Mathematisches Institut
der Universität München
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WS 2001/02
Set 2

Problem set for Advanced Algebra

(5) Let V be a finite dimensional vector space. Let $B = (v_i | i = 1, \dots, n)$ be a basis of V and $(v_i^* | i = 1, \dots, n)$ be the dual dual basis of the dual space V^*. Show that $\sum_{i=1}^n v_i \otimes v_i^* \in V \otimes V^*$ does not depend on the choice of the basis B and that

$$\forall v \in V : \quad \sum_i v_i^*(v) v_i = v$$

holds.

(Hint: Find an isomorphism $\mathrm{End}(V) \cong V \otimes V^*$ and show that id_V is mapped to $\sum_{i=1}^n v_i \otimes v_i^*$ under this isomorphism.)

(6) (a) Let M_R, ${}_RN$, M'_R, and ${}_RN'$ be R-modules. Show that the following is a homomorphism of abelian groups:

$$\mu : \mathrm{Hom}_R(M, M') \otimes_{\mathbb{Z}} \mathrm{Hom}_R(N, N') \ni f \otimes g \mapsto f \otimes_R g \in \mathrm{Hom}(M \otimes_R N, M' \otimes_R N').$$

(b) Find examples where μ is not injective and where μ is not surjective.

(c) Explain why $f \otimes g$ is a decomposable tensor whereas $f \otimes_R g$ is not a tensor.

(7) Give a complete proof of Theorem 1.22. In (5) show how $\mathrm{Hom}_T(M., N.)$ becomes an S-R-bimodule.

(8) Find an example of $M, N \in \mathbb{K}\text{-}\mathrm{Mod}\text{-}\mathbb{K}$ such that $M \otimes_{\mathbb{K}} N \not\cong N \otimes_{\mathbb{K}} M$.
(Hint: You may use $\mathbb{K} := L \times L$, ${}_{\mathbb{K}}M := {}_{\mathbb{K}}\mathbb{K}$, and $N_{\mathbb{K}} := \mathbb{K}_{\mathbb{K}}$. Define a right $\mathbb{K}$-structure on M by $(m, n)(a, b) := (ma, na)$ and a left $\mathbb{K}$-structure on N by $(a, b)(r, s) := (br, bs)$.)

Due date: Tuesday, 30.10.2001, 16:15 in Lecture Hall 138