xrightarrow{\nabla} A$ and the unit $\eta(1)$ is a ring and $\eta : \mathbb{K} \longrightarrow \mathrm{Cent}(A)$ is a ring homomorphism into the *center* of A, where $\mathrm{Cent}(A) := \{a \in A | \forall b \in A : ab = ba\}$.

Due date: Tuesday, 06.11.2001, 16:15 in Lecture Hall 138