Seminar Pareigis–Wess: WS 1996/97

Linearly Compact Algebraic Structures

1. Lecture: The Initial Topology

Definition 1.1. A *topological space* is a set X together with a system $\mathcal{T}$ of subsets of X with the following properties

(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,

(2) if $(O_i)_{i \in I}$ is a family of elements of $\mathcal{T}$ then $\bigcup_{i \in I} O_i \in \mathcal{T}$ holds,

(3) if O and O' are in $\mathcal{T}$ then $O \cap O'$ is in $\mathcal{T}$.

The system $\mathcal{T}$ is called the *topology* of X. The elements of $\mathcal{T}$ are called *open sets*.

Example 1.2. (1) The set of real numbers $\mathbb{R}$ is a topological space with the following topology: A subset O of $\mathbb{R}$ is called open if

$$\forall x \in O \ \exists \varepsilon > 0 \ \forall y \in \mathbb{R} : |\curvearrowright - \curvearrowleft| < \varepsilon \Rightarrow \curvearrowright \in \mathbb{O}.$$

This can be expressed as follows: for each point $x \in O$ there is an ε-interval $K_\varepsilon(x)$ that is contained in O.

(2) Every set X is a topological space with the discrete topology. The *discrete topology* is defined by the system $\mathcal{T}$ of all subsets of X, i.e. every subset of X is open.

(3) For every set X the set $\mathcal{T} = \{\emptyset, \mathcal{X}\}$ is a topology for X.

Definition 1.3. A subset A of a topological space X is called *closed* if its complement $X \setminus A$ is open.

The fact that arbitrary unions of open sets are open implies that arbitrary intersections of closed sets are closed. If we therefore define the *closure* $\bar{M}$ of a subset M of a topological space to be the intersection of all closed sets that contain M, we end up with a closed set.

Definition 1.4. Let X and Y be topological spaces with the topologies $\mathcal{T}_\mathcal{X}$ und $\mathcal{T}_\mathcal{Y}$. A map $f : X \to Y$ is called *continuous*, if

$$\forall O \in \mathcal{T}_\mathcal{Y} : \{^{-\infty}(\mathcal{O}) \in \mathcal{T}_\mathcal{X},$$

i.e. if the inverse image of any open set is open.

Theorem 1.5. *Let X be a set. Let $(X_i)_{i\in I}$ be a family of topological spaces with the corresponding topologies $\mathcal{T}_\rangle$. Given a map $f_i : X \to X_i$ for every $i \in I$. Then there exists precisely one topology $\mathcal{T}$ for X, called the* initial topology*, with the following properties:*

(1) *For every $i \in I$ the map f_i is continuous.*

(2) *(Universal property of the initial toplogogy)*
If Y is a topological space and $f : Y \to X$ is a map then f is continuous if and only if the composition $f_i \circ f$ is continuous for every $i \in I$.

Proof. Wir zeigen das in mehreren Schritten:

(1) (Eindeutigkeit)
Seien $\mathcal{T}$ und $\mathcal{T}'$ zwei Topologien auf X, die die beiden genannten Eigenschaften erfüllen. Wir betrachten die identische Abbildung

$$id_X : (X, \mathcal{T}) \to (\mathcal{X}, \mathcal{T}')$$

Weil $\mathcal{T}$ die Eigenschaft 1 hat, ist

$$f_i \circ id_X : (x, \mathcal{T}) \to \mathcal{X}_\rangle$$

stetig. Wiel $\mathcal{T}'$ die Eigenschaft 2 hat, ist

$$id_X : (X, \mathcal{T}) \to (\mathcal{X}, \mathcal{T}')$$

stetig. Nach Definition der Stetigkeit ist also für $O \in \mathcal{T}'$ das Urbild $id_X^{-1}(O) \in \mathcal{T}$. Insgesamt folgt: $\mathcal{T}' \subset \mathcal{T}$. Vertauschen von $\mathcal{T}$ und $\mathcal{T}'$ liefert: $\mathcal{T} = \mathcal{T}'$.

(2) Wir wenden uns jetzt der Frage nach der Existenz zu. Wir definieren $\mathcal{T}$ wie folgt: Eine Teilmenge O von X heiße offen, wenn es für jeden Punkt $x \in O$ endlich viele offene Mengen $O_{i_1} \subset X_{i_1}, \dots, O_{i_n} \subset X_{i_n}$ gibt, so daß gilt:

$$x \in \bigcap_{j=1}^{n} f_{i_j}^{-1}(O_{i_j}) \subset O$$

Wir prüfen zunachst, daß $\mathcal{T}$ eine Topologie ist. Eigenschaft 1.1.1 ist offensichtlich, ebenso Eigenschaft 1.1.2.

Sind O und O' aus $\mathcal{T}$ und ist $x \in O \cap O'$, so gibt es $O_{i_1}, \dots, O_{i_n}$ mit $x \in \bigcap_{j=1}^{n} f_{i_j}^{-1}(O_{i_j}) \subset O$ und $O'_{k_1}, \dots, O'_{k_n}$ mit $x \in \bigcap_{l=1}^{n} f_{k_l}^{-1}(O'_{k_l}) \subset O'$, wobei $O_{i_j} \subset X_{i_j}$ offen und $O'_{k_l} \subset X_{k_l}$ offen ist. Daraus folgt:

$$x \in \bigcap_{j=1}^{n} f_{i_j}^{-1}(O_{i_j}) \cap \bigcap_{l=1}^{m} f_{k_l}^{-1}(O'_{k_l}) \subset O \cap O'$$

Damit ist auch Eigenschaft 1.1.3 erfüllt.

(3) Wir zeigen nun, daß die in Schritt 2 definierte Topologie $\mathcal{T}$ die beiden behaupteten Eigenschaften hat. Ist $O \subset X_i$ offen, so ist $f_i^{-1}(O) \in \mathcal{T}$, f_i ist also stetig. Ist $f : Y \to X$ eine Abbildung, für die $f_i \circ f$ (für alle $i \in I$) stetig ist, und ist $O \subset X$ offen, so ist zu zeigen, daß $f^{-1}(O)$ offen ist. Ist $y \in f^{-1}(O)$, also $f(y) \in O$, so gibt es offene Mengen $O_{i_1}, \ldots, O_{i_n}$ in $X_{i_1}, \ldots, X_{i_n}$, so daß $f(y) \in \bigcap_{j=1}^{m} f_{i_j}^{-1}(O_{i_j}) \subset O$, also $y \in \bigcap_{j=1}^{n} f^{-1}(f_{i_j}^{-1}(O_{i_j})) \subset f^{-1}(O)$. Da $f_{i_j} \circ f$ stetig ist, ist $f^{-1}(f_{i_j}^{-1}(O_{i_j})) = (f_{i_j} \circ f)^{-1}(O_{i_j})$ offen. Also ist $f^{-1}(O)$ offen.

□

Example 1.6. Sei X ein topologischer Raum mit der Topologie $\mathcal{T}$ und U eine Teilmenge von X. Die Initialtopologie bezüglich der Inklusionsabbildung $\iota : U \to X, x \mapsto x$ ist eine Topologie auf U, die die Relativtopologie genannt wird. Sie kann auch so beschrieben werden: Eine Teilmenge U' von U ist offen in der Relativtopologie, wenn es eine offene Menge $X' \in \mathcal{T}$ von X gibt mit $U' = U \cap X'$, d. h. wenn sie der Durchschnitt von U und einer offenen Menge von X ist.

Example 1.7. Seien $(X_i)_{i\in I}$ topologische Räume, jeweils mit den Topologien $\mathcal{T}_{\rangle}$. Die Initialtopologie des cartesischen Produktes $\prod\limits_{i\in I} X_i$ bezüglich der Projektionen

$$\pi_j : \prod_{i\in I} X_i \to X_j, (x_i)_{i\in I} \mapsto x_j$$

heißt die Produkttopologie auf $\prod\limits_{i\in I} X_i$. So trägt etwa der $\mathbb{R}^{\ltimes} = \mathbb{R}\times\ldots\times\mathbb{R}$ die Produkttopologien der Topologien aus Beispiel 1.2.1. Die Stetigkeit der Diagonalabbildung

$$d : \mathbb{R} \to \mathbb{R}^{\nvDash}, \curvearrowleft \mapsto (\curvearrowleft, \curvearrowleft)$$

folgt also wegen der universellen Eigenschaft der Initialtopologie daraus, daß die Kompositionen $\pi_1 \circ d : \mathbb{R} \to \mathbb{R}, \pi_{\nvDash} \circ \; : \mathbb{R} \to \mathbb{R}$ mit den Projektionen $\pi_i : \mathbb{R}^{\nvDash} \to \mathbb{R}, (\curvearrowleft_{\mathbb{K}}, \curvearrowleft_{\nvDash}) \mapsto \curvearrowleft_{\beth}$ jeweils die identische Abbildung ergeben, die stetig ist.

Definition 1.8. Sei I eine Menge, $\leq$ eine Relation auf I. I heißt durch $\leq$ gerichtet, wenn gilt:

(1) $\forall i \in I : i \leq i$

(2) $\forall i, j, k \in I : i \leq j$ und $j \leq k \to i \leq k$

(3) $\forall i, j \in I \exists k \in I : i \leq k$ und $j \leq k$

Wir schreiben $i < j$, falls $i \leq j$, aber nicht $j \leq i$ gilt.

Example 1.9. Die natürlichen Zahlen sind gerichtet, z. B. gibt es zu den natürlichen Zahlen 2 und 3 die Zahl 5, für die gilt: $2 \leq 5$ und $3 \leq 5$. Die Menge aller endlichen Teilmengen einer Menge X ist durch die Inklusion gerichtet, denn sind Y und Z endliche Teilmengen von X, so erfüllt die Vereinigung $Y \cup Z$:

$$Y \leq Y \cup Z \text{ und } Z \leq Y \cup Z$$

Definition 1.10. Sei I eine durch $\leq$ gerichtete Menge. Für jedes $i \in I$ sei ein topologischer Raum X_i mit der Topologie $\mathcal{T}_\rangle$ gegeben. Weiter sei für je zwei Elemente $i, j \in I$ mit $i \leq j$ eine stetige Abbildung

$$f_{ij} : X_j \to X_i$$

gegeben, so daß gilt:

(1) $\forall i \in I : f_{ii} = id_{X_i}$

(2) $\forall i, j, k \in I : i \leq j \leq k \to f_{ik} = f_{ij} \circ f_{jk}$

Der projektive Limes der Familie $(X_i)_{i \in I}$ ist die Menge:

$$\varprojlim_{i \in I} X_i = \{(x_i)_{i \in I} \in \prod_{i \in I} X_i | \forall i, j \in I : i \leq j \to f_{ij}(x_j) = x_i\}$$

Offenbar gilt: $\prod_{i \in I} X_i \subset \varprojlim_{i \in I} X_i$ Der projektive Limes ist ein topologischer Raum mit der Relativtopologie der Produkttopologie. Seine Elemente heißen Fäden. Wir bezeichnen mit π_j die Abbildung

$$\pi_j : \varprojlim_{i \in I} X_i \to X_j, (x_i)_{i \in I} \mapsto x_j$$

Theorem 1.11. *(Universelle Eigenschaft des projektiven Limes)*

(1) $\forall i, j \in I : i \leq j \to \pi_i = f_{ij} \circ \pi_j$

(2) *Ist X ein topologischer Raum und ist für jedes $i \in I$ eine stetige Abbildung*

$$f_i : X \to X_i$$

gegeben, so daß gilt:

$$\forall i, j \in I : i \leq j \to f_i = f_{ij} \circ f_j,$$

so gibt es genau eine stetige Abbildung $f : X \to \varprojlim_{i \in I} X_i$, für die gilt:

$$\forall i \in I : f_i = \pi_i \circ f$$

Proof. Die erste Aussage folgt direkt aus der Definition. Die zweite Aussage erhält man, indem man für f setzt:

$$f : X \to \varprojlim_{i\in I} X_i, x \mapsto (f(x_i))_{i\in I}$$

Die Stetigkeit von f folgt aus der universellen Eigenschaft der Initialtopologie. Die Eindeutigkeit ist klar. $\square$

Example 1.12. Sei X ein topologischer Raum und sei I ein System von Teilmengen von X, das mit zwei Elementen auch deren Durchschnitt enthält. I ist dann gerichtet durch die umgekehrte Inklusion: Sind $A, B \in I$, so setze:

$$A \leq B :\Leftrightarrow A \supset B$$

Für $A \in I$ sei $X_A = A$, die Menge selbst, und für $A \leq B$ sei $f_{AB} : X_B \to X_A$ die Inklusionsabbildung. Der projektive Limes dieser Familie ist einfach ihr Durchschnitt, d. h. die Abbildung

$$\bigcap_{A\in I} A \to \varprojlim_{A\in I} X_A, x \mapsto (x)_{A\in I}$$

ist ein Homöomorphismus, d. h. eine bijektive stetige Abbildung, deren Umkehrabbildung ebenfalls stetig ist.

Example 1.13. Sei p eine Primzahl. Die natürlichen Zahlen $\mathbb{N} = \{\nVdash, \nvDash, \nVdash, \dots\}$ sind mit der gewöhnlichen Ordnung gerichtet. Für $n \in \mathbb{N}$ sei $X_n := \mathbb{Z}/\imath^{\ltimes}\mathbb{Z}$, versehen mit der diskreten Topologie. Für $n \leq m$ sei

$$f_{nm} : \mathbb{Z}/\imath^{\gtrdot}\mathbb{Z} \to \mathbb{Z}/\imath^{\ltimes}\mathbb{Z}$$

die kanonische Abbildung. Der projektive Limes $\varprojlim_{n\in\mathbb{N}}\mathbb{Z}/\imath^{\ltimes}\mathbb{Z}$ heißt die Menge der ganzen p-adischen Zahlen.

2. Linear topological vector spaces

Definition 2.1. (1) A group G together with a topology $\mathcal{T}$ is called a topological group if the following properties are satisfied:

(a) The map
$$G \times G \to G, (g, g') \mapsto gg'$$
is continuous. Here $G \times G$ is endowed with the product topology.

(b) The map
$$G \to G, g \mapsto g^{-1}$$
is continuous.

(2) A ring R together with a topology $\mathcal{T}$ is called a topological ring if the following properties are satisfied:

(a) R is a topological group with respect to addition.

(b) The map
$$R \times R \to R, (r, s) \mapsto rs$$
is continuous. Here $R \times R$ is again endowed with the product topology.

(3) Suppose that R is a topological ring. A module M over R together with a topology $\mathcal{T}$ is called a topological module if the following properties are satisfied:

(a) M is a topological group with respect to addition.

(b) The map
$$R \times M \to M, (r, m) \mapsto rm$$
is continuous. Here $R \times M$ is again endowed with the product topology.

Definition 2.2. (1) A field K together with a topology $\mathcal{T}$ is called a topological field if the following properties are satisfied:

(a) K is a topological ring.

(b) The map
$$K \setminus \{0\} \to K \setminus \{0\}, \lambda \mapsto \lambda^{-1}$$
is continuous. Here $K \setminus \{0\}$ is endowed with the relative topology.

(2) A topological vector space is a topological module over a topological field.

Theorem 2.3. (1) *Every field becomes a topological field if it is endowed with the discrete topology.*

(2) *If K is (in this sense) discrete and if V is a vector space over K, then the following assertions are equivalent:*

(a) *The map*

$$K \times V \to V, (\lambda, v) \mapsto \lambda v$$

is continuous.

(b) *For every $\lambda \in K$ the mapping*

$$V \to V, v \mapsto \lambda v$$

is continuous.

Proof. Products and subsets of discrete topological spaces are - together with the product topology resp. the relative topology - again discrete. Mappings from discrete topological spaces to any other topological spaces are always contionuous. This implies the first assertion.

Suppose that $\lambda \in K$. The mapping $V \to K \times V, v \mapsto (\lambda, v)$ is continuous by the universal property of the product topology. The mapping $K \times V \to V, (\lambda, v) \mapsto \lambda v$ is continuous by assumption. Therefore the composition of these mappings is continuous, as we have asserted.

For the other direction, suppose that $O \in V$ is open. We have to show that $\{(\lambda, v) \in K \times V | \lambda v \in O\}$ is open. Since we know that $\{v \in V | \lambda v \in O\}$, we have that

$$\{\lambda\} \times \{v \in V | \lambda v \in O\}\{(\lambda, v) \in K \times V | \lambda v \in O\}$$

is open, as we can see from the construction of the product topology. Therefore, we have that

$$\bigcup_{\lambda \in K} \{(\lambda, v) \in K \times V | \lambda v \in O\}$$

is open, as we had to show. □

Definition 2.4. A linear topological vector space is a topological vector space over a discrete field satisfying the following condition: For every neighbourhood U of the origin there is another neighbourhood U' of the origin which is contained in U and is a subspace of V. (Here and below 'subspace' means 'subvector space'. Recall that a neighbourhood of a point in a topological space is a set which contains an open set which in turn contains the point.)

Theorem 2.5. *Suppose that V is a linear topological vector space. Then the following assertions are equivalent:*

(1) *V is Hausdorff, that is, two distinct points have disjoint neighbourhoods.*

(2) *The intersection of all neighbourhoods of the origin is $\{0\}$.*

Proof. We denote the set of all neighbourhoods of the origin by $\mathcal{U}(\prime)$. If there were $v \in \bigcap_{U \in \mathcal{U}(\prime)} U$, $v \neq 0$, then we would have neighbourhoods U_v of v and U_0 of 0 such that $U_v \cap U_0 = \emptyset$ by the Hausdorff axiom. But this is a contradiction.

For the other implication, suppose that v and w are two distinct points. By assumption there is a neighbourhood U of the origin which does not contain $v - w$. Since V is linear topological, we can assume that U is a subspace. $v + U$ and $w + U$ then are neighbourhoods of v resp. w. We have $(v + U) \cap (w + U) = \emptyset$, because if there would be an element $x \in (v + U) \cap (w + U)$, i. e. $x = v + u_1 = w + u_2, u_1, u_2 \in U$, we would have $v - w = u_2 - u_1 \in U$, a contradiction. □

Definition 2.6. (1) Suppose that X is a set and that F is a system of subsets of X. F is called a filter basis if it is not void, no element of F is void and has the following property:

$$\forall A, B \in F \exists C \in F : C \subset A \cap B$$

(2) If X is a topological space and $x \in X$, a system F of neighbourhoods of x is called a basis of the neighbourhood system if for every neighbourhood U of x there is an element $A \in F$ such that $A \subset U$.

Theorem 2.7. *Suppose that V is a vector space and that F is a filter basis consisting of subspaces of V. Then there is a unique topology $\mathcal{T}$ on V having the following properties:*

(1) *F is a basis of the neighborhood system of the origin.*

(2) *V is a linear topological vector space.*

Proof. (1) We first show the existence of such a topology. We call a subset O open if we have:

$$\forall v \in O \exists U \in F v + U \subset O$$

The first two axioms in the definition of a topology are obviously satisfied. We show the third axiom: If O and O' are open and if $v \in O \cap O'$, we select $U \in F$ and $U' \in F$ satisfying $v + U \subset O$ and $v + U' \subset O'$. Since F is a filter basis, there exists $U'' \in F$ satisfying $U'' \subset U \cap U'$. We then have:

$$v + U'' \subset (v + U) \cap (v + U') \subset O \cap O'$$

Therefore $O \cap O'$ is open. We now have shown that we have defined a topology in this way.

(2) It is obvious that F is a basis of the neighbourhood system of the origin with respect to this topology. We show now that V has become a linear topological vector space. We first check the continuity of the addition. If O is an open set and $v + w \in O$, we have some $U \in F$ satisfying $v + w + U \subset O$. $(v + U) \times (w + U)$ then is an open set in $V \times V$ which maps to O under addition. Therefore, addition is continuous. Suppose now that we are given $\lambda \in K$ and that O is an open set containing λv for some $v \in V$. We then have some element $U \in F$ satisfying $\lambda v + U \subset O$. The mapping $w \mapsto \lambda w$ then maps the open set $v + U$ to O. The assertion now follows from Proposition 2.3.

(3) We shall now prove the uniqueness. If $\mathcal{T}'$ is another toplogy which possesses the stated properties, and if $O \in \mathcal{T}', \sqsubseteq \in \mathcal{O}$, we have that $-v+O$ is a neighbourhood of the origin. Therefore there is $U \in F$ with $U \subset -v + O$. But this implies $v + U \subset O$, i. e. $O \in \mathcal{T}$. We therefore have shown: $\mathcal{T}' \subset \mathcal{T}$.

If we have $O \in \mathcal{T}$ and $v \in O$, there exists $U \in F$ satisfying $v + U \subset O$. Since U is a $\mathcal{T}'$-neighbourhood of the origin, $v + U$ is a $\mathcal{T}'$- neighbourhood of v. Therefore O is a $\mathcal{T}'$-neighbourhood of v. This implies: $O \in \mathcal{T}'$.

□

Theorem 2.8. *Every finite dimensional linear topological Hausdorff space is finite dimensional.*

Proof. Let $dim\ V = n$. Suppose that U_1 is a subspace of V which is a neighbourhood of the origin. If $U_1 \neq \{0\}$, we have by 2.5.2 a second subspace of this kind satisfying $U_1 \cap U_2 \subset U_1$, and in case that $U_1 \cap U_2 \neq \{0\}$, there is another such subspace U_3 such that $U_1 \cap U_2 \supset U_1 \cap U_2 \cap U_3$. After at most $n + 1$ steps, this procedure terminates and we have: $U_1 \cap \ldots \cap U_{n+1} = \{0\}$. Therefore $\{0\}$ is a neighborhood of the origin, and therefore it is open. Therefore, for every vector $v \in V$, the set $\{v\} = v + \{0\}$ is open. ☐

Theorem 2.9. *Suppose that V is a linear topological vector space. If U is a subspace which is a neighborhood of the origin, then U is open and closed.*

Proof. If $u \in U$, then we have $u + U = U$, and therefore U is a neighbourhood of u. Therefore, U is a neighbourhood of all of its points, and therefore open. On the other hand, if $u \notin U$, we have $(u+U) \cap U = \emptyset$, because if there were $u'' \in (u+U) \cap U$, that is $u'' = u + u', u' \in U$, we would have $u = u'' - u' \in U$, a contradiction. Therefore, the complement of U contains with every point a whole neighbourhood of this point, and therefore it is open. ☐

Theorem 2.10. *Suppose that V is a linear topological vector space and that U is a subspace of V. Then the closure $\overline{U}$ of U is again a subspace.*

Proof. If we are given $\lambda, \mu \in K$ and $v, w \in \overline{U}$, then for every open subspace W of V there exist elements $v', w' \in U$ such that $v' \in v + U, w' \in w + U$. We then have $\lambda v' + \mu w' \in (\lambda v + \mu w) + U$, and therefore $\lambda v + \mu w \in \overline{U}$. ☐

Theorem 2.11. *Suppose that V is a Hausdorff linear topological vector space. Suppose that $v \in V$ and that D is the intersection of all simultaneously open and closed sets that contain v. Then we have: $D = \{v\}$.*

Proof. If U is a neighbourhood of the origin which is a subspace, it is by 2.9 open and closed. $v + U$ then is a open and closed set which contains v. Since by 2.5 the intersection of all such U is equal to $\{0\}$, the intersection of all $v + U$ is equal to $\{v\}$. But this intersection contains D. ☐

Remark 2.12. It can be deduced from the preceding proposition that V is totally disconnected in the sense that the connected component of every point consists only of this point. (Cf. [6], Satz 4.18, p. 47). A different example of a totally disconnected space are the rational numbers $\mathbb{Q}$.

Theorem 2.13. *Suppose that V is a vector space and that $(V_i)_{i \in I}$ is a family of linear topological vector spaces. We assume that for every $i \in I$ there is given a linear map*

$$f_i : V \to V_i$$

Then V is a linear topological vector space with respect to the corresponding initial topology.

Proof. We first show the continuity of the addition

$$+ : V \times V \to V, (v, w) \mapsto v + w$$

By the universal property of the initial topology, it suffices to prove the continuity of the mappings

$$V \times V \to V, (v, w) \mapsto f_i(v + w)$$

But this is just the composition of the continuous mappings

$$V \times V \to V_i \times V_i, (v, w) \mapsto (f_i(v), f_i(w))$$

and

$$V_i \times V_i \to V_i, (v, w) \mapsto v + w$$

We further have to show by 2.3 that for a given $\lambda \in K$ the mapping

$$V \to V, v \mapsto \lambda v$$

is continuous. Again by the universal property of the initial topology it is sufficient to prove that the mapping

$$V \to V_i, v \mapsto (f_i(\lambda v)$$

is continuous. But this is just the composition of f_i and the continuous mapping

$$V_i \to V_i, v \mapsto \lambda v$$

We have shown that V is a topological vector space. It remains to show that V possesses a basis of the neighbourhood system of the origin that consists of subspaces. If U is a neighbourhood of the origin, then by construction of the initial topology in the proof of 1.5 there exist open sets $O_{i_1}, \ldots, O_{i_n}$ in $V_{i_1}, \ldots, V_{i_n}$ satisfying:

$$0 \in \bigcap_{j=1}^{n} f_{i_j}^{-1}(O_{i_j}) \subset U.$$

We then have $0 = f_{i_j}(0) \in O_{i_j}$, and therefore there is an open subspace U_{i_j} of V_{i_j}, such that $U_{i_j} \in O_{i_j}$. This implies

$$0 \in \bigcap_{j=1}^{n} f_{i_j}^{-1}(U_{i_j}) \subset U,$$

and obviously $\bigcap_{j=1}^{n} f_{i_j}^{-1}(U_{i_j})$ is a subspace of V. □

Remark 2.14. In particular, it follows from the preceding proposition that a subspace of a linear topological vector space becomes a linear topological vector space via the relative topology, and that a product of linear topological vector spaces together with the product topology is a linear topological vector space.

Suppose that I is a set which is directed by $\leq$. Suppose further that for every $i \in I$ we are given a linear topological vector space V_i and for any two elements $i, j \in I$ satisfying $i \leq j$ a continuous linear map

$$f_{ij} : V_j \to V_i$$

such that 1.10.1 and 1.10.2 are satisfied. The projective limit

$$\varprojlim_{i \in I} V_i = \{(v_i)_{i \in I} \in \prod_{i \in I} V_i | \forall i, j \in I : i \leq j \to f_{ij}(v_j) = v_i\}$$

obviously is a subspace of the product. We now have by the preceding proposition that it is a linear topological vector space via the relative topology of the product topology.

3. Duality theory

Theorem 3.1. (1) *If $f : V \to V$ is a linear map between topological vector spaces, then f is continuous if and only if it is continuous at the origin.*

(2) *If $f : V \to K$ is a linear functional on the linear topological vector space V, it is continuous if and only if it vanishes on some open subspace.*

Proof. Suppose that f is continuous at the origin. We show that f is continuous at $v \in V$. For this purpose we write f as the composition of the three mappings $f = h \circ f \circ g$ where we define:

$$g : V \to V, v' \mapsto v' - v$$
$$h : W \to W, w \mapsto f(v) + w$$

g and h are continuous at every pointin particular g is continuous at v, f is continuous at $0 = g(v)$ amd h is continuous at $0 = f(0)$. Therefore $h \circ f \circ g$ is continuous at v.

(Recall that $f : X \to Y$ is called continuous at x if for every neighbourhood U of $f(x)$ $f^{-1}(U)$ is a neighbourhood of x.)

To prove the second statement, we observe that if f is continuous, then $ker f = f^{-1}(\{0\})$ is open, since K is discrete. on the other hand, if U is an open subspace on which f vanishes, then for every neighbourhood U' of the origin in K $f^{-1}(U')$ is a neighbourhood of the origin in V, since it contains U. □

Definition 3.2. Suppose that $\langle \cdot, \cdot \rangle : V \times W \to K$ is a nondegenerate bilinear form. For $\tilde{W}$ define:

$$\tilde{W}^{\perp} = \{v \in V | \forall w \in \tilde{W} : \langle v, w \rangle = 0\rangle$$

The set $\{\tilde{W}^{\perp} | \tilde{W}$ is a finite dimensional subspace of $W\}$ is a filter basis (Why?). The topology induced by this filter basis (cf. 2.7) is called the finite topology of V.

Obviously the evaluation map

$$\langle \cdot, \cdot \rangle : V^* \times V \to K, (f, v) \mapsto f(v)$$

is a nondegenerate bilinear form. V^*, and also V, therefore carry finite topologies.

Definition 3.3. Suppose that V is a linear topological vector space. We denote the set of continuous linear functionals by V'. V' is a subspace of the dual vector space V^*. (Why?)

Theorem 3.4. *(Trennungssatz) Suppose that V is a linear topological vector space and U is a closed subspace. If $v \notin U$, there is a continuous linear functional $f \in V'$ satisfying $f(v) = 1$ and $f(u) = 0$ for all $u \in U$.*

Proof. Since U is closed, $V \setminus U$ is open. Since $v \in V \setminus U$, there is an open subspace W such that $v + W \subset V \setminus U$. Then we have $v \notin U + W$, since if we could write $v = u + w, u \in U, w \in W$, we would have $v - w = u \in (v + W) \cap U$. We know from linear algebra that there is a linear functional $f \in V^*$ that satisfies $f(v) = 1, f(U + W) = 0$. Therefore we have in particular that $f(W) = 0$, and this implies that f is continuous by 3.1. □

Theorem 3.5. *(Fortsetzungssatz) Suppose that V is a linear topological vector space and that U is a subspace of V. If $f : U \to K$ is a continuous linear functional, then f can be extended to the whole vector space, that is, there is a continuous linear functional $g \in V'$ satisfying $f(u) = g(u)$ for all $u \in U$.*

Proof. If $f = 0$, define $g = 0$. If $f \neq 0$, there is u_0 such that $f(u_0) = 1$. We then have: $U = Ku_0 \oplus ker(f)$. Since f is continuous and K is discrete, $f^{-1}(\{1\})$ is an open neighbourhood of u_0 and therefore we have by the construction of the relative topology (cf. 1.6) an open subspace W of V such that $(u_0 + W) \cap U \subset f^{-1}(\{1\})$. We see that $u_0 \notin W + ker(f)$, because if we could write $u_0 = w + v, w \in W, v \in ker(f)$, we could conclude that $v = u_0 - w \in u_0 + W \subset f^{-1}(\{1\})$ and therefore $0 = f(v) = 1$. $W + ker(f) = \bigcup_{v \in ker(f)} v + W$ is an open subspace which is therefore closed. By the Trennungssatz we conclude that there is a continuous linear form $g \in V'$ satisfying $g(u_0) = 1$, but $g(W + ker(f)) = 0$. If $u \in U$, we write $u = \alpha u_0 + x, x \in ker(f)$. Then we have $f(u) = \alpha = g(u)$. □

Theorem 3.6. *Suppose that $\langle \cdot, \cdot \rangle : V \times W \to K$ is a nondegenerate bilinear form. Suppose that $w_1, \ldots, w_n \in W$ are linearly independent vectors. Then there are vectors $v_1, \ldots, v_n \in V$ satisfying:*

$$\forall i, j \leq n : \langle v_i, w_j \rangle = \delta_{ij}$$

Proof. We argue by induction on n, the case $n = 1$ being obvious. By the induction hypothesis, we can find vectors $v_1, \ldots, v_{n-1}$ such that we have:

$$\forall i, j \leq n-1 : \langle v_i', w_j \rangle = \delta_{ij}$$

For $v \in V$ form

$$v - \sum_{i=1}^{n-1} \langle v, w_i \rangle v_i'$$

We have $\langle v - \sum_{i=1}^{n-1} \langle v, w_i \rangle v_i', w_j \rangle = 0$ for $j = 1, \dots, n$. On the other hand, there is $v \in V$ such that $\langle v - \sum_{i=1}^{n-1} \langle v, w_i \rangle v_i', w_n \rangle \neq 0$, because if we would have for all vectors $v \in V$:

$$\begin{aligned} 0 &= \langle v - \sum_{i=1}^{n-1} \langle v, w_i \rangle v_i', w_n \rangle \\ &= \langle v, w_n \rangle - \sum_{i=1}^{n-1} \langle v, w_i \rangle \langle v_i', w_n \rangle \\ &= \langle v, w_n - \sum_{i=1}^{n-1} \langle v_i', w_n \rangle w_i \rangle \end{aligned}$$

we could conclude by the nondegeneracy the bilinear form that we have $w_n - \sum_{i=1}^{n-1} \langle v_i', w_n \rangle w_i = 0$ which contradicts the linear independence of $w_1, \dots, w_n$. We now select $v \in V$ such that $v \in V$ such that $\langle v - \sum_{i=1}^{n-1} \langle v, w_i \rangle v_i', w_n \rangle = 1$. Define

$$v_n = v - \sum_{i=1}^{n-1} \langle v, w_i \rangle v_i'$$

and for $i = 1, \dots n-1$:

$$v_i := v_i' - \langle v_i', w_n \rangle v_n$$

It is then easy to see that these vectors have the required property. □

Theorem 3.7. *Suppose that $\langle \cdot, \cdot \rangle : V \times W \to K$ is a nondegenerate bilinear form. Suppose that V carries the finite topology induced by W. Then the mapping*

$$W \to V', w \mapsto (v \mapsto \langle v, w \rangle)$$

is a bijection.

Proof. First we show that the map is well defined, that is, that for $w \in W$ the linear functional $V \to K, v \mapsto \langle v, w \rangle$ is continuous with respect to the finite topology. But this map vanishes on $(Kw)^\perp$ and therefore is continuous by 3.1. We now prove surjectivity. Suppose that $f : V \to K$ is continuous with respect to the finite topology. By 3.1 and 3.2 there is a finite dimensional subspace $\tilde{W} = Span(w_1, \dots, w_n)$ such that $f(\tilde{W}^\perp) = \{0\}$. We can assume that $w_1, \dots, w_n$ constitute a basis of $\tilde{W}$. By 3.6 there are vectors $v_1, \dots, v_n$ such that $\langle v_i, w_j \rangle = \delta_{ij}$. Define $w := \sum_{i=1}^n f(v_i) w_i$. The surjectivity follows if we can prove:

$$\forall v \in V : f(v) = \langle v, w \rangle$$

But for all $v \in V$ we have that $v - \sum_{i=1}^n \langle v, w_i \rangle v_i \in \tilde{W}^\perp$ and therefore

$$f(v - \sum_{i=1}^n \langle v, w_i \rangle v_i) = 0$$

which means $f(v) = \sum_{i=1}^n \langle v, w_i \rangle f(v_i) = \langle v, w \rangle$ The injectivity is clear. □

Theorem 3.8. *Suppose that V is a vector space. Endow the dual vector space V^* with the finite topology. Then the mapping*

$$V \to (V^*)', v \mapsto (f \mapsto f(v))$$

is a bijection.

Proof. This follows by applying the preceding proposition 3.7 to the bilinear form

$$V^* \times V \to K, (f, v) \mapsto f(v)$$

□

Theorem 3.9. *Suppose that $\langle \cdot, \cdot \rangle : V \times W \to K$ is a nondegenerate bilinear form. Suppose that V carries the finite topology. Then we have for every subspace U of V:*

$$\bar{U} = U^{\perp\perp}$$

Proof. This proceeds in steps.

(1) We first show that we have $\bar{U} \subset U^{\perp\perp}$. If $\tilde{W}$ is any subspace of W, then $\tilde{W}^\perp$ is closed, because if $v \notin \tilde{W}^\perp$, we have $w \in \tilde{W}$ such that $\langle v, w \rangle \neq 0$. $(Kw)^\perp$ is closed by 3.2 and 2.9, and because

$$v \in V \setminus (Kw)^\perp \subset V \setminus \tilde{W}^\perp$$

we see that $V \setminus \tilde{W}^\perp$ is open.

In particular, we see that $U^{\perp\perp}$ is closed, and since $U \subset= U^{\perp\perp}$, we have that $\bar{U} \subset U^{\perp\perp}$

(2) We now prove $U^{\perp\perp} \subset \bar{U}$. For this purpose, it is sufficient to prove that every neighbourhood of an element $u \in U^{\perp\perp}$ which is of the form $u + \tilde{W}^\perp$ for some finite dimensional subspace $\tilde{W}$ of W contains an element of U. We select a basis $w_1, \dots, w_m$ of $\tilde{W} \cap U^\perp$ which we complete to a basis $w_1, \dots, w_n$ of the whole

subspace $\tilde{W}$. The equivalence classes $\bar{w}_{m+1}, \dots, \bar{w}_n$ in $W/U^\perp$ are linearly independent. Since the bilinear form

$$U \times (W/U^\perp) \to K, (v, \bar{w}) \mapsto \langle v, w \rangle$$

is well defined and nondegenerate, we can apply 3.6 to obtain vectors $u_{m+1}, \dots, u_n \in U$ satisfying $\langle u_i, w_j \rangle = \delta_{ij}$ for $i, j = m+1, \dots, n$. We now set:

$$u' := \sum_{i=m+1}^{n} \langle u, w_i \rangle u_i$$

Then we have: $\langle u', w_j \rangle = \langle u, w_j \rangle$ for $j = m+1, \dots, n$. Since we have $w_1, \dots, w_m \in U^\perp, u \in U^{\perp\perp}$ and $u' \in U$, it follows that $\langle u, w_j \rangle = 0 = \langle u', w_j \rangle$ for $j = 1, \dots, m$. This implies $\langle u, w \rangle = \langle u', w \rangle$ for $w \in \tilde{W}$, i. e. $u' - u \in \tilde{W}^\perp$, which means that $u' \in u + \tilde{W}^\perp$

$\square$

4. Linearly compact vector spaces

Theorem 4.1. *Suppose that $\langle \cdot, \cdot \rangle : V \times W \to K$is a nondegenerate bilinear form. The finite topology induced by W on V is the initial topology with respect to the mappings*

$$\eta_w : V \to K, v \mapsto \eta_w(v) := \langle v, w \rangle$$

Proof. It is sufficient to prove that the finite topology satisfies the universal property of the initial topology, since this property determines the initial topology uniquely. Observe first that η_w is continuous by 3.1, since it vanishes on the open subspace $(Kw)^\perp$. Now suppose that X is a topological space and that $f : X \to V$ is a mapping for that every map $\eta_w \circ f$ is continuous. We have to show that f is continuous. If $O \subset V$ is open and $x \in f^{-1}(O)$, we have $f(x) \in O$ and therefore there is a finite dimensional subspace $\tilde{W}$ of W satisfying $f(x) + \tilde{W}^\perp \subset O$. Choose a basis $w_1, \dots, w_n$ of $\tilde{W}$. Since K is discrete and $\eta_w \circ f$ is continuous, we have that

$$U := \bigcap_{i=1}^{n} (\eta_{w_i} \circ f)^{-1}(\{\eta_{w_i}(f(x))\})$$

is an open subspace of X which obviously contains x. Our assertion will be proved if we can show that $U \subset f^{-1}(O)$. If we are given $y \in U$, we have for $i = 1, \dots, n$ that $\eta_{w_i}(f(y)) = \eta_{w_i}(f(x))$, that is $\langle f(y), w_i \rangle = \langle f(x), w_i \rangle$. This implies $f(y) - f(x) \in \tilde{W}^\perp$, therfore $f(y) \in f(x) + \tilde{W}^\perp$, therefore $y \in f^{-1}(O)$. $\square$

Theorem 4.2. *Suppose that V is a linear topological vector space with respect to the topology $\mathcal{T}$. The space V' induces on V the finite topology $\mathcal{T}_e$. We have: $\mathcal{T} \subset \mathcal{T}_e$.*

Proof. It is sufficient to prove that we have $U^\perp \in \mathcal{T}$ for every finite dimensional subspace U of V'. Suppose that $f_1, \dots, f_n$ is a basis of U. Then we have:

$$U^\perp = \bigcap_{i=1}^{n} ker f_i$$

and $ker f_i = f^{-1}(\{0\})$ is open, since f_i is continuous and K is discrete. $\square$

Theorem 4.3. *Suppose that V is a linear topological vector space with respect to the topology $\mathcal{T}$. Denote the finite topology on V that is induced by V' by $\mathcal{T}_e$. Then the following assertions are equivalent:*

(1) *U is closed with respect to $\mathcal{T}$.*

(2) *U is closed with respect to $\mathcal{T}_e$.*

(3) *$U = U^{\perp\perp}$, where the orthogonal complements are formed with respect to the bilinear form*

$$V' \times V \to K, (f, v) \mapsto f(v).$$

Proof. The second statement obviously implies the first, because we have:

$$U \quad \mathcal{T}_e-\text{closed} \;\Rightarrow\; V\backslash U \in \mathcal{T}_e \Rightarrow (4.2 :) \quad V\backslash U \in \mathcal{T} \;\Rightarrow\; U \quad \mathcal{T}-\text{closed}$$

We now show that the first statement implies the third. It is obvious that we have $U \subset U^{\perp\perp}$. If there were $u \in U^{\perp\perp}$ that is not contained in U, then we could conclude from the Trennungssatz that there is $f \in U^{\perp}$ satisfying $f(v) = 1$, a contradiction. It follows from 3.9 that the third statement implies the second. □

Definition 4.4. Suppose that V is a linear topological vector space. We shall say that V is linearly compact if for any family $(A_i)_{i\in I}$ of closed affine subspaces of V that satisfies $\bigcap_{i\in I} A_i = \emptyset$ there are already finitely many indices $i_1, \dots, i_H???n \in I$ such that $\bigcap_{j=1}^{n} A_{i_j} = \emptyset$.

(Recall that an affine subspace of V is a subset of the form $v + U$ for some element $v \in V$ and some subspace U of V.)

Remark 4.5. Recall that an arbitrary topological space X is compact if for any family $(A_i)_{i\in I}$ of closed subsets of X that satisfies $\bigcap_{i\in I} A_i = \emptyset$ there are already finitely many indices $i_1, \dots, i_n \in I$ such that $\bigcap_{j=1}^{n} A_{i_j} = \emptyset$. This is because the complements of the sets A_i form an open covering of X.

Theorem 4.6. *Suppose that V is linear compact vector space and that U is a closed subspace. Then U is linear compact with respect to the relative topology.*

Proof. If $(A_i)_{i\in I}$ is a family of closed affine subspaces of U that satisfies $\bigcap_{i\in I} A_i = \emptyset$, we have that all the A_i are also closed in V because U is itself closed. Therefore, we have by definition that there are finitely many indices $i_1, \dots, i_n \in I$ such that $\bigcap_{j=1}^{n} A_{i_j} = \emptyset$. □

Theorem 4.7. *Suppose that V and W are linear topological vector spaces and that $f : V \to W$ is a surjective, continuous, linear mapping. Then if V is linearly compact, W is also linearly compact.*

Proof. If $(A_i)_{i\in I}$ is a family of closed affine subspaces of W that satisfies $\bigcap_{i\in I} A_i = \emptyset$, then $(f^{-1}(A_i))_{i\in I}$ is a family sharing the same properties

(Why?). Therefore there are finitely many indices $i_1, \dots, i_n \in I$ such that $\bigcap_{j=1}^n f^{-1}(A_{i_j}) = \emptyset$. Therefore, we have

$$f^{-1}(\bigcap_{j=1}^n A_{i_j}) = \bigcap_{j=1}^n f^{-1}(A_{i_j}) = \emptyset,$$

which implies $\bigcap_{j=1}^n A_{i_j} = \emptyset$ since f is surjective. □

Theorem 4.8. *A linear compact, discrete vector space is finite dimensional.*

Proof. Suppose that $(v_i)_{i\in I}$ is a basis of the vector space V under consideration. Let us assume that V is infinite dimensional. We define:

$$V_i := \mathrm{Span}\{v_j | j \neq i\} \quad \text{and} \quad A_i := v_i + V_i$$

A_i is closed since V is discrete. If $v = \sum_{j=1}^n \lambda_j v_{i_j}$ is an arbitrary vector in V, and if $i \notin \{i_1, \dots, i_n\}$, we have $v \notin A_i$. This implies $\bigcap_{i\in I} A_i = \emptyset$. Since V is linearly compact, there are finitely many indices $k_1, \dots, k_m \in I$ such that $\bigcap_{j=1}^m A_{k_j} = \emptyset$. But we have that $v_{k_1} + \ldots + v_{k_m} \in \bigcap_{j=1}^m A_{k_j}$, a contradiction. □

Theorem 4.9. *Suppose that $(V_i)_{i\in I}$ is a family of linearly compact vector spaces. Then the cartesian product $\prod_{i\in I} V_i$ is linearly compact.*

Proof. We shall show more generally: If $\mathfrak{A}$ is a set of affine subspaces of $V := \prod_{i\in I} V_i$ such that the intersection of finitely many elements of $\mathfrak{A}$ is not void, then also the intersection of the closures of the elements of $\mathfrak{A}$ is not void:

$$\bigcap_{A\in\mathfrak{A}} \overline{A} \neq \emptyset$$

We shall proceed in steps.

(1) We consider the following set:

$$\mathfrak{M} := \{\mathfrak{A}' | \mathfrak{A} \subset \mathfrak{A}', \mathfrak{A}' \text{ is a set of affine subspaces of } V$$
$$\text{such that finitely many elements of } \mathfrak{A}' \text{ have nonvoid intersection}\}$$

$\mathfrak{M}$ is ordered by inclusion. If $\mathfrak{N}$ is a totally ordered subset of $\mathfrak{M}$, then $\bigcap_{\mathfrak{A}'\in\mathfrak{N}} \mathfrak{A}'$ is an upper bound for $\mathfrak{N}$. By Zorn's Lemma we conclude that $\mathfrak{M}$ contains a maximal element $\mathfrak{A}_{\max}$. If we could show that

$$\bigcap_{A\in\mathfrak{A}_{\max}} \overline{A} \neq \emptyset$$

we could conclude that also $\bigcap_{A\in\mathfrak{A}} \overline{A} \neq \emptyset$ since it is obvious that

$$\bigcap_{A\in\mathfrak{A}_{\max}} \overline{A} \subset \bigcap_{A\in\mathfrak{A}} \overline{A}$$

Therefore we can and will assume in our further considerations that $\mathfrak{A}$ already is maximal. This implies that the set $\mathfrak{M}$ contains only one element.

(2) Consider the set

$$\{\bigcap_{j=1}^{n} A_j | A_1, \dots, A_n \in \mathfrak{A}\}$$

This is a set of affine subspaces such that the intersection of finitely many of its elements is nonvoid and that contains $\mathfrak{A}$. By the maximality of $\mathfrak{A}$ this set is equal to $\mathfrak{A}$. This simply means that $\mathfrak{A}$ contains finite intersections of its elements, that is, $\bigcap_{j=1}^{n} A_j \in \mathfrak{A}$ whenever $A_1, \dots, A_n \in \mathfrak{A}$.

(3) Consider the projections

$$\pi_i : V \to V_i, (v_j)_{j \in I} \mapsto v_i$$

For $A \in \mathfrak{A}$, $\pi_i(A)$ is an affine subspace of V_i. Since the closure of an affine subspace is again an affine subspace (cf. 2.14), and V_i is linearly compact, we have:

$$\bigcap_{A \in \mathfrak{A}} \overline{\pi_i(A)} \neq \emptyset$$

Therefore we can select for every $i \in I$ some element $v_i \in \bigcap_{A \in \mathfrak{A}} \overline{\pi_i(A)}$. Define $v := (v_i)_{i \in I}$. Now suppose that we are given a neighborhood of v_i of the form $v_i + U$ where U is an open subspace of V_i. This neighbourhood then intersects $\pi_i(A)$. $\pi_i^{-1}(v_i + U)$ then is an affine subspace of V (cf. 4.7) which intersects every $A \in \mathfrak{A}$. Therefore $\mathfrak{A} \cup \{\pi_i^{-1}(v_i + U)\}$ is a set of affine subspaces of V with the property that the intersection of finitely many of its elements is nonvoid. Since it obviously contains $\mathfrak{A}$, we conclude by the maximality of $\mathfrak{A}$ that it must be equal to $\mathfrak{A}$, that is, we have $\pi_i^{-1}(v_i + U) \in \mathfrak{A}$. We can now conclude by the second step that for open subspaces $U_{i_1}, \dots, U_{i_k}$ of $V_{i_1}, \dots, V_{i_k}$ we have:

$$\bigcap_{j=1}^{n} \pi_{i_j}^{-1}(v_{i_j} + U_{i_j}) = v + \bigcap_{j=1}^{n} \pi_{i_j}^{-1}(U_{i_j}) \in \mathfrak{A},$$

and therefore this set also intersects every element of $\mathfrak{A}$. This implies $v \in \overline{A}$ for every $A \in \mathfrak{A}$.

$\square$

Theorem 4.10. *Suppose that V is an arbitrary vector space. Endow the dual vector space V^* with the finite topology. Then V^* is linearly compact.*

Proof. First of all observe that the set of all mappings from V to K is nothing else but the cartesian product $K^V = \prod_{v \in V} K$ and therefore can be endowed with the product topology of the discrete topology on K. The projections onto the components K are precisely the evaluation mappings. K^V is linearly compact by 4.9, since K is linearly compact. Secondly, observe that V^* is a closed subset of K^V, since we have

$$V^* = \bigcap_{v,w \in V} \{f \in K^V | f(v) + f(w) - f(v+w) = 0\}$$
$$\cap \bigcap_{v \in V, \lambda \in K} \{f \in K^V | f(\lambda v) - \lambda f(v) = 0\}$$

and this shows that V^* is the intersection of inverse images of the closed set $\{0\}$ in K under continuous mappings. We now can conclude by 4.6 that V^* together with the relative topology of the product topology is a linearly compact space. But by 4.1, this is precisely the finite topology! $\square$

5. The linearly compact topology

Theorem 5.1. *Suppose that U is an open subspace of the linearly compact vector space V. Then U has finite codimension.*

Proof. Suppose that the codimension of U is infinite. Then there exists a sequence $(v_i)_{i\in\mathbb{N}}$ such that the equivalence classes $(\bar{v}_i)_{i\in\mathbb{N}}$ in V/U are linearly independent. We define:

$$V_i := U + \mathrm{Span}\{v_j | j \neq i\} \quad \text{and} \quad A_i := v_i + V_i$$

V_i - and therefore A_i - is closed by 2.9. If $v = \sum_{j=1}^n \lambda_j v_j$, we have $v \notin A_{n+1}$. This implies $\bigcap_{i\in\mathbb{N}} A_i = \emptyset$. Since V is linearly compact, there exists $n \in \mathbb{N}$ such that $\bigcap_{j=1}^n A_j = \emptyset$. But we have that $v_1 + \ldots + v_n \in \bigcap_{j=1}^n A_j$, a contradiction. □

Remark 5.2. If we set $U = \{0\}$ in the above Proposition, we recover Proposition 4.8.

Theorem 5.3. *Suppose that V is a Hausdorff lineartopological vector space. If U is a subspace of V which is linearly compact with respect to the relative topology, then it is closed.*

Proof. It is sufficient to show: If $v \notin U$, we have also $v \notin \bar{U}$. If $v \notin U$, there is, for every $u \in U$, an open subspace V_u that satisfies $u \notin v + V_u$, since V is Hausdorff. Let: $A_u = (v + V_u) \cap U$. By 2.9 we see that A_u is closed in the relative topology. By construction, we have that $\bigcap_{u\in U} A_u = \emptyset$. Since U is linearly compact, we conclude that there are finitely many vectors $u_1, \ldots, u_n \in U$ such that:

$$(v + \bigcap_{i=1}^n V_{u_i}) \cap U = \bigcap_{i=1}^n A_{u_i} = \emptyset.$$

We therefore have constructed an open neighbourhood of v - namely $v + \bigcap_{i=1}^n V_{u_i}$ - that does not contain any point of U. Therefore, we have $v \notin \bar{V}$. □

Theorem 5.4. *Suppose that V is a Hausdorff linearly compact vector space. We endow the space V'^* with the finite topology induced by V'. Then the canonical map*

$$\theta : V \to V'^*, v \mapsto (f \mapsto f(v))$$

is a homeomorphism. Therefore every linearly compact Hausdorff space is linearly homeomorphic to some dual space.

Proof. We shall proceed in steps.

(1) We first show that θ is continuous. By the universal property of the initial topology, we argue by 1.2 and 4.1 that it is sufficient to prove the continuity of the composition of θ and the mappings

$$V'^* \to K, \phi \mapsto \phi(f)$$

for $f \in V'$. But this is obvious, since this composition is just f.

(2) We now prove that θ is an open map onto its image, that is, that the map

$$V \to \theta(V), v \mapsto \theta(v)$$

is open, where of course $\theta(V)$ is endowed with the relative topology. (Recall that a mapping is called open if the image of every open set is open. If the mapping is bijective, this is obviously equivalent to saying that the inverse map is continuous.) In our case it is sufficient to show that the image of every open subspace is open. (Why?) Now, if U is some open subspace of V, then we can conclude by 5.1 that U has finite codimension. We therefore can select linearly independent vectors $v_1, \dots, v_n \in V$ such that

$$V = U \oplus \operatorname{Span}(v_1, \dots, v_n)$$

Since U is also closed by 2.9, the bilinear form

$$U^\perp \times \operatorname{Span}(v_1, \dots, v_n) \to K, (f, v) \mapsto f(v)$$

is nondegenerate by the Trennungssatz 3.4. We therefore can use 3.6 to get continuous linear funtions $f_1, \dots, f_n \in V'$ which on the one hand vanish on U and on the other hand satisfy $f_i(v_j) = \delta_{ij}$. This implies that

$$\theta(U) = \theta(V) \cap \{\phi \in V'^* | \forall i \leq n : \phi(f_i) = 0\},$$

since if we have $\theta(v) \in \{\phi \in V'^* | \forall i \leq n : \phi(f_i) = 0\}$ for some $v = u + \sum_{i=1}^n \lambda_i v_i, u \in U$, we have $\lambda_i = f_i(v) = 0$ and therefore $v = u \in U$. This shows that $\theta(U)$ is open in the relative topology, since we have represented it as the intersection of $\theta(V)$ and the open set $\{\phi \in V'^* | \forall i \leq n : \phi(f_i) = 0\}$.

(3) We now that θ is surjective. We have that $\theta(V)$ is linearly compact by 4.7. Since V'^* is Hausdorff, $\theta(V)$ is closed by 5.3. We therefore can conclude from 4.3 that $\theta(V) = \theta(V)^{\perp\perp}$, where we are forming the orthogonal complements with respect to the bilinear form

$$V'^{*\prime} \times V'^* \to K, (\psi, \phi) \mapsto \psi(\phi)$$

But since the canonical map $V' \to V'^{*\prime}$ is a bijection by 3.8, it amounts to the same thing to form orthogonal complements with respect to the bilinear form

$$V' \times V'^{*} \to K, (f, \phi) \mapsto \phi(f)$$

But now it is obvious that $\theta(V)^{\perp} = \{0\}$, and therefore we have

$$\theta(V) = \theta(V)^{\perp\perp} = V"^{*},$$

and θ is surjective.

(4) We now prove that θ is injective. Since V is Hausdorff, we conclude from 2.5 and 2.9 that $\{0\}$ is a closed subspace. Therefore, if $v \in V$ is nonzero, we can find by the Trennungssatz 3.4 a continuous linear function f that satisfies $f(v) = 1$. But if we would have $\theta(v) = 0$, then we would also have $\theta(v)(f) = f(v) = 0$.

□

Theorem 5.5. *Suppose that V is a lineartopological vector space and that U is an open subspace. Let: $U^{\perp} := \{f \in V^{*} | \forall u \in U : f(u) = 0\}$. Then we have:*

(1) $U^{\perp} \subset V'$

(2) *$U^{\perp}$ is linearly compact with respect to the relative topology of the finite topology.*

Proof. The first assertion follows from 3.1. To prove the second assertion, we contemplate the mappings

$$\psi : U^{\perp} \to (V/U)^{*}, f \mapsto (\bar{v} \mapsto f(v))$$

which is a bijection with the inverse

$$\psi^{-1} : (V/U)^{*} \to U^{\perp}, f \mapsto (v \mapsto f(\bar{v}))$$

We show that these mappings are continuous if $(V/U)^{*}$ is endowed with the finite topology. But since the finite topology is the initial topology with respect to the evaluation mappings, the continuity of ψ follows from the continuity of the mappings $U^{\perp} \to K, f \mapsto f(v)$ and the continuity of ψ^{-1} follows from the continuity of

$$(V/U)^{*} \to K, f \mapsto f(\bar{v})$$

Since $(V/U)^{*}$ is linearly compact by 4.10, $U^{\perp}$ must be linearly compact, too. □

Theorem 5.6. *Suppose that V is a lineartopological vector space. Then the set*

$$\{U^\perp | U \text{ is a linearly compact subspace of } V\}$$

is a filter basis of V'.

Proof. If U and W are linearly compact subspaces of V, then $U + W$ is also linearly compact, since it is the image of the linearly compact space $U \times W$ under the continuous linear mapping

$$U \times W \to U + W, (u, w) \mapsto u + w$$

We then have: $(U + W)^\perp \subset U^\perp \cap W^\perp$. □

Definition 5.7. The topology on V' which is by 5.7 uniquely determined by the requirement that the set

$$\{U^\perp | \text{U is a linearly compact subspace of } V\}$$

is a basis of the neighbourhood system of the origin is called the linearly compact topology.

Theorem 5.8. *Suppose that V is a lineartopological vector space. Endow V' with the finite topology and V'' with the linearly compact topology. Define $\mathcal{T}_{\updownarrow\|}$ to be the initial topology with respect to the canonical mapping*

$$\theta : V \to V'', v \mapsto (f \mapsto f(v))$$

Then we have for the original topology $\mathcal{T}$ of V: $\mathcal{T} \subset \mathcal{T}_{\updownarrow\|}$.

Proof. It is sufficient to prove that if U is a $\mathcal{T}$-open subspace of V, then it is also $\mathcal{T}_{\updownarrow\|}$-open. Now we have by 5.5 that $U^\perp \subset V'$ is linearly compact, and therefore $U^{\perp\perp}$ is open. We therefore have that

$$U^{\perp\perp} = \theta^{-1}(U^{\perp\perp}) \subset V$$

is $\mathcal{T}_{\updownarrow\|}$-open. But since U is also $\mathcal{T}$-closed by 2.9, we have by 4.3 that $U = U^{\perp\perp}$. □

Theorem 5.9. *Suppose that V is a linearly compact vector space. Then the linearly compact topology on V' is discrete.*

Proof. Since V is linearly compact, $V^\perp = \{0\}$ is open. □

Remark 5.10. We have just completed the last cornerstone in our treatment of duality theory. Let us try a summary. As we already knew from linear algebra, taking the dual space twice does not lead us back to the original space in the infinite dimensional case. The original space rather occurs as a subspace of the bidual space, and the question is: Which subspace is this? In section 3, we have proposed the following answer to this question: The dual space comes with a natural topology, namely the finite topology, and the elements of the bidual space that belong to the original space are precisely those which are continuous with respect to this topology. In other words, the canonical mapping

$$V \to V^{*\prime}$$

is a bijection.

Now, we found us in the situation that we were told that we should not only watch out for vector spaces, but should also look for natural topologies that come along with these spaces, for example, the dual vector space and its finite topology. In section 4, we found out where the process of constructing the dual vector space actually had taken us: We found that every dual vector space is linearly compact and Hausdorff with respect to the finite topology. But shortly afterwards we observed that dual vector spaces are actually characterized by their property of being linearly compact. We observed this in the following way: In order to find our way back from the dual vector space to the original vector space, we should take the space of continuous linear functions. On the other hand, if we start out with a linearly compact vector space, take the space of continuous linear functions, and then take again the dual vector space, we also get back the original space which is therefore realized as the dual of some vector space.

We therefore had found a complete duality between all vector spaces on the one hand and all linearly compact vector spaces on the other hand. However, there was some asymmetry left: In one of the two paths between these two, we had to consider topologies, but not in the other one. But finally, we learned that this is not completely true: Every vector space comes with a particularly simple topology, namely the discrete topology, and taking the dual vector space is just taking the space of continuous linear mappings with respect to that topology. And on the other path there is also a natural topology on the space of all continuous linear mappings, namely the linearly compact topology, which gives us back our original discrete topology when applied to the dual of some vector space, because this dual is linearly compact.

6. Completeness

Definition 6.1. Let X be a set. A system $\mathcal{U}$ of subsets of $X \times X$ is called a uniformity if it has the following properties:

(1) $\forall U \in \mathcal{U} : \Delta \subset U$, where $\Delta = \{(x,x)|x \in X\}$ is the diagonal of X.

(2) $\forall U \in \mathcal{U} : U^{-1} \in \mathcal{U}$, where we define $U^{-1} := \{(x,y)|(y,x) \in U\}$

(3) Weak triangle inequality: $\forall U \in \mathcal{U}\ \exists V \in \mathcal{U} : V \circ V \subset U$, where for two subsets $V, W \subset X \times X$ we define

$$V \circ W := \{(x,z) \in X \times X | \exists y \in X : (x,y) \in W \text{ and } (y,z) \in V\}$$

(4) $\forall U, V \in \mathcal{U} : U \cap V \in \mathcal{U}$

(5) $\forall U \in \mathcal{U}\ \forall V \subset X \times X : U \subset V \Rightarrow V \in \mathcal{U}$

A uniform space is a set together with a uniformity.

Example 6.2. Suppose that X is a metric space. For $\epsilon > 0$ define:

$$\Delta_\epsilon = \{(x,y) \in X \times X | d(x,y) < \epsilon\}$$

Then the set

$$\mathcal{U} := \{U \subset X \times X | \exists \epsilon > 0 : \Delta_\epsilon \subset U\}$$

is a uniformity, called the metric uniformity. Here the axioms 6.1.1, 6.1.2, 6.1.4 and 6.1.5 are obvious. The weak triangle inequality follows from the triangle inequality: If $\Delta_\epsilon \subset U$, define $V := \Delta_{\epsilon/2}$. For $(x,z) \in V \circ V$, we then have $y \in X$ such that $d(x,y) < \frac{\epsilon}{2}$, $d(y,z) < \frac{\epsilon}{2}$ and therefore

$$d(x,z) \le d(x,y) + d(y,z) < \epsilon.$$

Therefore, we have $(x,z) \in U$.

Example 6.3. Suppose that V is a topological vector space. For $W \subset V$ define:

$$\Delta_W := \{(v,w) \in V \times V | w - v \in W\}$$

Then the set

$$\mathcal{U} := \{U \subset V \times V | \text{There is a neighbourhood of the origin such that } \Delta_W \subset U\}$$

is a uniformity on V: Axiom 6.1.1 is obvious. If $\Delta_W \subset U$, we have $\Delta_{-W} \subset U^{-1}$, therefore 6.1.2 holds. We have $\Delta_{W \cap W'} \subset \Delta_W \cap \Delta_{W'}$, therefore 6.1.4 holds. 6.1.5 is obvious. We now prove the weak triangle inequality: If W is a neighbourhood of the origin, then by the continuity of the addition we have two other neighbourhoods W', W'' of the origin such that $W' + W'' \subset W$. We therefore see that $\Delta_{W'} \circ \Delta_{W''} \subset \Delta_W$.

Theorem 6.4. *Suppose that X is a uniform space with uniformity $\mathcal{U}$. Define:*

$$\mathcal{T} := \{\mathcal{O} \subset \mathcal{X} \mid \forall \S \in \mathcal{O}\ \exists \mathcal{U} \in \mathcal{U} : \mathcal{U}[\S] \subset \mathcal{O}\}$$

where we set $U[x] := \{y \in X \mid (x, y) \in U\}$. Then $\mathcal{T}$ is a topology on $\mathcal{U}$, called the uniform topology.

Proof. This is obvious. (Why?) □

Remark 6.5. If V is a topological vector space, then the uniform topology that arises from the topology in Example 6.3 is the original topology. (Why?)

Theorem 6.6. *A uniform space is Hausdorff if and only if the intersection of all members of the uniformity is the diagonal.*

Proof. Assume first that the uniform space X is Hausdorff. If $(x, y) \notin \Delta$, i. e. $x \neq y$, choose disjoint neighbourhoods $U \in \mathcal{U}(x)$ and $V \in \mathcal{U}(y)$. By definition of the uniform topology, there are $U', V' \in \mathcal{U}$ such that $U'[x] \subset U$ and $V'[y] \subset V$. We then have, for example, $(x, y) \notin U'$.

Conversely, if $\bigcap_{U \in \mathcal{U}} U = \Delta$ and $x \neq y$, we can choose $U \in \mathcal{U}$ such that $(x, y) \notin U$. By assumption we can find $V \in \mathcal{U}$ such that $V \circ V \subset U$. If $z \in V[x] \cap V^{-1}[y]$, i. e. $(x, z) \in V$, $(z, y) \in V$, we have $(x, y) \in V \circ V \subset U$, a contradiction. Therefore, $V[x]$ and $V^{-1}[y]$ are disjoint neighbourhoods of x and y. (Why?) □

Definition 6.7. Suppose that X is a topological space. A net in X is a map from a directed set I to X. A net is also called a Moore-Smith sequence, in case of $I = \mathbb{N}$ simply a sequence. We say that a net $x : I \to X$ converges to $p \in X$ if we have:

$$\forall U \in \mathcal{U}(p)\ \exists i_0 \in I\ \forall i \geq i_0 : x_i \in U$$

Definition 6.8. Suppose that X is a topological space. A filter in X is a system $\mathcal{F}$ of subsets of X that satisfies the following properties:

(1) $\emptyset \notin \mathcal{F},\ \mathcal{X} \in \mathcal{F}$

(2) $\forall F_1, F_2 \in \mathcal{F} : \mathcal{F}_\infty \cap \mathcal{F}_\in \in \mathcal{F}$

(3) $\forall F \in \mathcal{F}\ \forall \mathcal{F}' \subset \mathcal{X} : \mathcal{F} \subset \mathcal{F}' \Rightarrow \mathcal{F}' \in \mathcal{F}$

We say that a filter $\mathcal{F}$ converges to $p \in X$ if $\mathcal{U}(p) \subset \mathcal{F}$.

Remark 6.9. If $\mathcal{B}$ is a filter basis, then

$$\mathcal{F}(\mathcal{B}) := \{\mathcal{F} \subset \mathcal{X} \mid \exists \mathcal{B} \in \mathcal{B} : \mathcal{B} \subset \mathcal{F}\}$$

is a filter, called the filter generated by $\mathcal{B}$. If $\mathcal{F}$ is a filter and $\mathcal{B}$ is a filter basis with $\mathcal{F} = \mathcal{F}(\mathcal{B})$, then $\mathcal{B}$ is called a filter basis of $\mathcal{F}$.

If $\mathcal{F}$ is a filter in X and $f : X \to Y$ is a map, then

$$\mathcal{B} = \{f(F) \mid F \in \mathcal{F}\}$$

is a filter basis. The filter $f(\mathcal{F}) := \mathcal{F}(\mathcal{B})$ is called the image filter of $\mathcal{F}$.

Theorem 6.10. *If X is a topological space and $(x_i)_{i\in I}$ is a net in X, then*

$$\mathcal{F} := \{\mathcal{F} \subset \mathcal{X} \mid \exists \rangle_\prime \in \mathcal{I}\ \forall \rangle \geq \rangle_\prime : \S_\rangle \in \mathcal{F}\}$$

is a filter in X, called the associated filter of the net. $(x_i)_{i\in I}$ converges to $p \in X$ if and only if $\mathcal{F}$ converges to p.

Proof. The axioms 6.8.1 and 6.8.3 are obvious. If we have $F_1, F_2 \in \mathcal{F}$, choose $i_0, j_0 \in I$ such that

$$\forall i \geq i_0 : x_i \in F_1 \qquad \text{and} \qquad \forall j \geq j_0 : x_j \in F_2$$

Since I is directed, we can choose $k_0 \in I$ such that $k_0 \geq i_0$ and $k_0 \geq j_0$. We then have: $\forall i \geq k_0 : x_i \in F_1 \cap F_2$. $\square$

Theorem 6.11. *Suppose that X is a topological space and that $\mathcal{F}$ is a filter in X. $\mathcal{F}$ is directed by reverse inclusion. Every map from $\mathcal{F}$ to X therefore is a net. For $p \in X$, the following assertions are equivalent:*

(1) *$\mathcal{F}$ converges to p.*

(2) *For every net $f : \mathcal{F} \to \mathcal{X}$ with the property:*

$$\forall F \in \mathcal{F} : \{(\mathcal{F}) \in \mathcal{F}$$

we have: f converges to p.

Proof.
(1. $\Rightarrow$ 2.) We have to prove:

$$\forall U \in \mathcal{U}(p)\ \exists F_0 \in \mathcal{F}\ \forall \mathcal{F} \in \mathcal{F} : \mathcal{F} \subset \mathcal{F}_\prime \Rightarrow \{(\mathcal{F}) \in \mathcal{U}$$

If $U \in \mathcal{U}(p) \subset \mathcal{F}$ is given, define $F_0 = U$.

(1. $\Rightarrow$ 2.) Suppose that $U \in \mathcal{U}(p)$. We have to prove: $U \in \mathcal{F}$. Suppose that this is not the case. Then we have $F \not\subset U$ for all $F \in \mathcal{F}$. Therefore, for every $F \in \mathcal{F}$ we can choose a point $x_F \in F \setminus U$. Since the net $(x_F)_{F\in\mathcal{F}}$ satisfies the assumption, it converges to p. This means:

$$\exists F_0 \in \mathcal{F}\ \forall \mathcal{F} \in \mathcal{F} : \mathcal{F} \subset \mathcal{F}_\prime \Rightarrow \S_\mathcal{F} \in \mathcal{U}$$

If we choose such an F_0, we can conclude in particular that $x_{F_0} \in U$, a contradiction. $\square$

Theorem 6.12. *In a Hausdorff topological space, a net or a filter converges to at most one point.*

Proof. If $\mathcal{F}$ is a filter which converges to x and y, and if $x \neq y$, we can choose neighbourhoods U resp. V of x resp. y such that $U \cap V = \emptyset$. We then have $U \in \mathcal{F}$ and $V \in \mathcal{F}$, and therefore $\emptyset = U \cap V \in \mathcal{F}$, a contradiction.

If $(x_i)_{i \in I}$ is a net that converges to x and to y, choose neighbourhoods U resp. V of x resp. y satisfying $U \cap V = \emptyset$ if $x \neq y$. By definition, we have $i_0 \in I$ such that $x_i \in U$ for $i \geq i_0$, and $j_0 \in I$ such that $x_j \in V$ for $j \geq j_0$. Since I is directed, there exists $k \in I$ such that $k \geq i_0$ and $k \geq j_0$, and therefore we have $x_k \in U \cap V = \emptyset$. $\square$

Definition 6.13. Suppose that X is a uniform space with uniformity $\mathcal{U}$.

(1) A net $(x_i)_{i \in I}$ is called a Cauchy net if:

$$\forall U \in \mathcal{U}\ \exists i_0 \in I\ \forall i, j \geq i_0 : (x_i, x_j) \in U$$

(2) A filter $\mathcal{F}$ in X is called a Cauchy filter if:

$$\forall U \in \mathcal{U}\ \exists F \in \mathcal{F} : \mathcal{F} \times \mathcal{F} \subset \mathcal{U}$$

Theorem 6.14. *Suppose that X is a uniform space with uniformity $\mathcal{U}$.*

(1) *A net $(x_i)_{i \in I}$ is a Cauchy net if and only if the associated filter is a Cauchy filter.*

(2) *If $\mathcal{F}$ is a Cauchy filter, then every net $f : \mathcal{F} \to \mathcal{X}$ with the property:*

$$\forall F \in \mathcal{F} : \{(\mathcal{F}) \in \mathcal{F}$$

is a Cauchy net.

Proof. (1) Suppose that $(x_i)_{i \in I}$ is a Cauchy net. If $U \in \mathcal{U}$, choose $i_0 \in I$ such that we have $(x_i, x_j) \in U$ for all $i, j \geq i_0$. For $F := \{x_i \mid i \geq i_0\}$ we then have $F \times F \subset U$. Conversely, if the associated filter is a Cauchy filter, we select for a given $U \in \mathcal{U}$ some $F \in \mathcal{F}$ with $F \times F \subset U$, and for this F some $i_0 \in I$ such that $x_i \in F$ for all $i \geq i_0$. Then it is obvious that we have:

$$\forall i, j \geq i_0 : (x_i, x_j) \in U$$

(2) Suppose that $\mathcal{F}$ is a Cauchy filter. If f is a net with the stated property, choose for every given $U \in \mathcal{U}$ some $F_0 \in \mathcal{F}$ that satisfies

$F_0 \times F_0 \subset U$. If then F, F' are two elements of $\mathcal{F}$ with $F, F' \subset F_0$, we have $(f(F), f(F')) \in F \times F' \subset U$. □

Theorem 6.15. *Suppose that X is a uniform space with uniformity $\mathcal{U}$. Then the following assertions are equivalent:*

(1) *Every Cauchy net converges.*

(2) *Every Cauchy filter converges.*

Uniform spaces with this property are called complete.

Proof. The implication (1. $\Rightarrow$ 2.) follows from Proposition 6.11 and Proposition 6.14. The implication (2. $\Rightarrow$ 1.) follows from Proposition 6.10 and Proposition 6.14. □

7. Completion

Version 2

Definition 7.1. Suppose that X and Y are uniform spaces with uniformities $\mathcal{U}$ and $\mathcal{V}$ respectively. A mapping $f : X \to Y$ is called uniformly continuous if we have:

$$\forall\, V \in \mathcal{V} : f_2^{-1}(V) \in \mathcal{U},$$

where f_2 denotes the map $f_2 : X \times X \to Y \times Y, (x, x') \mapsto (f(x), f(x'))$.

Theorem 7.2. *Uniformly continuous mappings are continuous.*

Proof. If W is a neighbourhood of $f(x)$, select $V \in \mathcal{V}$ such that $V[f(x)] \subset W$, and define $U := f_2^{-1}(v) \in \mathcal{U}$. Then we have $f(U[x]) \subset W$. $\square$

Theorem 7.3. *Suppose that X is a uniform space with uniformity $\mathcal{U}$.*

(1) *If $U \in \mathcal{U}$, we have $\overset{\circ}{U} \in \mathcal{U}$. Therefore, there is an open $V \in \mathcal{U}$ such that $V \subset U$.*

(2) *If $U \in \mathcal{U}$, there is a closed $V \in \mathcal{U}$ such that $V \subset U$.*

Theorem 7.4. *Suppose that X and Y are uniform spaces with uniformities $\mathcal{U}$ resp. $\mathcal{V}$. Suppose that D is a dense subset of X. Let $f : D \to Y$ be a map that is uniformly continuous with respect to the relative uniformity*

$$\mathcal{U}_D := \{U \cap (D \times D) | U \in \mathcal{U}\}.$$

If Y is Hausdorff and complete, then there is a uniformly continuous extension of f to X.

Proof. A map is the same as its graph:

$$f = \{(x, f(x)) | x \in D\} \subset D \times Y \subset X \times Y$$

Consider the closure $\bar{f}$ of f in $X \times Y$. We proceed in steps:

(1) We first prove:

$$\forall W \in \mathcal{V}\ \exists U \in \mathcal{U}\ \forall (x, y), (u, v) \in \bar{f} : (x, u) \in \mathcal{U} \Rightarrow (y, v) \in W$$

Suppose we are given $W \in \mathcal{V}$. We select $V \in \mathcal{V}$ such that $V \circ V \subset W$. By perhaps making V smaller we can assume by Proposition 7.3 that V is closed, and similarly we can assume that $V = V^{-1}$. (Why ?) Choose $U \in \mathcal{U}$ such that $f_2(U \cap (D \times D)) \subset V \times V$. By Proposition 7.3 we can assume that U is open and satisfies $U = U^{-1}$ by perhaps making U smaller. Now suppose we are given $(x, y), (u, v) \in \bar{f}$ such that $(x, u) \in U$. $U[x]$ and $U[u]$ are open (why ?), therefore $U[x] \cap U[u]$ is an open neighbourhood of u. Since D is dense, we have $z \in D$ such that $z \in U[x] \cap U[u]$. Since $U = U^{-1}$ we have $(x, z) \in U$ and $(z, u) \in U$.

For all $V' \in \mathcal{V}$ we have that $U[z] \times V'[y]$ is a neighbourhood of (x, y). Since $(x, y) \in \bar{f}$, we have $x' \in D$ such that $(x', f(x')) \in U[z] \times V'[y]$, therefore $V'[y] \cap f(D \cap U[z])$ is not empty. We conclude:

$$y \in \overline{f(D \cap U[z])} \subset \overline{V[f(z)]} = V[f(z)],$$

therefore we have $(y, f(z)) \in V$. Similarly, we can show $(u, f(z)) \in V = V^{-1}$, also $(y, v) \in V \circ V \subset W$.

(2) We prove now that $\bar{f}$ is a map. If we have $(x, y), (x, v) \in \bar{f}$, select $W, W' \in \mathcal{V}$ such that $W[y] \cap W'[v] = \emptyset$ if $y \neq v$. Determine for $W, W' \in \mathcal{V}$ $U, U' \in \mathcal{U}$ according to step 1. Since $(x, x) \in U, (x, x) \in U'$ we have $(y, v) \in W, (y, v) \in W'$. Therefore, we have $v \in W[y] \cap W'[v]$, a contradiction.

(3) Next, we prove that $\bar{f}$ is defined on the whole space X. Let $x \in X$. For every neighbourhood U of x choose a point $d_U \in D \cap U$. Define:

$$N : \mathcal{U}(x) \to Y, U \mapsto f(d_U)$$

N is a net in Y. We prove that N is a Cauchy net: If $V \in \mathcal{V}$, we select $U \in \mathcal{U}$ such that $f_2(U \cap (D \times D)) \subset V$. Determine $\tilde{U} \in \mathcal{U}$ such that $\tilde{U}^{-1} = \tilde{U}$ and $\tilde{U} \circ \tilde{U} \subset U$. Then, if we are given neighbourhoods U', U'' of x such that $U', U'' \subset \tilde{U}[x]$, we have $d_{U'}, d_{U''} \in \tilde{U}[x]$, therefore $(x, d_{U''}), (d_{U'}, x) \in \tilde{U}$, therefore $(d_{U'}, d_{U''}) \in U$, therefore $(N(u'), N(u'')) = f_2(d_{U'}, d_{U''}) \in V$.

Since Y is complete, the net N converges to a point $y \in Y$. Since the net $(d_U, f(d_U))_{U \in \mathcal{U}(x)}$ converges to (x, y), we have $(x, y) \in \bar{f}$, i. e. $\bar{f}(x) = y$.

(4) The fact that $\bar{f}$ is uniformly continuous follows from step 1.

□

Theorem 7.5. *Suppose that X is a uniform space with uniformity $\mathcal{U}$, and that $\mathcal{F}$ is a Cauchy filter in X. Then there is a Cauchy filter $\mathcal{F}_\prime \subset \mathcal{F}$ which is the smallest one in the sense that if $\mathcal{G}$ is another Cauchy filter that satisfies $\mathcal{G} \subset \mathcal{F}$, then $\mathcal{F}_\prime \subset \mathcal{G}$.*

Proof. For $U \in \mathcal{U}$ and $M \subset X$, we define

$$U[M] = \{y \in X | \exists x \in M : (x, y) \in U\} = \bigcup_{x \in M} U[x]$$

Now we define: $\mathcal{F}_\prime := \{\mathcal{V} \subset \mathcal{X} | \exists \mathcal{U} \in \mathcal{U} \exists \mathcal{M} \in \mathcal{F} : \mathcal{U}[\mathcal{M}] \subset \mathcal{V}\}$.

(1) We first prove that $\mathcal{F}_\prime$ is a Cauchy filter. Obviously, $\mathcal{F}_\prime$ is a filter. If $U \in \mathcal{U}$, select $V \in \mathcal{U}$ such that $V = V^{-1}$ and $V \circ V \circ V \subset U$. Choose $M \in \mathcal{F}$ that satisfies $M \times M \subset V$. We prove: $V[M] \times V[M] \subset U$. If $y, z \in V[M]$, there are $u, v \in M$ such that $(y, u) \in V, (v, z) \in V$. Since $(u, v) \in V$ we have:

$$(y, z) = (v, z) \circ (u, v) \circ (y, u) \in V \circ V \circ V \subset U$$

(2) We prove that $\mathcal{F}_\prime \subset \mathcal{F}$. If $V \in \mathcal{F}_\prime$, we have $U \in \mathcal{U}, M \in \mathcal{F}$ such that $U[M] \subset V$, therefore $M \subset V$, therefore $V \in \mathcal{F}$.

(3) Suppose that $\mathcal{G}$ is a Cauchy filter satisfying $\mathcal{G} \subset \mathcal{F}$. To prove $\mathcal{F}_\prime \subset \mathcal{G}$, it is sufficient to prove $U[M] \in \mathcal{G}$ for $U \in \mathcal{U}$ and $M \in \mathcal{F}$. Choose $N \in \mathcal{G}$ such that $N \times N \subset U$. Since $N \in \mathcal{F}$ we have $M \cap N \neq \emptyset$. If $x \in M \cap N$, we have $\{x\} \times N \subset U$, therefore $N \subset U[x] \subset U[M]$, therefore $U[M] \in \mathcal{G}$.

□

Remark 7.6. If we have in Proposition 7.5 $\mathcal{F} = \{\mathcal{M} \subset \mathcal{X} | \mathcal{X} \in \mathcal{M}\}$ for some fixed $x \in X$ (This is a Cauchy filter!), then the construction shows that we have $\mathcal{F}_\prime = \mathcal{U}(\S)$, the neighbourhood system of x.

Theorem 7.7. *Suppose that X is a uniform space with uniformity $\mathcal{U}$. Suppose that $D \subset X$ is dense in X. We suppose that for every Cauchy filter $\mathcal{F}$ in D the filter*

$$\mathcal{F}_\mathcal{X} := \{\mathcal{M} \subset \mathcal{X} | \exists \mathcal{F} \in \mathcal{F} : \mathcal{F} \subset \mathcal{M}\}$$

converges. Then X is complete.

Proof. Suppose that $\mathcal{F}$ is a Cauchy filter in X. As in Proposition 7.5 define $\mathcal{F}_\prime := \{\mathcal{M} \subset \mathcal{X} | \exists \mathcal{U} \in \mathcal{U} \exists \mathcal{F} \in \mathcal{F} : \mathcal{U}[\mathcal{F}] \subset \mathcal{M}\}$. Every $M \in \mathcal{F}_\prime$ contains a point of D, because if $U \in \mathcal{U}, F \in \mathcal{F}$ satisfy $U[F] \subset M$, we can assume by Proposition 7.3 that U and therefore $U[F] = \bigcup_{x \in F} U[x]$ is open. In $U[F]$, and therefore in M, there is a point of D. This implies that

$$\mathcal{F}_\mathcal{D} := \{\mathcal{M} \subset \mathcal{D} | \exists \mathcal{F} \in \mathcal{F}_\prime : \mathcal{M} = \mathcal{F} \cap \mathcal{D}\}$$

is a filter in D which is obviously a Cauchy filter. By assumption the filter $(\mathcal{F}_\mathcal{D})_\mathcal{X}$ converges to $x \in X$. We prove now that $\mathcal{F}_\prime$ converges to x, too. If U is a neighbourhood of x, there is by Proposition 7.3 a closed element $V \in \mathcal{U}$ such that $V[x] \subset U$. Since we have $V[x] \in (\mathcal{F}_{\mathcal{D}})_{\mathcal{X}}$, we can choose $F \in \mathcal{F}_{\prime}$ such that $F \cap D \subset V[x]$. As at the beginning of the proof we can find an open $U' \in \mathcal{U}$ and some $F' \in \mathcal{F}$ such that $U'[F'] \subset F$. Since $U'[F']$ is open, we have:

$$U'[F'] \subset \overline{U'[F'] \cap D} \subset \overline{F \cap D} \subset \overline{V[x]} = V[x]$$

This implies $U \in \mathcal{F}_{\prime}$, therefore $U \in \mathcal{F}$. This means that $\mathcal{F}$ converges to x. □

Theorem 7.8. *Suppose that X is a uniform space. Then there exists a Hausdorff complete uniform space $\hat{X}$, called the Hausdorff completion of X, and a uniformly continuous mapping $\iota : X \to \hat{X}$ that has the following universal property:*

If $f : X \to Y$ is a second uniformly continuous map to a Hausdorff complete uniform space Y, then there exists a unique uniformly continuous map $\hat{f} : \hat{X} \to Y$ satisfying $f = \hat{f} \circ \iota$. Furthermore, we have:

(1) *$\iota(X)$ is dense in $\hat{X}$.*
(2) *If X is Hausdorff, then ι is injective.*

Proof. We define $\hat{X}$ to be the set of all minimal Cauchy filters in X. If $\mathcal{U}$ denotes the uniformity of X, define for $U \in \mathcal{U}$:

$$\hat{U} = \{(\mathcal{F}, \mathcal{G}) \in \hat{\mathcal{X}} \times \hat{\mathcal{X}} | \exists \mathcal{F} \in \mathcal{F} \cap \mathcal{G} : \mathcal{F} \times \mathcal{F} \subset \mathcal{U}\}$$

Furthermore, define:

$$\hat{\mathcal{U}} = \{V \subset \hat{X} \times \hat{X} | \exists U \in \mathcal{U} : \hat{U} \subset V\}$$

The proof will be carried out in steps.

(1) We prove first that $\hat{\mathcal{U}}$ is a uniformity.
  - Axiom 6.1: For $U \in \mathcal{U}$ and for a minimal Cauchy filter $\mathcal{F}$, there is $F \in \mathcal{F}$ such that $F \times F \subset U$. Therefore, we have $(\mathcal{F}, \mathcal{F}) \in \hat{\mathcal{U}}$.
  - Axiom 6.2: Obviously, we have: $\hat{U} = \widehat{(U^{-1})} = (\hat{U})^{-1}$
  - Axiom 6.3: Suppose that $U \in \mathcal{U}$. Determine $V \in \mathcal{U}$ such that $V \circ V \subset U$. ??? We prove: $\hat{V} \circ \hat{V} \subset \hat{U}$. If $(\mathcal{F}, \mathcal{H}) \in \hat{\mathcal{V}} \circ \hat{\mathcal{V}}$, then there is $\mathcal{G} \in \hat{\mathcal{X}}$ such that $(\mathcal{F}, \mathcal{G}) \in \hat{\mathcal{V}}, (\mathcal{G}, \mathcal{H}) \in \hat{\mathcal{V}}$. Therefore, there is $M \in \mathcal{F} \cap \mathcal{G}, \mathcal{N} \in \mathcal{G} \cap \mathcal{H}$ such that $M \times M \subset V, N \times N \subset V$. We have $M \cap N \in \mathcal{G}$, therefore $M \cap N \neq \emptyset$. Choose $x \in M \cap N$. If $(y, z) \in (M \cup N) \times (M \cup N)$ then we have $(y, x) \in V, (x, z) \in V$ and therefore

$(y, z) \in V \circ V \subset U$. This implies $M \cup N \in \mathcal{F} \cap \mathcal{H}$, where $(M \cup N) \times (M \cup N) \subset U$, and therefore we have $(\mathcal{F}, \mathcal{H}) \in \hat{\mathcal{U}}$.

- Axiom 6.4: Obviously, we have for $U, V \in \mathcal{U}$: $(\widehat{U \cap V}) \subset \hat{U} \cap \hat{V}$
- Axiom 6.5: Obvious.

(2) We now prove that $\hat{X}$ is Hausdorff. Suppose that we have $\mathcal{F}, \mathcal{G} \in \hat{\mathcal{X}}$ such that $(\mathcal{F}, \mathcal{G}) \in \hat{\mathcal{U}}$ for all $U \in \mathcal{U}$. By Proposition 6.6.6 is is sufficient to prove $\mathcal{F} = \mathcal{G}$. Define:

$$\mathcal{H} = \{\mathcal{M} \subset \mathcal{X} | \exists \mathcal{F} \in \mathcal{F}\ \exists \mathcal{G} \in \mathcal{G} : \mathcal{F} \cup \mathcal{G} \subset \mathcal{M}\}$$

$\mathcal{H}$ is a filter, since if we have $F, F' \in \mathcal{F}, \mathcal{G}, \mathcal{G}' \in \mathcal{G}$, we have $(F \cap F') \cup (G \cap G') \subset (F \cup G) \cap (F' \cup G')$. $\mathcal{H}$ is also a Cauchy filter, because for all $U \in \mathcal{U}$ there is $M \in \mathcal{F} \cap \mathcal{G}$ such that $M \times M \subset U$, and we have $M \in \mathcal{H}$. Since $\mathcal{H} \subset \mathcal{F}, \mathcal{H} \subset \mathcal{G}$ it follows from minimality: $\mathcal{F} = \mathcal{H} = \mathcal{G}$.

(3) We define:

$$\iota : X \to \hat{X}, x \mapsto \mathcal{U}(x)$$

ι is well defined by remark 7.6. We prove that ι is uniformly continuous. Suppose that $U \in \hat{U}$. Determine $V \in \mathcal{U}$ such that $V = V^{-1}$ and $V \circ V \circ V \subset U$. It is sufficient to prove that $\iota_2(V) \subset \hat{U}$. If $(x, y) \in V$, we define $F := V[x] \cup V[y] \in \mathcal{U}(x) \cap \mathcal{U}(y)$. We have $F \times F \subset U$, since one of the four possibilities:

$$\begin{array}{ll} (x, x') \in V, (x, y') \in V & (x, x') \in V, (y, y') \in V \\ (y, x') \in V, (x, y') \in V & (y, x') \in V, (y, y') \in V \end{array}$$

will be satisfied, and in any case we have $(x', y') \in U$. From this it follows that we have: $(\mathcal{U}(x), \mathcal{U}(y)) \in \hat{U}$.

(4) We prove that $\iota(X)$ is dense in $\hat{X}$. Suppose that $\mathcal{F} \in \hat{\mathcal{X}}$. It is sufficient to prove that for $U \in \mathcal{U}$ the set $\hat{U}[\mathcal{F}]$ intersects the set $\iota(X)$. Select $F \in \mathcal{F}$ satisfying $F \times F \in U$. From the construction of minimal Cauchy filters in Proposition 7.5 we conclude that F contains some inner point x. Then we have $F \in \mathcal{F} \cap \mathcal{U}(\S)$, therefore $(\mathcal{F}, \mathcal{U}(\S)) \in \hat{\mathcal{U}}$, therefore $\iota(x) = \mathcal{U}(x) \in \hat{U}[\mathcal{F}]$.

(5) We prove: $\forall U \in \mathcal{U} : \iota_2^{-1}(\hat{U}) \subset U$. This follows from the implications: $(x, y) \in \iota_2^{-1}(\hat{U}) \Rightarrow (\mathcal{U}(x), \mathcal{U}(y)) \in \hat{U} \Rightarrow \exists O \in \mathcal{U}(x) \cap \mathcal{U}(y) : O \times O \subset U \Rightarrow (x, y) \in U$.

(6) Suppose that $\mathcal{F}$ is a Cauchy filter in $\iota(X)$. Then

$$\mathcal{F}' := \{\mathcal{M} \subset \mathcal{X} | \exists \mathcal{F} \in \mathcal{F} : \iota^{-\infty}(\mathcal{F}) \subset \mathcal{M}\}$$

is a filter in X, and also a Cauchy filter by step 5. By Proposition 7.5 there is a minimal Cauchy filter $\mathcal{F}'_{\prime}$ satisfying $\mathcal{F}'_{\prime} \subset \mathcal{F}'$. We consider the Cauchy filter

$$\mathcal{F}'' := \{\mathcal{M} \subset \iota(\mathcal{X}) | \exists \mathcal{F} \in \mathcal{F}'_{\prime} : \iota(\mathcal{F}) \subset \mathcal{M}\}$$

This filter leads to the following filter on the whole completion:

$$\mathcal{F}''' := \{\mathcal{M} \subset \hat{\mathcal{X}} | \exists \mathcal{F} \in \mathcal{F}'' : \mathcal{F} \subset \mathcal{M}\}$$

By Proposition 7.7 is suffices to prove that $\mathcal{F}'''$ converges. Now we can say that $\mathcal{F}'''$ also is:

$$\mathcal{F}''' = \{\mathcal{M} \subset \hat{\mathcal{X}} | \exists \mathcal{F} \in \mathcal{F}'_{\prime} : \iota(\mathcal{F}) \subset \mathcal{M}\}$$

We prove that $\mathcal{F}'''$ converges to $\mathcal{F}'_{\prime}$ by proving that we have: $\forall U \in \mathcal{U} : \ \hat{U}[\mathcal{F}'_{\prime}] \in \mathcal{F}'''$, that is: $\forall U \in \mathcal{U}\ \exists \tilde{F} \in \mathcal{F}'_{\prime} : \ \iota(\tilde{\mathcal{F}}) \subset \hat{\mathcal{U}}[\mathcal{F}'_{\prime}]$, that is: $\forall U \in \mathcal{U}\ \exists \tilde{F} \in \mathcal{F}'_{\prime}\ \forall \S \in \tilde{\mathcal{F}} : \ (\mathcal{F}'_{\prime}, \mathcal{U}(\S)) \in \hat{\mathcal{U}}$. Now, if $U \in \mathcal{U}$, we can choose by Proposition 7.3 some open $V \in \mathcal{U}$ such that $V = V^{-1}$ and $V \circ V \circ V \subset U$. Since $\mathcal{F}'$ is a Cauchy filter, there is $F' \in \mathcal{F}'$ satisfying $F' \times F' \subset V$. Define: $\tilde{F} := V[F'] \in \mathcal{F}'_{\prime}$. $\tilde{F}$ is open. If $(x, y) \in \tilde{F} \times \tilde{F}$, we have $x', y' \in F'$ such that $(x', x) \in V, (y', y) \in V$. Since we have also $(x', y') \in V$, we have $(x, y) \in U$, therefore $\tilde{F} \times \tilde{F} \subset U$. Since we have for $x \in \tilde{F}$ that $\tilde{F} \in \mathcal{U}(x)$, we conclude: $(\mathcal{F}'_{\prime}, \mathcal{U}(\S)) \in \hat{\mathcal{U}}$.

(7) We now prove the universal property of $\hat{X}$. Suppose that $f : X \to Y$ is a uniformly continuous mapping into a Hausdorff complete uniform space. We define a map

$$\hat{f}_0 : \iota(X) \to Y$$

as follows: For $x \in X$, $\mathcal{U}(x)$ is a Cauchy filter, therefore

$$\mathcal{F}_{\mathcal{X}} := \{\mathcal{M} \subset \mathcal{Y} | \exists \mathcal{U} \in \mathcal{U}(\S) : \ \{(\mathcal{U}) \subset \mathcal{M}\}$$

is a Cauchy filter, too. By Proposition 6.6.12 $\mathcal{F}_{\mathcal{X}}$ converges to a unique point that we denote by $\hat{f}_0(\iota(x))$. This point is of course $f(x)$. We prove that $\hat{f}_0$ is uniformly continuous: Suppose that V is a member of the uniformity of Y. Choose $U \in \mathcal{U}$ such that $f_2(U) \subset V$. If we have $(\iota(x), \iota(y)) \in \hat{U}$, then we have by step 5 that $(x, y) \in U$ and therefore $(\hat{f}_0)_2(\iota(x), \iota(y)) = (f(x), f(y)) \in V$. By Proposition 7.4, there is a unique extension $\hat{f}$ of $\hat{f}_0$ to the whole space $\hat{X}$.

(8) In a Hausdorff space, we obviously have $x = y$ if $\mathcal{U}(x) = \mathcal{U}(y)$.

$\square$

8. Completion of lineartopological vector spaces

Definition 8.1. Suppose that V is a lineartopological vector space. Define: $I := \{U | U \text{ is an open subspace of } V\}$. I is directed with respect to reverse inclusion:

$$U \leq W :\Leftrightarrow W \subset U$$

A net $N : I \to V$ is called centered if we have:

$$\forall\, U, W \in I : W \subset U \Rightarrow N(u) - N(w) \in U$$

Theorem 8.2. *Suppose that V is a lineartopological vector space.*

(1) *Every centered net is a Cauchy net.*

(2) *V is complete if and only if every centered net converges.*

Proof. To prove the first statement, we prove for every centered net:

$$\forall U \in \mathcal{U}(0)\ \exists U_0 \in I\ \forall W, W' \in I : W \subset U_0 \text{ and } W' \subset U_0 \Rightarrow (N(W), N(W')) \in \Delta_U$$

If $U \in \mathcal{U}(0)$, choose $U_0 \in I$ such that $U_0 \in U$. If W, W' are open subspaces satisfying $W \subset U_0$ and $W' \subset U_0$, we have $N(U_0) - N(W) \in U_0$, $N(U_0) - N(W') \in U_0$, and therefore $N(W') - N(W) \in U_0 \subset U$.

To prove the second statement, suppose that $N : J \to V$ is a Cauchy net. For every $U \in I$ determine $j_U \in I$ such that:

$$\forall i, j \geq j_0 : N(j) - N(i) \in U$$

and define $M(U) := N(j_U)$. We prove that M is a centered net: If $U, W \in I$ satisfy $W \subset U$, determine $j \in I$ such that $j \geq j_U$ and $j \geq j_W$. Then we have for that $N(j) - N(j_U) \in U$, $N(j) - N(j_W) \in U$, therefore

$$M(U) - M(W) = (N(j_U) - N(j)) - (N(j_W) - N(j)) \in U$$

Now we can conclude by assumption that M converges to some point $v \in V$. We prove that N converges to v, too. It is sufficient to prove:

$$\forall U \in I\ \exists i_0 \in J\ \forall i \geq i_0 : N(i) - v \in U$$

If we are given $U \in I$, then we have for $i \geq j_U$ that $N(i) - N(j_U) \in U$ and also that $N(i) - v \in U$. $\square$

Definition 8.3. Suppose that V is a lineartopological vector space and that U is a subspace of V. The set

$$\{(W + U)/U \mid W \text{ is an open subspace of } V\}$$

is a filter basis. (Why?) The linear topology which is by Proposition 2.6 uniquely determined by this set is called the linear quotient topology of V/U.

Theorem 8.4. *Suppose that V is a lineartopological vector space and that U is a subspace of V. Denote the canonical projection by $\pi : V \to V/U$.*

(1) *π is open, i. e. π maps open sets to open sets, and continuous.*

(2) *If U is closed, then V/U is Hausdorff.*

(3) *If U is open, then V/U is discrete.*

Proof. Observe first that π maps an open subspace W to the open subspace $(W+U)/U$. Since we have $\pi(v+W) = \bar{v}+\pi(W)$ we see that π maps open affine subspaces to open affine subspaces. Since every open set is the union of open affine subspaces, π is open. The continuity follows from Proposition 3.1.

To prove the second statement, assume that V/U is not Hausdorff. Then we have by Proposition 2.5 a vector $\bar{v} \in (V/U) \setminus \{\bar{0}\}$ such that

$$\bar{v} \in \bigcap_{U \subset W \subset V,\ W \text{open subspace}} (W/U)$$

Then we have $v \notin U$, but $v \in W$ for all open subspaces W satisfying $U \subset W$. If W is an open subspace, then we have $v \in W + U$, therefore $(v + W) \cap U \neq \emptyset$, therefore $v \in \bar{U}$.

The last statement holds because in this case $\{\bar{0}\} = \pi(U)$ is open. $\square$

Remark 8.5. We can conclude from Proposition 8.4 that the linear quotient topology coincides with the ordinary quotient topology. (Cf. [3], Chap. 3, Theorem 8, p. 95.)

Theorem 8.6. *Suppose that V is a lineartopological vector space. Denote the set of all open subspace by I. I is directed by reverse inclusion. For $U, W \in I$ define:*

$$f_{UW} : V/W \to V/U, \bar{v} \mapsto \bar{v}$$

Define $\hat{V} := \varprojlim_{U \in I} V/U$, the projective limit of the family $(V/U)_{U \in I}$, and define:

$$\iota : V \to \hat{V}, v \mapsto (\bar{v})_{U \in I}$$

Then we have:

(1) *$\hat{V}$ is Hausdorff.*

(2) *$\hat{V}$ is complete.*

(3) *If $f : V \to X$ is a uniformly continuous map into a complete Hausdorff uniform space X, then there is a uniquely determined uniformly continuous map $g : \hat{V} \to X$ such that $g \circ \iota = f$.*

Proof. (1) For $U \in I$ denote by $\pi_U : \hat{V} \to V/U$ the projection on the component V/U. Suppose that $\hat{v} \in \hat{V}$ is an element that is contained in every open subspace of $\hat{V}$. By Proposition 2.5 it is sufficient to prove $\hat{v} = 0$. But, since $\hat{v} \in \pi_U^{-1}(\{0\})$, if $\hat{v} = (\bar{v}_U)_{U \in I}$, we have $\bar{v}_U = 0$, and this means $\hat{v} = 0$. This proves the first assertion.

(2) Denote the set of all open subspaces of $\hat{V}$ by J. Suppose that $N : J \to \hat{V}$ is a centered net. For $U \in I$ define:

$$\bar{v}_U := \pi_U(N(\pi_U^{-1}(\{0\})))$$

and $\hat{v} := (\bar{v}_U)_{U \in I}$.

(3) We prove now: $W, U \in I,\ W \subset U \Rightarrow \pi_W^{-1}(\{0\}) \subset \pi_U^{-1}(\{0\})$. This is easy: If $\hat{w} = (\bar{w}_X)_{X \in I} \in \pi_W^{-1}(\{0\}$, we have $\bar{w}_W = 0$ and therefore $\bar{w}_U = f_{UW}(\bar{w}_W) = 0$, which means that $\hat{w} = (\bar{w}_X)_{X \in I} \in \pi_U^{-1}(\{0\})$.

(4) We prove now: $\hat{v} \in \hat{V}$, that is, if $U \subset W$, then $f_{UW}(\bar{v}_W) = \bar{v}_U$:

$$\begin{aligned}
&U \subset W \Rightarrow W \subset U \Rightarrow \pi_W^{-1}(\{0\}) \subset \pi_U^{-1}(\{0\}) \\
&\Rightarrow N(\pi_W^{-1}(\{0\})) - N(\pi_U^{-1}(\{0\})) \in \pi_U^{-1}(\{0\}) \\
&\Rightarrow \pi_U(N(\pi_W^{-1}(\{0\}))) - \pi_U(N(\pi_U^{-1}(\{0\}))) = 0 \\
&\Rightarrow \bar{v} - f_{UW}(\bar{v}_W) = 0
\end{aligned}$$

(5) We prove now that N converges to $\hat{v}$: For this, it is sufficient to prove that $\pi_U \circ N$ converges to $\bar{v}_U$, because the topology of the projective limit is the initial topology with respect to the projections (cf. [3], Chap. 3). We therefore have to prove:

$$\forall W \in I,\ U \subset W\ \exists j_0 \in J\ \forall j \geq j_0 : \pi_U(N(j)) - \bar{v}_U \in W/U$$

Suppose we are given $W \in I$ and $U \subset W$. Define: $j_0 := \pi_U^{-1}(\{0\})$. If $X \in J$ satisfies $X \subset \pi_U^{-1}(\{0\})$, then we have, since N is centered, that

$$N(\pi_U^{-1}(\{0\})) - N(X) \in \pi_U^{-1}(\{0\})$$

and therefore $\bar{v}_U - \pi_U(N(X)) = 0 \in W/U$. This proves the second assertion.

(6) We prove that $\iota(V)$ is dense in $\hat{V}$. If $\hat{v} = (\bar{v}_U)_{U\in I} \in \hat{V}$, then by construction of the initial topology every neighbourhood of $\hat{v}$ contains a neighbourhood of the form

$$\hat{v} + \bigcap_{i=1}^{n} \pi_{U_i}^{-1}(W_i/U_i),$$

where W_i is an open subspace of V that contains U_i. Now define: $U := \bigcap_{i=1}^{n} U_i$ and $v := v_U$. We prove that:

$$\iota(v) \in \hat{v} + \bigcap_{i=1}^{n} \pi_{U_i}^{-1}(W_i/U_i)$$

But now, since $U \subset U_i$, we have:

$$\pi_{U_i}(\iota(v) - \hat{v}) = \bar{v} - f_{UU}(\bar{v}_U) = 0 \in W_i/U_i$$

(7) Define: $A = \overline{\{0\}} \subset V$ to be the closure of the origin inside V. We prove that $A = ker\ \iota$. Since ι is continuous by the universal property of the initial topology and $\hat{V}$ is Hausdorff, we see that $\{0\} \subset \hat{V}$ is closed and therefore $ker\ \iota$ is also closed, which implies $A \subset ker\ \iota$. On the other hand, if $\iota(v) = 0$, we have that $\pi_U(\iota(v)) = \bar{v} = 0$ and therefore $v \in U$ for all open subspaces $U \in I$. That means $0 \in v + U$ for all $U \in I$, i. e. $v \in \overline{\{0\}}$.

(8) We prove now that the map

$$V/A \to \iota(V), v \mapsto \iota(v)$$

is a linear homeomorphism. By the universal property of the initial topology, for the proof of continuity it is sufficient to show that for $U \in I$ the map $V/A \to V/U$, $\bar{v} \mapsto \bar{v}$ is continuous, and this is clear from Proposition 3.1 (cf. [3], Chap. 3, Theorem 9, p. 95). We prove next that the map is open. For this, it is sufficient to prove for every $U \in I$:

$$\iota(U) = \iota(V) \cap \pi_U^{-1}(\{0\})$$

The inclusion "$\subset$" is obvious. If $\iota(v) \in \pi_U^{-1}(\{0\})$, we have that $\pi_U(\iota(v)) = \bar{v} = 0$, and therefore $v \in U$.

(9) If $f : V \to X$ is a uniformly continuous map to a complete Hausdorff uniform space X, then f can be factorized over V/A, that is, the map

$$\bar{f} : V/A \to X,\ \bar{v} \mapsto f(v)$$

is well defined. This is because if we have $v \in V$, we have $\bar{v} = v + A = \overline{\{v\}}$. Since X is a Hausdorff space, $\{f(v)\}$ is a closed set, and by the continuity of f we conclude that $f^{-1}(\{f(v)\})$ is also closed, which implies $v + A \subset f^{-1}(\{f(v)\})$. $\bar{f}$ is also uniformly continuous. (Why?) We can now conclude from the last step that there is a unique uniformly continuous map

$$f' : \iota(V) \to X$$

that satisfies $f'(\iota(v)) = f(v)$ for every $v \in V$. Since $\iota(V)$ is dense in $\hat{V}$, we conclude from Proposition 6.19 that there is a unique uniformly continuous extension

$$g : \hat{V} \to X$$

of f', and this proves our theorem.

□

Remark 8.7. From Proposition 8.6 we can conclude that the projective limit considered above is uniformly equivalent to the Hausdorff completion of the preceding section. (Why?) Here, we call a bijection between two uniform spaces a uniform equivalence if the map itself and its inverse are uniformly continuous.

Definition 8.8. Suppose that $V_1, \dots, V_n$ are lineartopological vector spaces.

(1) By Proposition 2.7, there is a unique linear topology on $\bigotimes_{i=1}^n V_i$ for which the set

$$\{\sum_{j=1}^n \bigotimes_{i=1}^n U_{ij} \mid U_{ij} \text{ is an open subspace of } V_i, U_{ij} = V_i \text{ if } i \neq j\}$$

is a basis of the neighbourhood system of the origin of $\bigotimes_{i=1}^n V_i$. This topology is called the tensor product topology.

(2) The completion of the tensor product with respect to the tensor product topology is called the completed tensor product, it is denoted by $\hat{\bigotimes}_{i=1}^n V_i$.

Remark 8.9. The set in Definition 8.8 really is a filter basis. since we have:

$$\sum_{j=1}^n \bigotimes_{i=1}^n (U_{ij} \cap U'_{ij}) \subset (\sum_{j=1}^n \bigotimes_{i=1}^n U_{ij}) \cap (\sum_{j=1}^n \bigotimes_{i=1}^n U'_{ij})$$

Theorem 8.10. *Linearly compact vector spaces are complete.*

Proof. (1) Suppose that $\mathcal{F}$ is a Cauchy filter in V. For every open subspace U of V determine $F_U \in \mathcal{F}$ such that $F_U \times F_U \subset \Delta_U$.

(2) If $x \in F_U$, we have $x + U = F_U + U$, because for every $y + u \in F_U + U$ where $y \in F_U$, $u \in U$, we have $y - x \in U$ and therefore:

$$y + u = x + (y - x) + u \in x + U$$

(3) From the first step and Proposition 2.9 we conclude that $F_U + U$ is closed. If $U_1, \dots, U_n$ are open subspaces of V, define: $U := \bigcap_{i=1}^n U_i$, $F := F_U \cap \bigcap_{i=1}^n F_{U_i} \in \mathcal{F}$. Then we have $F + U \subset F_{U_i} + U_i$ for all $i \leq n$, and therefore $F_{U_i} + U_i \neq \emptyset$. Since V is linearly compact, we see that $\bigcap_{U \text{ open subspace}} (F_U + U) \neq \emptyset$. Select $v \in V$ such that $v \in F_U + U$ for every open subspace U. We claim that $\mathcal{F}$ converges to v. If U is an open subspace of V and $x \in F_U$, we have $x - v \in U$, therefore $x \in v + U$, therefore $F_U \subset v + U$, therefore $v + U \in \mathcal{F}$. This proves that every neighbourhood of v is contained in $\mathcal{F}$.

□

Theorem 8.11. *Suppose that V and W are vector spaces. Then we have:*

$$(V \otimes W)^* \cong V^* \hat{\otimes} W^*$$

Proof. $(V \otimes W)^*$ is Hausdorff and complete by Proposition 4.10 and Proposition 7.10. Consider the map:

$$\iota : V^* \otimes W^* \to (V \otimes W)^*,\ f \otimes g \mapsto (v \otimes w \mapsto f(v)g(w))$$

It is sufficient to prove that $\iota(V^* \otimes W^*)$ is dense and that ι is a homeomorphism onto its image. (Why?)

(1) We prove that $\iota(V^* \otimes W^*)$ is dense in $(V \otimes W)^*$. Suppose that $f \in (V \otimes W)^*$ and that $U \subset V \otimes W$ is a finite dimensional subspace. We have to prove that

$$(f + U^\perp) \cap \iota(V^* \otimes W^*) \neq \emptyset$$

There are finite dimensional subspaces $V_1 \subset V$, $W_1 \subset W$ such that $U \subset V_1 \otimes W_1$ (Why?). If $v_1, \dots, v_n$ is a basis of V_1, $w_1, \dots, w_m$ a basis of W_1, there are by Proposition 3.6 $f_1, \dots, f_n \in V^*$ and $g_1, \dots, g_n \in W^*$ such that:

$$f_i(v_j) = \delta_{ij} \qquad g_i(w_j) = \delta_{ij}$$

Consider $g := \sum_{i=1}^n \sum_{j=1}^m f(v_i \otimes w_j) f_i \otimes g_j$. Since we have:

$$(f - \iota(g))(v_k \otimes w_l) = f(v_k \otimes w_l) - \sum_{i,j} f(v_i \otimes w_j) f_i(v_k) g_j(w_l) = 0$$

we have $f - \iota(g) \in (V_1 \otimes W_1)^\perp \subset U^\perp$, and therefore we see that $\iota(f) \in f + U^\perp$.

(2) We prove that ι is continuous. By the universal property of the initial topology and due to the fact that every tensor is the sum of decomposable tensors, is suffices by Proposition 4.1 that for $v \in V$, $w \in W$ the map

$$V^* \otimes W^* \to K,\ f \otimes g \mapsto f(v)g(w)$$

is continuous. But since this map vanishes on the open subspace $\{v\}^\perp \otimes W^* + V^* \otimes \{w\}^\perp$, it is continuous by Proposition 3.1.

(3) We prove: If $O \subset V^* \otimes W^*$ is open, then $\iota(O)$ is open in $\iota(V)$. We can assume by Proposition 2.9 that O is an open subspace, even more that $O = V_1^\perp \otimes W^* + V^* \otimes W_1^\perp$ for two finite dimensional subspaces $V_1 \subset V$, $W_1 \subset W$. Then we have:

$$\iota(O) = \iota(V) \cap (V_1 \otimes W_1)^\perp$$

and therefore is open.

$\square$

9. Infinite Galois theory

Definition 9.1. Suppose that G is a topological group.

(1) G is called normaltopological if every neighbourhood of the unit element contains a further neighbourhood which is a normal subgroup.

(2) G is called projectively finite if it is normaltopological, Hausdorff and compact.

Theorem 9.2. *Suppose that G is a group and that F is a filter basis consisting of normal subgroups. Then there is a unique topology $\mathcal{T}$ on G such that:*

(1) *F is a basis of the neighbourhood system of the unit element.*

(2) *G is normaltopological.*

Proof. (Cf. Proposition 2.7) We prove the uniqueness first. The neighbourhoods of 1 are precisely the sets that contain a set that belongs to F. The neighbourhood of an arbitrary element $g \in G$ are the g-translates of the neighbourhoods of the unit element. Since a set is open if and only if it is the neighbourhood of all its points, the topology $\mathcal{T}$ is unique. We now prove the existence. We shall call a set $O \subset G$ open if it satisfies:

$$\forall g \in O \, \exists U \in F : gU \subset O$$

The axioms 1.1.1.1 and 1.1.1.2 are obvious. If O and O' are open and if $g \in O \cap O'$, we can select $U, U' \in F$ such that $gU \in O$ and $gU' \in U'$. Now, by the definition of a filter basis there exists $U'' \in F$ such that $U'' \subset U \cap U'$. Therefore we have $gU'' \subset O \cap O'$ and $O \cap O'$ is open.

It remains to prove that G is a topological group. Suppose that G is an open set and that we are given $g, g' \in G$ such that $gg' \in O$. Select $U \in F$ that satisfies $gg'U \in O$. Then we have:

$$(gU)(g'U) = gg'(g'^{-1}Ug')U = gg'UU \subset gg'U \subset O$$

since U is a normal subgroup. Therefore, multiplication is continuous. If O is open and $g^{-1} \in O$, determine $U \in F$ satisfying $g^{-1}U \subset O$. We then have:

$$(gU)^{-1} = U^{-1}g^{-1} = Ug^{-1} = g^{-1}gUg^{-1} = g^{-1}U \subset O,$$

since U is a normal subgroup. Therefore, forming inverses is a continuous operation. □

Definition 9.3. Suppose that $K \subset L$ is a field extension.

(1) The Galois group of this field extension is defined as:

$$G(L/K) := \{\sigma \in Aut(L) \mid \forall x \in K : \sigma(x) = x\},$$

the set of field automorphisms that restrict to the identity in K.

(2) The extension $K \subset L$ is called Galois if the following properties hold:

(a) $K \subset L$ is algebraic, i. e. every element of L is the root of a polynomial with coefficients in K.

(b) $K \subset L$ is separable.

(c) $K \subset L$ is normal, i. e. every irreducible polynomial with coefficients in K that has a root in L already splits over L.

Theorem 9.4. *Suppose that $K \subset L$ is a Galois extension. If $E \subset L$ is a finite subset, then there is an intermediate field $K \subset P \subset L$ such that $E \subset P$ and $K \subset P$ is a finite Galois extension.*

Proof. If $E = \{e_1, \dots, e_n\}$, denote the minimum polynomials of $e_1, \dots, e_n$ by $p_1, \dots, p_n$. Define: $p = \prod_{i=1}^{n} p_i$. p splits over L. We denote the roots of p in L by $x_1, \dots, x_m$. $P := K[x_1, \dots, x_m]$ is a finite Galois extension that contains E. □

Definition 9.5. Suppose that $K \subset L$ is a Galois extension. The unique normaltopological topology on $G(L/K)$ for which

$$F := \{G(L/K) \mid K \subset P \subset L \text{ is an intermediate field such that } K \subset P \text{ is a finite Galois extension}\}$$

is a basis of the neighbourhood system of the identity is called the Krull topology on $G(L/K)$.

Remark 9.6. There are two points to note about this definition:

(1) F is a filter basis: If P and Q are intermediate fields that are finite dimensional Galois over K, we select K-bases $p_1, \dots, p_n$ of P and $q_1, \dots, q_m$ of Q. If R is an intermediate field which according to Proposition 9.4 is finite-dimensional Galois over K and contains $p_1, \dots, p_n$ and $q_1, \dots, q_m$, we have $P \subset R$, $Q \subset R$ and therefore $G(L/R) \subset G(L/P) \cap G(L/Q)$.

(2) If P is a finite Galois intermediate field, then $G(L/P)$ is a normal subgroup of $G(L/K)$: Suppose that $x \in P$ and $\sigma \in G(L/K)$. Denote by $p \in K[t]$ the minimum polynomial of x. Since $K \subset L$ is Galois, we see that $p \in P$ splits into linear factors. Denote by $x = x_1, \dots, x_n \in P$ the roots of p. Since $\sigma(x)$ is a root of p, we have $\sigma(x) \in P$. This proves $\sigma(P) \subset P$, and this implies easily that $\sigma \circ \tau \circ \sigma^{-1} \in G(L/P)$ if $\tau \in G(L/P)$.

Theorem 9.7. *Suppose that $K \subset L$ is a Galois extension. We regard L as a topological space with the discrete topology. Then the Krull topology on $G(L/K)$ coincides with the relative topology of the product topology on $L^L = Map(L, L)$.*

Proof. (Cf. Proposition 4.1) We prove first that the inclusion $\iota : G(L/K) \to L^L$ is continuous, where $G(L/K)$ is endowed with the Krull topology and L^L with the product topology. By the universal property of the product topology it is sufficient to prove that for every $x \in L$ the evaluation

$$G(L/K) \to L,\ \sigma \mapsto \sigma(x)$$

is continuous, which is equivalent to prove that the set

$$\{\sigma \in G(L/K) \mid \sigma(x) = y\}$$

is open. By Proposition 9.4 there is a intermediate field P which is finite-dimensional Galois over K and contains x. If we have $\sigma(x) = y$, we see that the elements of $\sigma G(L/K)$ also map x to y.

On the other hand, we have to show that ι is an open map onto $\iota(G(L/K))$. Suppose that P is an intermediate field which is finite-dimensional Galois over K, and select a K-basis $p_1, \dots, p_n$ of P. The assertion follows if we can prove that we have for $\sigma \in G(L/P)$:

$$\sigma G(L/P) = \{\tau \in G(L/K) \mid \forall i \leq n : \sigma(p_i) = \tau(p_i)\}$$

Namely, the left hand side is a typical open set in the Krull topology, and the right hand side is open in the relative topology. But the equality is obvious, because for $\tau \in G(L/K)$ we have:

$$\tau \in G(L/P) \Leftrightarrow \forall i \leq n : \tau(p_i) = p_i$$

□

Theorem 9.8. *Suppose that $K \subset L$ is a Galois extension.*

(1) *$G(L/K)$ is closed in L^L.*

(2) *$G(L/K)$ is projectively finite.*

Proof. (1) In a first step we prove that a field homomorphism which leaves K pointwise fixed is automatically surjective. If $x \in L$ is given, we know from Proposition 9.4 that there is an finite Galois intermediate field P that contains x. As we have proved in Remark 9.6, σ maps P to P. Since the restriction to P is a bijection, we see that x is contained in the image of σ.

(2) We now prove that $G(L/K)$ is closed. By the preceding step, it is possible to write $G(L/K)$ in the form:

$$\begin{aligned} G(L/K) &= \bigcap_{x,y\in L} \{\sigma \in L^L \mid \sigma(xy) - \sigma(x)\sigma(y) = 0\} \\ &= \bigcap_{x,y\in L} \{\sigma \in L^L \mid \sigma(x+y) - \sigma(x) - \sigma(y) = 0\} \\ &= \bigcap_{x\in K} \{\sigma \in L^L \mid \sigma(x) - x = 0\} \end{aligned}$$

This represents $G(L/K)$ as the intersection of closed sets. This is because for example the first set is the preimage of the closed set $\{0\}$ under the continuous mapping

$$G(L/K) \to L,\ \sigma \mapsto \sigma(xy) - \sigma(x)\sigma(y) = 0$$

(3) We prove that $G(L/K)$ is projectively finite. $G(L/K)$ is Hausdorff as a subset of a product of Hausdorff spaces (cf. [3], p. 92) and normaltopological by construction. It remains to show that $G(L/K)$ is compact. For $x \in X$, consider the orbit of x:

$$B_x := \{\sigma(x) \mid \sigma \in G(L/K)\}$$

We claim that the orbit is finite. Namely, if μ denotes the minimum polynomial of x and $x = x_1, \dots, x_n$ are the roots of μ, we see that $\sigma(x)$ is again a root of μ, and therefore $B_x \subset \{x_1, \dots, x_n\}$. Now we know by Tychonow's theorem (cf. [3], Chap. 5, Theorem 13, p. 143) that the product

$$\prod_{x\in L} B_x \subset \prod_{x\in L} L = L^L$$

is a compact set. Since the inclusion

$$G(L/K) \to \prod_{x\in L} B_x,\ \sigma \mapsto (\sigma(x))_{x\in L}$$

shows that $G(L/K)$ is a closed subset of a compact set, we see that $G(L/K)$ is itself compact.

□

Theorem 9.9. *Suppose that $K \subset L$ is a Galois extension and that $K \subset P \subset L$ is an intermediate field. Then $P \subset L$ is a Galois extension.*

Proof. It is obvious that L is algebraic over P. We prove that the extension $P \subset L$ is separable, i. e. that for every element $x \in L$ the minimum polynomial μ_P over P is separable. If μ_K denotes the minimum polynomial of x over K, then μ_K splits over L because the extension $K \subset L$ is normal:

$$\mu_K = \prod_{i=1}^{n}(t - x_i)$$

where $x = x_1$. Since the extension $K \subset L$ is separable, the roots $x_1, \dots, x_n$ of μ_K are all distinct. But since $\mu_P | \mu_K$, we see that the roots of μ_P form a subset of the roots of μ_K and therefore are distinct, too.

Now we prove that the extension $P \subset L$ is normal. Suppose that $p \in P[t]$ is an irreducible polynomial that has a root x in L. By Proposition 9.4 we know that there is a finite Galois intermediate field $K \subset Q \subset L$ that contains x as well as the coefficients of p. Define $R := P \cap Q$. From the Galois theory of finite field extensions we know that $R \subset Q$ is Galois. Since $p \in R[t]$ is still irreducible and has a root in Q, it splits over Q and in particular over L. □

Theorem 9.10. *Suppose that $K \subset L$ is a Galois extension. Suppose that $K \subset P \subset L$ is an intermediate field and that*

$$\sigma : P \to L$$

is a field homomorphism which leaves K pointwise fixed. Then σ can be extended to the whole field L, i. e. there is $\tau \in G(L/K)$ such that:

$$\forall x \in P : \sigma(x) = \tau(x)$$

Proof. We assume that this is known under the additional assumption that $K \subset L$ is finite (cf. [2], Theorem 7, p. 35). We look at the set:

$$\mathcal{M} := \{(Q, \rho) \mid P \subset Q \subset L \text{ is an intemediate field, } \rho : Q \to L \text{ is a field homomorphism with } \rho|_P = \sigma\}$$

$\mathcal{M}$ is partially ordered by the ordering:

$$(Q, \rho) \leq (Q', \rho') :\Leftrightarrow Q \subset Q' \text{ and } \rho'|_Q = \rho$$

If $((Q_i, \rho_i))_{i \in I}$ is a totally ordered family in $\mathcal{M}$, we see that $Q := \bigcup_{i \in I} Q_i$ is a subfield of L and the unique map $\rho : Q \to L$ that satisfies:

$$\forall i \in I \, \forall x \in Q_i : \rho(x) = \rho_i(x)$$

is a field homomorphism. We see that every totally ordered subfamily of $\mathcal{M}$ has an upper bound. Therefore we can conclude by Zorn's Lemma that $\mathcal{M}$ has a maximal element (R, τ).

We claim that $R = L$. Assume this is false. Select $x \in L \setminus R$. Denote the minimum polynomial of x over K by μ, and let $x = x_1, \dots, x_n$ be the roots of μ. If $S = \tau(R)$, then $R[x_1, \dots, x_n]$ is a splitting field of μ if μ is regarded as a polynomial with coefficients in R, and $S[x_1, \dots, x_n]$ is a splitting field of μ if μ is regarded as a polynomial with coefficients in S. By the isomorphism theorem for splitting fields (cf. [2], loc. cit.), there is an isomorphism

$$\omega : R[x_1, \dots, x_n] \to S[x_1, \dots, x_n]$$

that extends τ. This contradicts the maximality of (R, τ). $\square$

Theorem 9.11. *(Fundamental theorem of infinite Galois theory) Suppose that $K \subset L$ is a Galois extension. Define:*

$$\mathcal{G} := \{H \subset G(L/K) \mid H \textit{ is a closed subgroup}\}$$
$$\mathcal{K} := \{P \mid K \subset P \subset L \textit{ is an intermediate field}\}$$

Then the mappings

$$A : \mathcal{K} \to \mathcal{G},\ P \mapsto A(P) := G(L/P)$$
$$I : \mathcal{G} \to \mathcal{K},\ H \mapsto Fix(H) := \{x \in L \mid \forall \sigma \in H : \sigma(x) = x\}$$

are bijections that are mutually inverse.

Proof. (1) We know from Proposition 9.9 that, for any intermediate field $K \subset P \subset L$, the extension $P \subset L$ is again Galois. Therefore, we can conclude by Theorem 9.8 that $G(L/P)$ is closed in L^L and therefore in $G(L/K)$. This means that the mapping A is well defined.

(2) We prove that $I \circ A = id_{\mathcal{K}}$. It is obvious that we have $P \subset I(A(P))$. Suppose that there is $x \in I(A(P)) \setminus P$. By Proposition 9.4 we can find an intermediate field $P \subset U \subset L$ such that $P \subset U$ is a finite Galois extension and $x \in U$. By the fundamental theorem of finite Galois theory we conclude that there is $\sigma \in G(U/P)$ that satisfies $\sigma(x) \neq x$. By the extension theorem 9.10 there is $\tau \in G(L/P)$ that satisfies $\tau|_U = \sigma$, and therefore $\tau(x) \neq x$.

(3) Finally, we prove that $A \circ I = id_{\mathcal{G}}$. Suppose that $H \subset G(L/K)$ is a closed subgroup and define $P := I(H)$. It is clear that $H \subset A(P)$. If $\sigma \in A(P)$, we have to prove that $\sigma \in H$. Since H is

closed, it is sufficient to prove that $\sigma \in \bar{H}$. That means that we have to prove that $\sigma G(L/Q) \cap H \neq \emptyset$ for every intermediate field $K \subset Q \subset L$ which is finite Galois over K. From Theorem 9.4 we conclude that there is an intermediate field $P \subset R \subset L$ which is finite Galois over P, since by Proposition 9.9 the extension $P \subset L$ is itself Galois. We prove the stronger assertion that $\sigma G(L/R) \cap H \neq \emptyset$.

As in Remark 9.6 we can see that $\tau \in G(L/P)$ maps R to R. Therefore we have a group homomorphism

$$H \to G(R/P),\ \tau \mapsto t|_R$$

The image of this homomorphism is a subgroup $U \subset G(R/P)$ whose field of fixpoints is P by the very definition of P. By the fundamental theorem of finite Galois theory, this implies that $U = G(R/P)$, which means that the restriction homomorphism is surjective, which in turn means that every automorphism in $G(R/P)$ can be extended to a homomorphism contained in H. In particular, this applies to $\sigma|_R$, that is, we have $\tau \in H$ that satisfies:

$$\forall z \in R : \sigma(x) = \tau(x)$$

which means $\sigma^{-1} \circ \tau \in G(L/R)$ and therefore $\tau = \sigma(\sigma^{-1}\tau) \in \sigma G(L/R) \cap H$.

□

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