CHAPTER 7

Wess–Zumino–Witten Model

In this chapter, we give a construction of what is probably the best known example of a modular functor. This modular functor is based on the category of integrable representations of an affine Lie algebra and appears naturally in the Wess–Zumino–Witten model of conformal field theory; abusing the language, we will call it the *WZW modular functor*. The literature devoted to this measures in hundreds of papers; the most prominent among them are [**KZ**], [**MS1**], [**TUY**], [**BFM**]. For more detailed exposition of conformal field theory in general and WZW model in particular, we refer the reader to [**FMS**] and references therein.

The main goal of this chapter is to prove the following result. Fix a simple complex Lie algebra $\mathfrak{g}$, and let $\mathcal{O}_k^{\text{int}}$ be the category of integrable modules of level $k \in \mathbb{Z}_+$ over the corresponding affine Lie algebra $\widehat{\mathfrak{g}}$.

THEOREM 7.0.1. *The category $\mathcal{O}_k^{\text{int}}$ has a structure of a modular tensor category.*

Of course, in this form the theorem is not very precise since we have not defined the tensor product (which is usually called the *fusion product*, and denoted $\dot{\otimes}$, to distinguish it from the usual tensor product of vector spaces). We will give a precise definition later (see Corollary 7.9.11).

Another important result, which, unfortunately, we will not prove, is the following. Recall that in Section 3.3 we defined a structure of a modular tensor category on a certain subquotient $\mathcal{C}^{\text{int}}(\mathfrak{g}, \varkappa)$ of the category of representations of the quantum group $U_q\mathfrak{g}$, $q = e^{\pi \mathrm{i}/m\varkappa}$.

THEOREM 7.0.2 ([**F**]). *The category $\mathcal{O}_k^{\text{int}}$ is equivalent to the category $\mathcal{C}^{\text{int}}(\mathfrak{g}, \varkappa)$ as a modular tensor category for $\varkappa = k + h^\vee$, where $h^\vee$ is the dual Coxeter number for $\mathfrak{g}$.*

Because of the importance of these two theorems, we will comment here on their history. They have appeared in somewhat vague form in physics literature in the 1980s. The accurate construction of the tensor structure on $\mathcal{O}_k^{\text{int}}$ first appeared in [**MS1**]; however, Moore and Seiberg did not give a complete proof.

To the best of our knowledge, there are three known proofs of Theorem 7.0.1. The first one, which we present in this chapter, is based on the use of the notion of modular functor. The corresponding modular functor (which, as we mentioned above, naturally appears in the Wess–Zumino–Witten model of conformal field theory) is defined in terms of the spaces of coinvariants. The crucial step in proving that these spaces satisfy the axioms of a modular functor is checking the gluing axiom, which was done by Tsuchiya, Ueno, and Yamada [**TUY**]. Another proof of the gluing axiom can be obtained by suitably modifying the proof for the minimal models given in [**BFM**].

The second proof of Theorem 7.0.1 was given by Finkelberg [**F**], who based his approach on the series of papers of Kazhdan and Lusztig [**KL**]. They proved that for negative integer level k, the category $\mathcal{O}_k$ is a ribbon category, which is equivalent to the category $\mathcal{C}(\mathfrak{g}, \varkappa)$ of representations of the quantum group $U_q\mathfrak{g}$. Therefore, this category contains a subquotient category which is equivalent to the MTC $\mathcal{C}^{\mathrm{int}}(\mathfrak{g}, \varkappa)$. Combining this result with a certain duality between the categories $\mathcal{O}_k$ and $\mathcal{O}_{-2h^\vee-k}$, Finkelberg showed that this subquotient is dual to the category $\mathcal{O}_k^{\mathrm{int}}$, thus establishing simultaneously Theorems 7.0.1 and 7.0.2.

Finally, the third proof of Theorem 7.0.1, based on the theory of vertex operator algebras, was recently given by Huang and Lepowsky [**HL**].

Unfortunately, none of these proofs is easy. Finkelberg's proof is based on a 250 pages long series of papers [**KL**], which is very tersely written; few people (if any at all) have expertise and patience to follow all the details of this proof. Similarly, the proof of Huang and Lepowsky is heavily based on a number of their previous papers on vertex operator algebras, which can sometimes get rather technical. The modular functor approach seems to be the easiest of all three, but it still requires all the formalism of modular functors and their relation with tensor categories (which took the previous 140 pages of this book) and some non-trivial algebraic geometry used in [**TUY**], also not an easy reading.

The proof given in this chapter is based on the modular functor approach; however, our proof of the gluing axiom follows the ideas of [**BFM**] rather than [**TUY**]. This proof was never published before; however, for the most part it closely follows the arguments in [**BFM**], so all the credit belongs to Beilinson, Feigin, and Mazur. Modifying their arguments for WZW model was rather straightforward; according to private communications from Beilinson and Feigin, they intended to include the proof for WZW model in the final version of the manuscript. Unfortunately, it is not clear when (and if) such a final version appears, so we include this proof here.

7.1. Preliminaries on affine Lie algebras

The aim of this subsection is just to fix the notation, we refer to the book of Kac [**K1**] for a comprehensive treatment.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$. We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let $\langle \cdot, \cdot \rangle$ be an invariant bilinear form on $\mathfrak{g}$ normalized so that $\langle \alpha, \alpha \rangle = 2$ for long roots of $\mathfrak{g}$. We will use the same notations (and notions) as in Section 1.3.

Let $\mathfrak{g}((t)) \equiv \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((t))$ be the *loop algebra* of $\mathfrak{g}$. Then the *affine Lie algebra* of $\mathfrak{g}$ is

$$\widehat{\mathfrak{g}} = \mathfrak{g}((t)) \oplus \mathbb{C}K \tag{7.1.1}$$

with commutation relations

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg + \langle a, b \rangle \operatorname{Res}_0(df\, g)K, \qquad [K, \widehat{\mathfrak{g}}] = 0.$$

For brevity, we often use the notation $x[n] = x \otimes t^n, x \in \mathfrak{g}$.

We let $\widehat{\mathfrak{g}}^+ = t\mathfrak{g}[[t]]$, $\widehat{\mathfrak{g}}^- = t^{-1}\mathfrak{g}[t^{-1}]$. We have a decomposition of $\widehat{\mathfrak{g}}$ into subalgebras

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}^+ \oplus \mathfrak{g} \oplus \mathbb{C}K \oplus \widehat{\mathfrak{g}}^-.$$

We will be interested in $\widehat{\mathfrak{g}}$-modules of level $k \in \mathbb{C}$, i.e., modules V such that $K|_V = k\,\mathrm{id}_V$; this is equivalent to considering modules over $U(\widehat{\mathfrak{g}})_k = U\widehat{\mathfrak{g}}/U\widehat{\mathfrak{g}}(K-k)$.

We will denote by $\mathcal{O}_k$ the category of $\widehat{\mathfrak{g}}$-modules of level k which have weight decomposition with finite-dimensional weight subspaces, such that the action of $\widehat{\mathfrak{g}}^+$ is locally nilpotent and the action of $\mathfrak{g}$ is integrable.

Of special interest for us are two classes of modules from $\mathcal{O}_k$: Weyl modules and integrable modules. Weyl module $V_\lambda^k, \lambda \in P_+$, is defined by

$$V_\lambda^k = \operatorname{Ind}_{\mathfrak{g}\oplus\widehat{\mathfrak{g}}^+\oplus\mathbb{C}K}^{\widehat{\mathfrak{g}}} V_\lambda, \tag{7.1.2}$$

where V_λ is the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight λ, which we consider as a module over $\mathfrak{g}\oplus\widehat{\mathfrak{g}}^+\oplus\mathbb{C}K$ by letting $\widehat{\mathfrak{g}}^+$ act as 0 and K act as $k\,\mathrm{id}$. The Weyl module is free over $\widehat{\mathfrak{g}}^-$.

If $k \notin \mathbb{Q}$, then Weyl modules are irreducible and the category $\mathcal{O}_k$ is semisimple. We will be mostly interested in the other extreme case $k \in \mathbb{Z}_+$. In this case, we can also consider integrable highest-weight modules. We will denote by $\mathcal{O}_k^{\mathrm{int}} \subset \mathcal{O}_k$ the subcategory of integrable modules, i.e., such modules that for every root α, $n \in \mathbb{Z}$, the action of $e_\alpha[n]$ is locally nilpotent. It is known that $\mathcal{O}_k^{\mathrm{int}}$ is semisimple with simple objects $L_\lambda^k, \lambda \in P_+^k$, where P_+^k is the *positive Weyl alcove*

$$P_+^k = \{\lambda \in P_+ \mid (\lambda, \theta^\vee) \le k\}, \tag{7.1.3}$$

see [**K1**]. (Note that P_+^k is the same set which we denoted by C in Section 3.3.) The modules L_λ^k are irreducible and can be described as the quotient $L_\lambda^k = V_\lambda^k/Z_\lambda$, where Z_λ is the unique maximal proper submodule of V_λ^k. It is known that Z_λ is generated by one vector: $Z_\lambda = U\widehat{\mathfrak{g}}(e_\theta[-1])^{a+1}v_{\lambda,k}$, where $a = k - (\lambda, \theta^\vee)$.

It is useful to note that both V_λ^k and L_λ^k have a natural $\mathbb{Z}_-$-grading (sometimes called the *homogeneous grading*), defined by $\deg v_{\lambda,k} = 0, \deg a[n] = n, a \in \mathfrak{g}, n \in \mathbb{Z}$. It is easy to see that homogeneous components of V_λ^k (and, in fact, any module in the category $\mathcal{O}_k$) are finite-dimensional.

Finally, we will define the duality in the category $\mathcal{O}_k$ by $DV = (V^*)^\natural$, where V^* is the restricted dual to V, i.e. the direct sum of the dual spaces to homogeneous components of V, and $\natural$ is defined as follows: for a $\widehat{\mathfrak{g}}$ module W, the module $W^\natural$ coincides with W as a vector space, and the action of $\widehat{\mathfrak{g}}$ is twisted by the automorphism

$$\natural\colon x[n] \mapsto (-1)^n x[-n], \qquad K \mapsto -K. \tag{7.1.4}$$

It is easy to see that D is an anti-automorphism of the category $\mathcal{O}_k$ which preserves $\mathcal{O}_k^{\mathrm{int}}$. In particular, for an integrable module L_λ^k, DL_λ^k is also an irreducible integrable module, whose top homogeneous component is V_λ^*. It is (non-canonically) isomorphic to $L_{-w_0(\lambda)}^k$.

7.2. Reminders from algebraic geometry

In this section we briefly list some facts from algebraic geometry which will be used below. All of them are quite standard, so a reader who has even basic knowledge of algebraic geometry over $\mathbb{C}$ can safely skip this section.

All varieties considered in this section are considered with analytic topology; as before, we use the words "manifold" and "non-singular variety" as synonyms. For a variety S, we denote by $\mathcal{O}_S$ the structure sheaf of S (i.e., the sheaf of analytic functions on S). We assume that the reader is familiar with the notion of a $\mathcal{O}$-module and a coherent $\mathcal{O}$-module. As usual, for a point $s \in S$ we define by $\mathcal{O}_{S,s}$ the local ring at s, i.e. the ring of germs of analytic functions at s, and by m_s the maximal ideal of this ring, which consists of functions vanishing at s. We also denote by $\widehat{\mathcal{O}}_{S,s}$ the completion of the local ring with respect to topology given by the powers of the maximal ideal. In particular, if $\dim S = 1$, $s \in S$ is a regular point, and t is a *local parameter at* s, i.e., an analytic function in a neighborhood of s such that $t(s) = 0, (dt)_s \neq 0$, then $\widehat{\mathcal{O}}_{S,s} \simeq \mathbb{C}[[t]]$.

For an $\mathcal{O}_S$-module $\mathcal{F}$ we define its fiber at point $s \in S$ to be $\mathcal{F}_s/m_s\mathcal{F}_s$. In particular, if $\mathcal{F}$ is the sheaf of sections of a vector bundle F, then in this way one recovers the fibers of F. We will say that an $\mathcal{O}$-module $\mathcal{F}$ is *lisse* if it is the sheaf of sections of a finite-dimensional vector bundle. Note that every lisse sheaf is coherent, but converse is not true.

In general, for an open subset $U \subset S$ and a sheaf $\mathcal{F}$ on S, we denote by $\mathcal{F}(U)$ the vector space of sections of $\mathcal{F}$ over U. However, in the case when $U = C \setminus D$, where C is compact and D is a divisor, and $\mathcal{F}$—an $\mathcal{O}$-module over C, we will denote by $\mathcal{F}(C - D)$ the space of *meromorphic* sections of $\mathcal{F}$ over C which are regular outside of D. We hope it won't cause confusion.

We will use the following well known facts about complex curves. As before, all the curves are assumed to be compact and non-singular (unless specified otherwise), but not necesarily connected.

THEOREM 7.2.1 (Riemann-Roch). *Let C be a connected complex curve, and $p_1, \dots, p_n, q$—distinct points of C ($n \geq 0$). Let us fix the principal parts of Laurent expansions $(f)_i \in \mathbb{C}((t_i))/\mathbb{C}[[t]]$ near p_i. Then there exists a function $f \in \mathcal{O}(C - \{p_1, \dots, p_n, q\})$ which has given principal parts of Laurent expansion at p_i and has a pole at q. Moreover, the order of pole at q can be bounded by a constant which only depends on the order of poles at p_i and the genus of the curve C.*

This theorem can be generalized to curve which may have ordinary double point singularities and may be disconnected. In this case, we have to allow poles at a collection of points $q_1, \dots, q_m$ such that on every component of C there is at least one of the points q_i.

THEOREM 7.2.2. *Let C be a complex curve (possibly disconnected and singular). Let $q \in C$ be a regular point, and t—a local parameter at q. Then the vector space*

$$\mathbb{C}((t))/\mathbb{C}[[t]] + \mathcal{O}(C - q),$$

is finite dimensional. Moreover, there exists $N \in \mathbb{Z}_+$ which only depends on the topology of C such that

$$\mathcal{O}(C - q) + \mathbb{C}[[t]] \supset t^{-N}\mathbb{C}[t^{-1}] + \mathbb{C}[[t]].$$

7.3. Conformal blocks: definition

In this section, we will define the vector spaces of coinvariants; later we will show that these vector spaces satisfy the axioms of a modular functor. The basic references for this section are [**TUY**], [**Be**] (with minor changes).

Fix a compact nonsingular complex curve C (not necessarily connected), a finite dimensional simple Lie algebra $\mathfrak{g}$, and a positive integer k.

Let $p_1, \dots, p_n$ be distinct points on C with local coordinates $t_1, \dots, t_n$ (recall that a local coordinate at a point p is a holomorphic function t in a neighborhood of p such that $t(p) = 0, (dt)_p \neq 0$). We will always assume that on every connected component of C there is at least one point. Let $V_1, \dots, V_n \in \mathcal{O}_k$ be some $\widehat{\mathfrak{g}}$-modules associated to these points.

We will use the notations

$$\vec{p} = (p_1, \dots, p_n),$$
$$V = V_1 \otimes \dots \otimes V_n.$$

In particular, if $V_i = L^k_{\lambda_i}$ are integrable modules, we will use the notation

$$\vec{\lambda} = (\lambda_1, \dots, \lambda_n), \quad L^k_{\vec{\lambda}} = L^k_{\lambda_1} \otimes \dots \otimes L^k_{\lambda_n}.$$

Consider the Lie algebra

$$\mathfrak{g}(C - \vec{p}) = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}(C - \vec{p}) \tag{7.3.1}$$

of $\mathfrak{g}$-valued functions on C which are regular outside the points $p_1, \dots, p_n$ and meromorphic at these points. We have Lie algebra homomorphisms

$$\gamma_i \colon \mathfrak{g}(C - \vec{p}) \to \mathfrak{g}((t))$$

given by Laurent expansion around the point p_i in the local coordinate t_i. This does not give a Lie algebra homomorphism $\mathfrak{g}(C - \vec{p}) \to \widehat{\mathfrak{g}}$ because of the central term in definition of $\widehat{\mathfrak{g}}$. However, by the Residue Theorem,

$$\vec{\gamma} = \gamma_1 \oplus \dots \oplus \gamma_n \colon \mathfrak{g}(C - \vec{p}) \to \mathfrak{g}((t)) \oplus \dots \oplus \mathfrak{g}((t))$$

can be lifted to a homomorphism

$$\vec{\gamma} \colon \mathfrak{g}(C - \vec{p}) \to U(\widehat{\mathfrak{g}})_k \otimes \dots \otimes U(\widehat{\mathfrak{g}})_k, \quad \vec{\gamma}(x) = \sum_{i=1}^{n} 1 \otimes \dots \otimes \gamma_i(x) \otimes \dots \otimes 1.$$

In particular, $\mathfrak{g}(C - \vec{p})$ acts on V.

DEFINITION 7.3.1. The space of *conformal blocks* is the vector space of coinvariants

$$\tau(C, \vec{p}, V) := V_{\mathfrak{g}(C - \vec{p})} = V / \mathfrak{g}(C - \vec{p}) V.$$

We will write $\tau(C, \vec{p}, \vec{t}, V)$ when we need to show the dependence on the choice of local parameters $\vec{t} = (t_1, \dots, t_n)$.

It is easy to see that the construction above also makes perfect sense if we allow t_i be *formal* local parameters at p_i, i.e., $t_i \in \widehat{\mathcal{O}}_{p_i}, (dt_i)_{p_i} \neq 0$. Note that once t_i is chosen, one has $\widehat{\mathcal{O}}_{p_i} = \mathbb{C}[[t_i]]$.

LEMMA 7.3.2 (Beauville [**Be**]). *Let $\vec{p}$, V be as above, and let $q \in C - \vec{p}$, $\lambda \in P^k_+$. As before, let V_λ be the corresponding finite-dimensional $\mathfrak{g}$-module, and let V^k_λ be the Weyl module over $\widehat{\mathfrak{g}}$. Then the inclusion $V_\lambda \hookrightarrow V^k_\lambda$ induces an isomorphism*

$$(V \otimes V_\lambda)_{\mathfrak{g}(C - \vec{p})} \xrightarrow{\sim} (V \otimes V^k_\lambda)_{\mathfrak{g}(C - \vec{p} - q)} = \tau(C, \vec{p} \cup q, V \otimes V^k_\lambda), \tag{7.3.2}$$

where $\mathfrak{g}(C - \vec{p})$ acts on V_λ via the evaluation map $a \otimes f \mapsto f(q)a$, $a \in \mathfrak{g}$, $f \in \mathcal{O}(C - \vec{p})$.

PROOF. Since the natural embedding $V \otimes V_\lambda \hookrightarrow V \otimes V^k_\lambda$ is clearly $\mathfrak{g}(C - \vec{p})$ equivariant, it induces a map from the left hand side of (7.3.2) to the right hand side.

By the Riemann–Roch formula, there exists a function z on C regular outside $\vec{p} \cup q$ and having a simple pole at the point q. Then

$$\mathcal{O}(C - \vec{p} - q) = \mathcal{O}(C - \vec{p}) \oplus \bigoplus_{i=1}^{\infty} \mathbb{C} z^{-i},$$

therefore $\mathfrak{g}(C-\vec{p}-q)\simeq\mathfrak{g}(C-\vec{p})\oplus\widehat{\mathfrak{g}}^-$.

By definition, V^k_λ is a free $U(\widehat{\mathfrak{g}}^-)$-module isomorphic to $U(\widehat{\mathfrak{g}}^-)V_\lambda$; hence, $V_\lambda\simeq(V^k_\lambda)_{\widehat{\mathfrak{g}}^-}$. Then (7.3.2) follows by tensoring with V and taking coinvariants with respect to $\mathfrak{g}(C-\vec{p})$. □

LEMMA 7.3.3. *Let C be connected, and let V_i be quotients of Weyl modules: $V_i=V^k_{\lambda_i}/I_i$ (the ideals I_i may be zero, maximal, or anything in between). Assume also that at least one of V_i is integrable, i.e., equal to $L^k_{\lambda_i}$. Then the natural surjection $V=V_1\otimes\cdots\otimes V_n\twoheadrightarrow L^k_{\lambda_1}\otimes\cdots\otimes L^k_{\lambda_n}=L^k_{\vec{\lambda}}$ gives rise to an isomorphism*

$$\tau(C,\vec{p},V)\xrightarrow{\sim}\tau(C,\vec{p},L^k_{\vec{\lambda}}). \tag{7.3.3}$$

PROOF. It suffices to prove that

$$(L^k_{\lambda_1}\otimes V_2\otimes\cdots\otimes V_{n-1}\otimes V^k_{\lambda_n})_{\mathfrak{g}(C-\vec{p})}=(L^k_{\lambda_1}\otimes V_2\otimes\cdots\otimes V_{n-1}\otimes L^k_{\lambda_n})_{\mathfrak{g}(C-\vec{p})}.$$

Let $Z=\{v\in V^k_{\lambda_n}\mid L_{\lambda_1}\otimes\cdots\otimes V_{n-1}\otimes v\subset \operatorname{Im}\mathfrak{g}(C-\vec{p})\}$. Obviously, this is a submodule in $V^k_{\lambda_n}$; our goal is to prove that $V^k_{\lambda_n}/Z$ is integrable. This is equivalent to the following statement: for every root α and $v\in V^k_{\lambda_n}$, one has $(e_\alpha[-1])^Nv\in Z$ for $N\gg 0$ (in fact, it suffices to check this for $\alpha=\theta$). We leave it to the reader to check that if we choose $f\in\mathbb{C}((t))$ such that f has first order pole at 0, then the above condition is equivalent to $(e_\alpha f)^Nv\in Z$ for $N\gg 0$ (in other words, the notion of an integrable module does not depend on the choice of local parameter).

Now let $f\in\mathcal{O}(C-p_1-p_n)$ be a function which has a first order pole at p_n. By the Riemann–Roch theorem, such a function exists if we allow it to have a pole of sufficiently high order at p_1. Since $L^k_{\lambda_1}$ is integrable, and f is regular at $p_2,\dots,p_{n-1}$, we easily see that action of $e_\alpha f$ on $L^k_{\lambda_1}\otimes\cdots\otimes V_{n-1}$ is locally nilpotent. Therefore, for any $v_1\in L^k_{\lambda_1},\dots,v_n\in V^k_{\lambda_n}$, one has $v_1\otimes\cdots\otimes v_{n-1}\otimes(e_\alpha f)^Nv_n\in\operatorname{Im}\mathfrak{g}(C-\vec{p})$. But this exactly means that $(e_\alpha f)^Nv_n\in Z$ for $N\gg 0$. □

This theorem can be rewritten in more invariant terms. For a module $V\in\mathcal{O}_k$, denote by V^{int} its maximal integrable quotient (it is easy to see that it is well-defined). Then the previous lemma immediately implies the following corollary.

COROLLARY 7.3.4. *Let $V_i\in\mathcal{O}^I_kNT$, and at least one of V_i is integrable. Then*

$$\tau(C,\vec{p},V_1\otimes\cdots\otimes V_n)=\tau(C,\vec{p},V_1^{\mathrm{int}}\otimes\cdots\otimes V_n^{\mathrm{int}}).$$

COROLLARY 7.3.5. *Let $V=V_1\cdots\otimes \mathtt{V_n},\mathtt{V_i}\in\mathtt{O}^{\mathrm{int}}_{\mathtt{k}}$. Then the embedding $\mathbb{C}=V_0\hookrightarrow L^k_0$ induces an isomorphism*

$$\tau(C,\vec{p},V)\simeq\tau(C,\vec{p}\cup q,V\otimes L^k_0). \tag{7.3.4}$$

PROOF. This follows from Lemmas 7.3.2 and 7.3.3:

$$(V\otimes L^k_0)_{\mathfrak{g}(C-\vec{p}-q)}\simeq(V\otimes V^k_0)_{\mathfrak{g}(C-\vec{p}-q)}\simeq(V\otimes\mathbb{C})_{\mathfrak{g}(C-\vec{p})}.$$

□

Having proved these results, we can prove now the following proposition.

PROPOSITION 7.3.6. *If $V=V_1\cdots\otimes\mathtt{V_n},\mathtt{V_i}\in\mathtt{O}^{\mathrm{int}}_{\mathtt{k}}$, then the spaces of coinvariants $\tau(C,\vec{p},V)$ are finite dimensional.*

PROOF. We may assume that C is connected. Combining Lemma 7.3.2 and 7.3.3, we see that it suffices to prove the statement for $n = 1, V_1 = L^k_\lambda$. It follows from Theorem 7.2.2 that $\widehat{\mathfrak{g}}^+ + \mathfrak{g}(C-p) \supset \widehat{\mathfrak{g}}^+ + t^{-N}\widehat{\mathfrak{g}}^-$ for $N \gg 0$. Therefore, it suffices to prove that the vector space

$$W_N = L^k_\lambda / t^{-N}\widehat{\mathfrak{g}}^- V_\lambda$$

is finite-dimensional.

To prove this, note that one has a well-defined action of $\widehat{\mathfrak{g}}^{\le 0} = \mathfrak{g}[t^{-1}]$ on W_N, which factors through the finite-dimensional quotient $\mathfrak{a} = \widehat{\mathfrak{g}}^{\le 0}/t^{-N}\widehat{\mathfrak{g}}^{\le 0}$. Obviously, $W_N = (U\mathfrak{a})v_{\lambda,k}$. On the other hand, $\mathfrak{a}$ is generated by $e_\alpha, f_\alpha,\ e_\alpha t^{-1}$, and all of these generators act nilpotently on W_N. Thus, all we need is to prove the following lemma.

LEMMA 7.3.7. *If $\mathfrak{a}$ is a finite-dimensional Lie algebra with generators $x_1, \dots, x_n$, and W is a cyclic $\mathfrak{a}$-module such that the action of x_i in W is locally nilpotent, then W is finite-dimensional.*

To prove this lemma, we pass from the module W over $U\mathfrak{a}$ to the corresponding graded module $Gr\, W$ over $Gr(U\mathfrak{a}) = S(\mathfrak{a})$. Consider the variety $S = Supp(Gr\, W) \subset \mathfrak{a}^*$. Then it follows from the nilpotency condition that x_i, considered as a function on $\mathfrak{a}^*$, vanishes on S. By Gabber's integrability theorem [**Gab**], if x, y vanish on S, then $[x, y]$ also vanishes. Therefore, $S = \{0\}$. But every finitely generated module over the polynomial ring, which has a finite support, is finite-dimensional. This proves the lemma, and thus, the proposition. □

As an illustration, consider the simplest case $C = \mathbb{P}^1$.

PROPOSITION 7.3.8. *Let $C = \mathbb{P}^1$, $p_1, \dots, p_n$—distinct points on C.*
(i) *Let $V^k_{\vec\lambda} = V^k_{\lambda_1} \otimes \dots \otimes V^k_{\lambda_n}$, and $V_{\vec\lambda} = V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}$. Then the homomorphism*

$$(V_{\vec\lambda})_{\mathfrak{g}} \to \tau(C, \vec p, V^k_{\vec\lambda})$$

obtained by restricting the natural map $V^k_{\vec\lambda} \to V^k_{\vec\lambda}/\mathfrak{g}(C - \vec p)V^k_{\vec\lambda}$, is an isomorphism.

(ii) *Let z be a global coordinate on $\mathbb{P}^1$; assume that $z(p_i)$ is finite. Define the endomorphism $T\colon V_{\vec\lambda} \to V_{\vec\lambda}$ by*

$$T(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^n v_1 \otimes \cdots \otimes z(p_i)\, e_\theta v_i \otimes \cdots \otimes v_n$$

Then one has an isomorphism

$$(V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n})_{\mathfrak{g} \oplus \mathbb{C}T^{k+1}} \simeq \tau(\mathbb{P}^1, \vec p, L^k_{\vec\lambda}).$$

PROOF. Part (i) is proved in the same way as Lemma 7.3.2, if we also note that for one point, $\mathfrak{g}(\mathbb{P}^1 - p) = \mathfrak{g} \oplus \widehat{\mathfrak{g}}^-$. As for part (ii), it can be deduced from the fact that $L^k_\lambda = V^k_\lambda / U\widehat{\mathfrak{g}}(e_\theta[-1])^{a+1}v_{\lambda,k}$. □

Let us relate this description with the one usually given in the physics literature. As before, let $C = \mathbb{P}^1$ with global coordinate z, and let the marked points be $0, z_1, \dots, z_n, \infty$ with the local parameters $z, z - z_i, -1/z$ respectively. Let us assign to the points 0 and ∞ some $\mathcal{O}_k$-modules V_0, V_∞ respectively and assign to the points $z_1, \dots, z_n$ Weyl modules $V^k_{\lambda_1}, \dots, V^k_{\lambda_n}$. Then, by Lemma 7.3.2, we can replace in the definition of coinvariants $V^k_{\lambda_i}$ by V_{λ_i} and the algebra $\mathfrak{g}(\mathbb{P}^1 - \{0, z_i, \infty\})$ by $\mathfrak{g}(\mathbb{P}^1 - \{0,\infty\}) = \mathfrak{g}[z, z^{-1}]$. Thus

$$\begin{aligned}(7.3.5)\quad &\tau(\mathbb{P}^1, 0, z_1, \dots, z_n, \infty, V_0, \dots, V_\infty)\\ &= (V_0 \otimes V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n} \otimes V_\infty)/\big((x[n])_0 + \sum z_i^n x_i + (-1)^n (x[-n])_\infty\big)\end{aligned}$$

where $n \in \mathbb{Z}, x \in \mathfrak{g}$, and notation x_i means x acting on V_{λ_i}, etc. We can pass to the dual space τ^* which will be a subspace in

$$\mathrm{Hom}_{\mathbb{C}}(V_0 \otimes V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n} \otimes V_\infty, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V_0 \otimes V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}, \widehat{DV_\infty})$$

where $\widehat{W}$ is the completion of a $W \in \mathcal{O}_k$ with respect to the homogeneous grading. Rewriting the coinvariance condition, we get

$$(7.3.6)\quad \begin{aligned}\tau^* &= \{\Phi : V_0 \otimes V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n} \to \widehat{DV_\infty} \mid \Phi(x[n] + \sum z_i^n x_i) = x[n]\Phi\}\\ &= \mathrm{Hom}_{\mathfrak{g}[t,t^{-1}]}(V_0 \otimes V_{\lambda_1}(z_1) \otimes \dots V_{\lambda_n}(z_n), \widehat{DV_\infty}),\end{aligned}$$

where, as before, $V(z)$ is the evaluation representation.

For the case $\mathfrak{g} = \mathfrak{sl}_2, n = 1$ the dimensions of these spaces (which, as we will show below, play the role of multiplicity coefficients N^k_{ij} for the modular category $\mathcal{O}^{\mathrm{int}}_k$) were calculated in [**TK**]; their answer agrees with the formula for $U_q(\mathfrak{sl}_2), q = e^{\pi \mathrm{i}/(k+2)}$ given in (3.3.24)—as expected from Theorem 7.0.2.

REMARK 7.3.9. It is a natural question to generalize the definition of coinvariants, which can be viewed as Lie algebra homology in degree zero $H_0(\mathfrak{g}(C - \vec{p}), V)$ and consider all homology spaces $H_*(\mathfrak{g}(C-\vec{p}), V)$. To the best of out knowledge, this approach was first suggested by B. Feigin. One of the first results in this direction, proved in [**Tel**], is the vanishing theorem: if V_i are Weyl modules, then all higher homology vanish. In particular, this theorem allows one to calculate dimensions of the vector spaces of coinvariants $\tau(C, \vec{p}, L^k_{\vec{\lambda}})$, by writing for each of $L^k_{\lambda_i}$ a resolution consisting of Weyl modules, and then using the fact that for the Weyl modules, dimension of the space of coinvariants is known (see Lemma 7.3.8). This answer coincides with the dimension of the spaces of homomorphisms in the category of representations of quantum group at root of unity (see Proposition 3.3.23).

The meaning of the higher homology spaces ("higher conformal blocks") $H_i(\mathfrak{g}(C-\vec{p}), V)$ when V_i are integrable and the role they play in conformal field theory is still unclear.

7.4. Flat connection

In the previous section, we have defined and studied some properties of the vector spaces of coinvariants for a given curve C with marked points and chosen local parameters at these points. Now, let us study what happens with these spaces when we change the local parameters, or move the points. Let us assume that we have a smooth family of pointed curves $C_s, s \in S$ over a smooth base S. As mentioned above, it means that we have a smooth manifold C_S with a proper flat smooth morphism $\pi : C_S \to S$ such that each fiber $C_s = \pi^{-1}(s)$ is a complex curve; we also have n non-intersecting sections $p_i : S \to C_S$, and local parameters t_i, which are functions in a neighborhood of $p_i(S) \subset C_S$ such that $p_i(S)$ is the zero locus of t_i, and $dt_i \neq 0$ on $p_i(S)$. Such a data defines on each fiber a structure of a pointed complex curve, with a local parameter at each puncture; as before, we will assume taht on each connected component of C_s there is at least one marked point. Similarly to the construction of the previous section, it is convenient to allow t_i to be formal parameter, i.e. an element of the completed local ring $\widehat{\mathcal{O}}_{C_S,p_i(S)} \simeq \mathcal{O}_S[[t_i]]$.

We will denote by Θ_S the sheaf of vector fields on S. We will also denote by $\mathcal{O}(C_S - \vec{p}(S))$ the sheaf on S whose sections over $U \subset S$ are by definition meromorphic functions over $\pi^{-1}(U) \subset C_S$ which are regular outside of $p_i(S)$; when $S = \{point\}$, this coincides with the definition in the previous section. In a similar way, we define $\mathfrak{g}(C_S - \vec{p}(S)), \Theta(C_S - \vec{p}(S))$—all of them are sheaves on S.

Throughout this section, let us fix a family C_S as above, choose integrable $\widehat{\mathfrak{g}}$-modules $V_1, \dots, V_n \in \mathcal{O}_k^{\text{int}}$, and let $V = V_1 \otimes V_n$. Then for every point $s \in S$ we can define the vector space of coinvariants

$$\tau_s = \tau(C_s, \vec{p}(s), V) = V/\mathfrak{g}(C_s - \vec{p}(s))V. \tag{7.4.1}$$

The main goal of this section is to prove the following theorem.

THEOREM 7.4.1. *Under the above assumptions, the vector spaces τ_s form a vector bundle τ_S over S which carries a natural projectively flat connection. The assignment $S \mapsto \tau_S$ is functorial in S: for every map $\psi\colon S' \to S$ and a family C_S over S as before, there is a canonical isomorphism $\tau_{S'} = \psi^*(\tau_S)$, where $C_{S'} := \psi^*(C_S)$.*

We remind that a connection is called projectively flat if $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is an operator of multiplication by a function for any two vector fields X, Y on S. The failure of the connection to be flat is, of course, related with the central term in the definition of $\widehat{\mathfrak{g}}$: for $k = 0$, the connection is flat (but of little interest, since the only integrable module of level 0 is $L_0^k = \mathbb{C}$). We will discuss this later.

The remaining part of this section is devoted to the construction of the flat connection and the proof of the theorem. For simplicity, we will assume that $n = 1$; the general case can be treated similarly. Our exposition follows [**BFM**] (somewhat simplified).

LEMMA 7.4.2. *The vector spaces τ_s form a $\mathcal{O}_S$-coherent sheaf over S, i.e., there exists a coherent sheaf τ_S such that $\tau_s = \tau_S/I_s\tau_S$, I_s being the ideal of functions vanishing at s.*

PROOF. Let $V_S = \mathcal{O}_S \otimes V$ (usual algebraic vector product, no completions); this is an $\mathcal{O}_S$-module, which carries an $\mathcal{O}_S$-linear action of the $\mathcal{O}_S$-module $\mathfrak{g}(C_S - p(S))$. Define the sheaf

$$\tau_S = V_S/\mathfrak{g}(C_S - p(S))V_S. \tag{7.4.2}$$

It is obvious that localizing τ_S at $s \in S$, we get the vector space of coinvariants τ_s. The coherency of τ_S can be proved in a way similar to the proof of finite-dimensionality of the spaces $\tau(C)$ in the previous section, using the following lemma.

LEMMA 7.4.3. *Let A be a finite-dimensional vector bundle of Lie algebras over S which is generated (as a Lie algebra) by sections $x_1, \dots, x_n$. Denote by $\mathcal{A}$ the sheaf of sections of A. Let $\mathcal{W}$ be an $\mathcal{O}_S$-module with an $\mathcal{O}_S$-linear action of $\mathcal{A}$. Assume that $\mathcal{W}$ is locally cyclic (i.e., locally there exists a section $w_0 \in \mathcal{W}$ such that $\mathcal{W} = \mathcal{A}w_0$) and action of x_i is locally nilpotent: for every section w, one has $x_i^N w = 0$ for $N \gg 0$. Then $\mathcal{W}$ is $\mathcal{O}_S$-coherent.*

To prove this lemma, it suffices to note that by Gabber's theorem, $Supp(\mathcal{W})$ is the zero section of the bundle A^*, and that every module over $\mathcal{O}_S[x_1, \ldots, x_m]$ whose support is given by $x_i = 0$, is $\mathcal{O}_S$-coherent. $\square$

We will show that the sheaf τ_S has a natural structure of a twisted $\mathcal{D}_S$-module, i.e., a projective action of the sheaf Θ_S of vector fields on S which is compatible with the $\mathcal{O}_S$-module structure: $\xi(\phi\tau) = (\xi\phi)\tau + \phi(\xi\tau), \xi \in \Theta_S, \phi \in \mathcal{O}_S$. Since it is well known that every $\mathcal{O}$-coherent twisted $\mathcal{D}$-module is in fact a sheaf of sections of a vector bundle with a projectively flat connection, this will establish the theorem.

To construct an action of Θ_S on the sheaf of coinvariants, let us first consider the case when we have a fixed curve C with a marked point p, and S is the set of all possible choices of a formal local parameter t at p. This set has a natural structure of a projective limit of the smooth manifolds $S^{(N)} = \{N\text{-jets of local parameters at } p\}$. We have a tautological family of curves $C_S = C \times S$ over S, with the same marked point p and with the formal local parameter determined by $s \in S$.

This S is a torsor over the pro-Lie group (i.e., a projective limit of Lie groups) $K_0 = \operatorname{Aut} \mathbb{C}[[t]]$ of changes of local parameter. This group can be explicitly described as the group of power series of the form $a_1 t + a_2 t^2 + \ldots, a_1 \neq 0$, with the group operation being composition; it acts on the set of formal local parameters in an obvious way. The corresponding Lie algebra $\mathcal{T}_0 = \operatorname{Lie} K_0$ is given by $\mathcal{T}_0 = t\mathbb{C}[[t]]\partial_t$ (see [**TUY**, Section 1.4] for precise statements). Therefore, the tangent space to S at every point can be identified with $\mathcal{T}_0$. or, equivalently, $\mathcal{T}_0$ is the space of all K_0 left-invariant vector fields on S. Thus, to define an action of Θ_S on the bundle of coinvariants, one needs to define an action of $\mathcal{T}_0$.

Therefore, we see that the key step in this case would be to define an action of $\mathcal{T}_0 = t\mathbb{C}[[t]]\partial_t$ on V. In the general case, we will in fact need an action of a larger Lie algebra $\mathcal{T} = \mathbb{C}((t))\partial_t$, which is usually called the *Witt algebra*. It has a natural (topological) basis $L_n = -t^{n+1}\partial_t$, $n \in \mathbb{Z}$, with the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{7.4.3}$$

The subalgebra $\mathcal{T}_0$ is generated by L_n with $n \geq 0$. Similarly, we will also use the subalgebras $\mathcal{T}_1 = t^2\mathbb{C}[[t]]\partial_t$, $\mathcal{T}_{-1} = \mathbb{C}[[t]]\partial_t$ generated (as topological Lie algebras) by L_n with $n \geq 1$ (respectively, $n \geq -1$).

It is indeed possible to define a projective action of $\mathcal{T}$ on $\widehat{\mathfrak{g}}$-modules. This is known as the Sugawara construction. We formulate this result as a proposition, referring the reader to [**K1**] for details and the proof.

PROPOSITION 7.4.4. *One can define elements $L_n, n \in \mathbb{Z}$, in a certain completion of $U(\widehat{\mathfrak{g}})_k$ which have the following properties*:

(i) *In every module V from the category $\mathcal{O}_k$, the action of L_n is well-defined, and*

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3 - m}{12}c, \tag{7.4.4}$$

where

$$c = \frac{k \dim \mathfrak{g}}{k + h^\vee}. \tag{7.4.5}$$

(ii) *The operator L_n has degree n with respect to the homogeneous grading, and*

$$[L_n, a[m]] = -m\, a[m+n], \qquad a \in \mathfrak{g}. \tag{7.4.6}$$

(iii) *In the Weyl module* V_λ^k *(and thus, in* L_λ^k*), the operator* L_0 *acts by*

$$L_0 v = (\Delta_\lambda - \deg v)v, \qquad \Delta_\lambda = \frac{\langle \lambda, \lambda + 2\rho\rangle}{2(k+h^\vee)}. \tag{7.4.7}$$

Part (i) of this proposition can be reformulated as follows. Let

$$Vir = \mathbb{C}((t))\partial_t \oplus \mathbb{C}c \tag{7.4.8}$$

as before, this vector space has topological basis $c, L_n = -t^{n+1}\partial_t, n \in \mathbb{Z}$. We define the structure of Lie algebra on Vir by (7.4.4) (it can also be defined in a coordinate-free way, with the central term given as a residue of the $f'''g$). This algebra is called the *Virasoro algebra* and plays a central role in conformal field theory; by definition, it is a central extension of the Witt algebra $\mathbb{C}((t))\partial_t$. Thus, part (i) claims that every module $V \in \mathcal{O}_k$ is naturally a module over Vir with the central charge equal to $k \dim \mathfrak{g}/(k+h^\vee)$.

Note that when restricted to $\mathcal{T}_{-1} = \mathbb{C}[[t]]\partial_t$, the central term in (7.4.4) vanishes; thus, $\mathcal{T}_{-1}$ is a subalgebra in Vir and therefore acts on V. Hence, the same construction also defines an action $\mathcal{T}_0$ on V. Considering $\mathcal{T}_0$ as the Lie algebra of left-invariant vector fields on the set S of all choices of local parameter at p, one easily sees that this action can be uniquely extended to the action of the sheaf Θ_S of all vector fields on S on the sheaf $V_S = \mathcal{O}_S \otimes V$.

Let us now consider the general case, when not only the local parameter but also the the curve itself is allowed to vary.

First of all, let C be a complex curve, and t—a formal parameter at the point $p \in C$. Denote by $\Theta(C-p)$ the space of meromorphic vector fields on C which are holomorphic outside of p. Then we have a Lie algebra homomorphism $\gamma_p\colon \Theta(C-p) \to \mathcal{T}$ obtained by expanding a vector field in a neighborhood of p in power series in t. Similarly, if we have several marked points $p_1, \dots, p_n$, we can define a map

$$\gamma_{\vec{p}} = \bigoplus \gamma_{p_i}\colon \Theta(C-\vec{p}) \to \mathcal{T} \oplus \cdots \oplus \mathcal{T}. \tag{7.4.9}$$

On the other hand, Sugawara construction gives a projective action of the direct sum $\mathcal{T} \oplus \cdots \oplus \mathcal{T}$ on $V = V_1 \otimes \ldots \otimes V_n$; thus, we get a projective action of $\Theta(C-\vec{p})$ on V, which we will also denote by $\gamma_{\vec{p}}$.

LEMMA 7.4.5. (i) *The action of* $\Theta(C-\vec{p})$ *on* V*, given by* $\gamma_{\vec{p}}$*, is a true action, not a projective one.*

(ii) *The actions of* $\Theta(C-p)$ *and* $\mathfrak{g}(C-p)$ *on* V *agree as follows*:

$$[\gamma_{\vec{p}}(\xi), a\otimes f] = a \otimes \xi(f), \quad \xi \in \Theta(C-p), a\otimes f \in \mathfrak{g}\otimes\mathcal{O}_S.$$

(iii) *The induced action of* $\Theta(C-\vec{p})$ *on the space of coinvariants* $V_{\mathfrak{g}(C-\vec{p})}$ *is zero.*

PROOF. Part (i) follows from the fact that the central term in (7.4.4) can be written as a residue, and from the fact that the sum of residues of a meromorphic 1-form is equal to zero. The proof of part (ii) is immediate from (7.4.6). As for part (iii), the simplest way to prove it is to note that $\Theta(C-\vec{p})$ is a simple Lie algebra (see [**BFM**]), and therefore has no non-trivial finite-dimensional representations. Of course, this is a very artificial proof. A more natural proof can be obtained from the theory of chiral algebras. For readers familiar with this theory, we point out that the Sugawara construction in fact shows that the generating function $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ is a field in the vertex operator algebra (=chiral algebra on a formal punctured disk) generated by the Kac–Moody currents $a(z) = \sum_{n\in\mathbb{Z}}(at^n)z^{-n-1}$, $a \in \mathfrak{g}$ (see, e.g., [**K2**]); similarly, the Lie algebra $\Theta(C-p)$ is a subalgebra in the chiral algebra associated with the curve $C-p$. But since this chiral algebra is generated (in an appropriate sense) by the Kac–Moody currents, and these currents act on the space of coinvariants by zero, this whole chiral algebra acts by zero. Details can be found in [**Gai**]. □

Part (iii) of the lemma may seem discouraging. Note, however, that what we are looking for is an action of Θ_S on the bundle of coinvariants, not an action of Θ_C, so we do not have a problem with the fact that $\Theta(C-p)$ acts by zero. In fact, it will be useful to us.

In order to define an action of Θ_S, we will first lift a vector field on S to a vector field on C_S, and then restrict to a formal neighborhood of p.

Let θ be a vector field on S. Let us lift it to a vector field $\tilde{\theta}$ on $C_S - p(S)$. Such a lifting is always possible, which follows from the fact that $\pi : C_S - p(S) \to S$ is affine, and therefore defines an exact functor on $\mathcal{O}$-coherent sheaves (this is where we need to allow poles at $p(S)$!).

Let us consider the vector field $\tilde{\theta}$ in a neighborhood of one of the sections $p_i(S)$ ("marked point"). Then the choice of local coordinate t_i allows us to define the notion of horizontal vector field: a vector field v in a punctured neighborhood of $p_i(S)$ is horizontal if $v(t) = 0$. Then we can define "vertical" component $\gamma_p(\tilde{\theta})$ by

$$\tilde{\theta} = \gamma_{p_i}(\tilde{\theta}) + \tilde{\theta}^{\text{horiz}}, \quad \tilde{\theta}^{\text{horiz}}(t) = 0.$$

Note that while one can easily define the notion of a vertical vector filed on C_S (v is vertical if its projection to S is zero), the notion of horizontal vector field, nad thus, of "vertical component" $\gamma_{p_i}(\tilde{\theta})$ depends on the choice of local parameter t_i. If we choose local coordinates x_i on S, so that $\theta = \sum f_i(x)\partial_{x_i}$, then (x_i, t) give a coordinate system in a neighborhood of $p_i(S)$, and we can write $\tilde{\theta} = g(x,t)\partial_t + \sum f_i(x)\partial_{x_i}$. Then $\gamma_{p_i}(\tilde{\theta}) = g(x,t)\partial_t$. The function $g(x,t)$ can have poles at $t_i = 0$, so it can be viewed as a local section of $\mathcal{O}_S((t_i))$, and thus $\gamma_{p_i}(\tilde{\theta}) \in \mathcal{O}_S \otimes \mathcal{T}$.

Repeating this for all points p_i, we define

$$\gamma_{\vec{p}}(\tilde{\theta}) = \sum \gamma_{p_i}(\tilde{\theta}) \in \mathcal{O}_S \otimes (\mathcal{T} \oplus \cdots \oplus \mathcal{T}) \tag{7.4.10}$$

(for $S = \{pt\}$, this coincides with the definition (7.4.9)).

Now, let us define the action of $\tilde{\theta}$ on $V_S = V \otimes \mathcal{O}_S$ by

$$\tilde{\theta}(fv) = (\theta(f))v + f\sum_i \gamma_{p_i}(\tilde{\theta})v,$$

where $\gamma_{p_i}(\tilde{\theta})$ acts on V_i by the Sugawara construction.

LEMMA 7.4.6. *The above defined action of $\tilde{\theta}$ on V_S has teh following properties:*

1. *It is compatible with the structure of $\mathcal{O}_S$-module: for $f \in \mathcal{O}_S, v \in V_S$, one has $\tilde{\theta}(fv) = (\theta(f))v + f\tilde{\theta}(v)$.*
2. *It is compatible with the action of $\mathfrak{g}(C_S - \vec{p}_S)$ on V_S: if $f \in \mathcal{O}_{C_S - \vec{p}(S)}, x \in \mathfrak{g}$, then $[\tilde{\theta}, fx] = (\tilde{\theta}(f))x$.*

PROOF. The first part immediately follows from the definition; the second one follows from Theorem 7.4.5(ii). □

It immediately follows from part (ii) of this lemma that we have a well-defined action of $\tilde{\theta}$ on the bundle of coinvariants $\tau_S = V_S/\mathfrak{g}(C_S - \vec{p}_S)V_S$.

PROPOSITION 7.4.7. *The induced action of $\tilde{\theta}$ on the bundle of coinvariants depends only on θ and not on the choice of lifting $\tilde{\theta}$. It defines a projective action of the Lie algebra Θ_S on the bundle of coinvariants, which agrees with the structure of $\mathcal{O}_S$-module.*

PROOF. The only non-trivial statement is the independence of the choice of lifting. It follows from the fact that any two liftings differ by a vertical vector field. On the other hand, it follows from Theorem 7.4.5(iii) that vertical fields act by zero. □

This completes the proof of Theorem 7.4.1. □

More careful analysis also allows one to calculate explicitly the failure of the connection to be flat. Using the language of twisted $\mathcal{D}$-modules developed in Section 6.6 and the notion of determinant line bundle Q_S defined in Section 6.7, the result can be formulated as follows:

THEOREM 7.4.8. *Under the assumptions of Theorem* 7.4.1*, the sheaf τ_S carries a natural structure of a $\mathcal{D}_{Q^c}$-module, where c is the Virasoro central charge defined by* (7.4.5).

We do not give a proof of this theorem, referring the reader to [**BS**]. The proof is based on the fact that the central extension defining the Virasoro algebra can be defined using the action of the Lie algebra of vector fields on the space $\mathbb{C}((\vec{t})) = \oplus_i \mathbb{C}((t_i))$ and the "universal" cocycle defined by the the subspace $\mathbb{C}[[\vec{t}]] = \oplus_i \mathbb{C}[[t_i]] \subset \mathbb{C}((\vec{t}))$. This cocycle was first discovered by Tate [**Ta**] and rediscovered under different names by many authors (see [**BS**], [**ACK**]). On the other hand, it is well known that for a connected smooth curve C one has $\mathbb{C}((\vec{t}))/(\mathbb{C}[[\vec{t}]]+\mathcal{O}(C-\vec{p})) = H^1(C,\mathcal{O})$. This gives a relation between this cocycle and the determinant line bundle (recall that $Q_s = \det(H^1(C_s,\mathcal{O}))$). Details can be found in [**BS**] or [**BFM**].

EXAMPLE 7.4.9. Let us calculate this flat connection explicitly in the case when the curve C is fixed but the point p is allowed to move. Let u be a local coordinate on C, i.e. a biholomorphic map $u\colon C^0 \to U$, where C^0 is some open subset of C, and U an open subset of $\mathbb{C}$. We will denote by z a global coordinate on $\mathbb{C}$ and thus, on U. Let us define the following family of punctured curves over U: $C_U = C \times U$, $p(z) = u^{-1}(z)$, and the local parameter at p given by $t = u - z$ (considered as a function on $C \times U$). Note that both (z, u) and (z, t) can be considered as local coordinates on $C \times U$.

In this case, every vector field $f(z)\partial_z$ on U admits a canonical horizontal lifting to $C \times U$; in terms of the coordinate system (z, u) this lifting is given by $f(z)\partial_z \mapsto f(z)\partial_z + 0 \cdot \partial_u$. When we rewrite this in terms of (z, t), we get $f(z)(\partial_z - \partial_t)$. Therefore, the action of such a vector field on the bundle of coinvariants is given by $(f\partial_z)(\phi v) = f(\partial_z\phi)v + f\phi L_{-1}v$ (recall that $L_{-1} \in Vir$ corresponds to $-\partial_t$). In other words, the corresponding flat connection on U is induced from the connection on $V \otimes \mathcal{O}_S$ given by

$$\nabla = d + L_{-1}dz.$$

It is easy to see that for several points, we get

$$\nabla = d + \sum_i (L_{-1})_i dz_i, \tag{7.4.11}$$

where $(L_{-1})_i$ stands for L_{-1} acting in V_i.

Note that in this case every vector field on S can be lifted to a regular vector field on C_S. Therefore, we only need to use the Sugawara construction for the fields from $\mathbb{C}[[t]]\partial_t = \mathcal{T}_{-1}$. Since the central term in (7.4.4) vanishes when restricted to $\mathcal{T}_{-1}$, we get a true action, not a projective one.

Let us consider even more special case than in the previous example, namely when $C = \mathbb{P}^1$, with marked points $z_1, \dots, z_n \neq \infty$ and local parameters given by $t_i = z - z_i$. This defines a family of curves over $X_n = \mathbb{C}^n \setminus$diagonals. Assign to these points Weyl modules $V^k_{\lambda_1}, \dots, V^k_{\lambda_n}$. Then, by Proposition 7.3.8, the vector bundle of coinvariants $\tau(\mathbb{P}^1, z_1, \dots, z_n, V^k_{\lambda_1}, \dots, V^k_{\lambda_n})$ is a quotient of the trivial vector bundle with the fiber $(V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n})_{\mathfrak{g}}$ over X_n. Therefore, the construction above defines a flat connection in this quotient bundle. Passing to the dual vector bundle, we see get a flat connection in the vector subbundle

$$\big(\tau(\mathbb{P}^1, z_1, \dots, z_n; V^k_{\lambda_1}, \dots, V^k_{\lambda_n})\big)^* \subset \big(V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}\big)^*_{\mathfrak{g}} = (V^*_{\lambda_1} \otimes \dots \otimes V^*_{\lambda_n})^{\mathfrak{g}}$$

THEOREM 7.4.10 ([**KZ**]). *The flat connection described above coincides with the restriction of the KZ connection in $V^*_{\lambda_1} \otimes \dots \otimes V^*_{\lambda_n}$, defined by* ($\mathrm{KZ}_n$).

A proof of this theorem can be found in the original paper [**KZ**] (only recommended for those familiar with the basics of conformal field theory). This proof is also repeated in a number of sources, for example, in [**EFK**], in a language more familiar to mathematicians. This theorem and comparison of the gluing isomorphisms, which we will do later, will be used to show that for $k \notin \mathbb{Q}$ the functor of coinvariants defined above for genus zero curves coincides with the modular functor defining Drinfeld's category—see Theorem 7.9.12. In particular, this modular functor can be defined in a way which doesn't refer to the affine Lie algebras at all. Note, however, that for $k \notin \mathbb{Q}$ this modular functor can not be extended to positive genus.

EXAMPLE 7.4.11. Let $C, \vec{p}, \vec{t}$ be as before. Choose one of the points p_j and consider the family of curves $C \times \mathbb{C}^\times$ over $\mathbb{C}^\times$, with the the marked points $p_i(z) = p_i$ and local parameters $t_i(x, z) = t_i(x), x \in C, z \in \mathbb{C}$, except for $i = j$ when we set $t_i(x, z) = t_i(x)/z$. By the construction of this section, the corresponding vector bundle of coinvariants τ has a canonical flat connection. An easy calculation, similar to the one in Example 7.4.9, shows that this connection is induced from the connection

$$\nabla = d + (L_0)_j \frac{dz}{z}$$

in the trivial vector bundle with fiber $V_1 \otimes \dots \otimes V_n$. In particular, the monodromy of this connection around $z = 0$ is given by $e^{2\pi \mathrm{i} L_0}$, so if V_j is an irreducible module with highest weight λ, the monodromy operator is constant and equals $e^{2\pi \mathrm{i}\Delta_\lambda}$.

Note that if we pass from 1-jet of local parameter to tangent vector, we see that the tangent vector is given by $z\partial_{t_j}$, and thus, as z goes around the origin counterclocwise, so does the tangent vector. Recalling the relation between modular functor and tensor categories, we see that in the tensor category corresponding to the WZW modular functor, the universal twist is given by

$$\theta_{L^k_\lambda} = e^{2\pi \mathrm{i} \Delta_\lambda} \operatorname{id}_{L^k_\lambda} \tag{7.4.12}$$

(compare with Remark 3.1.20), which agrees with the formulas for universal twist in Drinfeld's category (Theorem 2.2.7) and in the category of representations of a quantum group Exercise 2.2.6—which is another argument confirming equivalence of these categories.

In fact, this vector bundle on $\mathbb{C}^\times$ admits a canonical extension to a vector bundle on $\mathbb{P}^1$, and the connection has logarithmic singularities at $0, \infty$. Indeed, we can assume that $V_j = L^k_\lambda$. Denote $V = \otimes_{i \neq j} V_i$. The fiber of τ at point $z \in \mathbb{C}^\times$ is given by $\tau_z = W_z / \mathfrak{g}(C - \vec{p}) W_z$, where $W_z = V \otimes L^k_\lambda$ does not depend on z. Note that the subspace $\mathfrak{g}(C - \vec{p}) W_z$ depends on z, since the choice of local coordinate at p_j depends on z. Let us choose a different trivialization of the vector bundle $V \otimes L^k_\lambda$, namely, let us identify

$$\begin{aligned} V \otimes L^k_\lambda &\to (V \otimes L^k_\lambda)_z, \\ v \otimes v_j &\mapsto z^{\deg v_j} v \otimes v_j. \end{aligned}$$

In other words, in this trivialization constant sections are given by $z^{\deg v_j} v \otimes v_j$. Then one easily sees that in this trivialization, the subspace $\mathfrak{g}(C - \vec{p}) W_z$ does not depend on z; thus, it also gives a trivialization of the vector bundle of coinvariants on $\mathbb{C}^\times$, and in this trivialization the flat connection is given by $\nabla = d + \Delta_\lambda dz/z$. Therefore, this gives an extension of our vector bundle with a flat connection to $\mathbb{P}^1$, and the connection has logarithmic singularities at $0, \infty$.

Note that for this definition of extension to $z = 0$, a function of the form $f(z)(v_1 \otimes \dots \otimes v_j \otimes \dots \otimes v_n)$ defines a section holomorphic at 0 iff $z^{-\deg v_j} f(z)$ is regular at $z = 0$ (we assume that v_j is homogeneous).

7.5. From local parameters to tangent vectors

In the previous section, we have studied properties of the vector spaces of coinvariants for a curve C with marked points and chosen local parameters at these points, or a family of such curves. In this section we will show that the vector space of coinvariants only depends on the 1-jet of local parameter: if t_i, t'_i are different choices of local parameter at p_i such that $d_{p_i} t_i = d_{p_i} t'_i$, then the vector spaces $\tau(C, \vec{p}, \vec{t}, \mathcal{L})$ and $\tau(C, \vec{p}, \vec{t'}, \mathcal{L})$ are canonically isomorphic, and similarly for families of curves.

Let us start with the case when we only have one curve C; as before, for simplicity we assume that it has only one marked point p. Let us fix a non-zero tangent vector $v \in T_p C$ and consider only such formal local parameters t at p that $\partial_v t = 1$; the set of formal local parameter form a pro-variety M. We want to show that for such local parameters t, the vector spaces $\tau(C, p, t, \mathcal{L})$ can be canonically identified. In order to do that, consider the family of curves $C_M = C \times M$ over M, with a marked point p (which does not depend on m) and the local parameter at $p \in C_m$ defined by $m \in M$. As discussed in the previous section, this defines a canonical flat connection on the bundle of coinvariants $\tau(C, p, t, \mathcal{L})$. We will show that this vector bundle with a flat connection is trivial. Indeed, it is easy to see that M is a torsor over the group

$$K_1 = \{k \in \operatorname{Aut}\mathbb{C}[[t]] \mid (k(t))'(0) = 1\}.$$

This group can be explicitly described as the group of all formal power series of the form $1 + \sum_2^\infty a_i t^i$, with the group operation being substitution of one series into another. The corresponding Lie algebra is $\operatorname{Lie} K = \mathcal{T}_1 = t^2\mathbb{C}[[t]]\partial_t$.

Now the triviality of the flat connection follows from the following two easy lemmas whose proofs are omitted.

LEMMA 7.5.1. *Let a manifold M be a torsor over a Lie group K, and E be a vector bundle with a flat connection over M. Then this flat connection is trivial iff the action of* $\operatorname{Lie} K$ *by vector fields on E can be lifted to an action of K on E.*

LEMMA 7.5.2. *The action of* $\operatorname{Lie} K_1 = \mathcal{T}_1$ *on an integrable module $\mathcal{L}$, defined by the Sugawara construction, can be integrated to an action of K_1 on $\mathcal{L}$.*

Combining these two lemmas, we get that in our case, the flat connection on the bundle of conformal blocks is trivial, and thus all the spaces $\tau(C, p, t, \mathcal{L})$ are canonically isomorphic. Therefore, we can define the space of coinvariants $\tau(C, p, v, \mathcal{L})$ as the space of global flat sections of the bundle $\tau(C, p, t, \mathcal{L})$ on M.

REMARK 7.5.3. Note that the action of $\mathcal{T}_0$ usually can not be integrated to the action of $\operatorname{Aut}\mathbb{C}[[t]]$. Indeed, in $\operatorname{Aut}\mathbb{C}[[t]]$ one has $e^{2\pi \mathrm{i} L_0} = 1$, but in a highest weight $\widehat{\mathfrak{g}}$ module with highest weight λ, one has

$$e^{2\pi \mathrm{i} L_0} =: \theta_\lambda = e^{2\pi \mathrm{i} \Delta_\lambda}$$

which is not equal to 1 unless $\Delta_\lambda \in \mathbb{Z}$. Therefore, we do need to specify a 1-jet of local parameter.

Now let us consider families of curves. Let $C_S, p(S)$ be a family of curves with a fixed 1-jet of local parameter t at $p(S)$. If we fix a formal local parameter t at $p(S)$ with given 1-jet, then, by the construction of the previous section, we get a vector bundle of coinvariants with a flat connection over S. Let us show that these vector bundles for different choices of t can be canonically identified.

Using the same idea as in the case $S = \{point\}$, consider the pro-variety $M = \{(s,t) \mid s \in S\}$; obviously, M is a principal K_1-bundle over S. The family C_S over S defines a family C_M over M and therefore defines a bundle of coinvariants τ_M with a flat connection over M. Our goal is to show that this flat connection is trivial along the fibers of the projection $M \to S$. A convenient framework for such proofs is provided by the formalism of Harish–Chandra pairs.

DEFINITION 7.5.4. A *Harish–Chandra pair* is a pair $(\mathfrak{g}, K)$, where $\mathfrak{g}$ is a Lie algebra, and K is a Lie group with the Lie algebra $\operatorname{Lie} K = \mathfrak{k} \subset \mathfrak{g}$. We also assume that we are given an action Ad of K on $\mathfrak{g}$ which agrees with both the standard Ad action of K on $\mathfrak{k}$ and ad action of $\mathfrak{k}$ on $\mathfrak{g}$.

As usual, we define a module V over a Harish–Chandra pair $(\mathfrak{g}, K)$ to be a vector space which has an action of both $\mathfrak{g}$ and K, and these actions agree on $\mathfrak{k}$.

These definitions can be suitably reformulated if we want to replace a Lie algebra $\mathfrak{g}$ by the sheaf of vector fields on a manifold M (or, more generally, by a Lie algebroid over M—see [**BFM**]). Let us assume that we have a manifold M with a free action of a Lie group K such that M is a principal K-bundle over a manifold S. We denote by $p : M \to S$ the projection. Denote by Θ_M the sheaf of vector fields on M. Then for every $U \subset M$, we have a natural embedding $\mathfrak{k} \subset \Theta_M(U)$, which is a Lie algebra homomorphism. We also have an adjoint action of K on Θ_X. Therefore, the pair (Θ_M, K) is a natural sheaf analogue of a Harish-Chandra pair.

DEFINITION 7.5.5. Let M, K, Θ_M be as above. A finite-dimensional (Θ_M, K)-*module* is a finite-dimensional vector bundle V with a flat connection over M with an action of K on V, which agrees an obvious sense with both the action of K on M and with the action of $\mathfrak{k} \subset \Theta_M$ by vector fields on V.

(A not necessarily finite-dimensional (Θ_M, K)-module can be defined in a similar way, replacing "vector bundle with a flat connection" by "$\mathcal{D}$-module.)

Our main reason in developing this technique is the following lemma.

LEMMA 7.5.6. *Any finite-dimensional (Θ_M, K)-module V defines a vector bundle with a flat connection V^K on $S = M/K$.*

PROOF. For every $s \in S$, define the vector space $V_s^K = (\Gamma(M_s, V))^K$, where $M_s = p^{-1}(s)$ is the fiber of the projection $p\colon M \to S$. It is easy to see that these vector spaces form a vector bundle over S of the same dimension as the original bundle V (it suffices to choose locally a section of the projection to show this). Note that any section ϕ of this bundle is killed by the vertical vector fields; thus, the quotient $\Theta_M/\Theta_M^v = \Theta_S$ acts on V^K. $\square$

Now we have all the prerequisites to prove the following theorem.

THEOREM 7.5.7. *Let C_S be a family of pointed curves over a smooth base S, and let $L_1^k, \dots, L_n^k$ be some integrable modules assigned to these points. Then we have a bundle of coinvariants τ_S over S which carries a natural projectively flat connection, and this bundle is functorial in S in the same sense as in Theorem* 7.4.1.

PROOF. Take $M = \{(s,t)\}, s \in S, t$–a local parameter at $p \in C_s$ with given differential. Obviously, M is a K^n–torsor over S, where $K = \mathrm{Aut}_1\,\mathbb{C}[[t]]$ and we have a tautological family C_M of curves over M with marked points and a local parameters at these points. By the construction of the previous section, this defines a vector bundle with a projectively flat connection over M. By Lemma 7.5.2, this connection is integrates to an action of K. Therefore, by Lemma 7.5.6, we have a flat connection on $S = M/K$. $\square$

COROLLARY 7.5.8. *For a fixed finite set A and a collection of modules $L_a^k \in \mathcal{O}_k^{\mathrm{int}}$, we have a vector bundle of coinvariants $\tau(\{L_a^k\})$ over the moduli stack $M_{*,A}$, which carries a natural projectively flat connection.*

As in Theorem 7.4.8, we can also explicitly describe the failure of the connection to be flat by saying that the sheaf of sections of the vector bundle $\tau(\{L_a^k\})$ is a $\mathcal{D}_{Q^c}$-module.

7.6. Families of curves over formal base

This section introduces some technical notions which will be used later for proving the gluing axiom for the WZW modular functor. Namely, we will generalize most of the results regarding the bundle of coinvariants to the case where the base is an infinitesimal neighborhood of a divisor D.

Throughout this section, we fix a non-singular variety S and a smooth divisor $D \subset S$. We also choose (locally) a function q on S such that the equation of D is $q = 0$, and $dq \neq 0$ on D. All our defintions and theorems will be local in S.

The main subject of this section is the study of the n-th infinitesimal neighborhood $D^{(n)}$ of D in S, where n is a fixed non-negative integer. As before, we will not really define $D^{(n)}$; instead, we will define the structure sheaf of $D^{(n)}$, $\mathcal{O}$-modules on $D^{(n)}$, family of curves over $D^{(n)}$, etc.

DEFINITION 7.6.1. The structure sheaf of $D^{(n)}$ is the sheaf of algebras $\mathcal{O}_D^{(n)}$ on D defined by $\mathcal{O}_D^{(n)} = \mathcal{O}_S/q^{n+1}\mathcal{O}_S$.

One also defines in an obvious way a notion of $\mathcal{O}_D^{(n)}$-module; it is called lisse if it is locally free module of finite rank. Every sheaf $\mathcal{F}$ over S defines a sheaf $\mathcal{F}^{(n)}$ over $D^{(n)}$ in an obvious way: $\mathcal{F}^{(n)} = \mathcal{F}_D/q^{n+1}\mathcal{F}_D$. It is easy to see that if $\mathcal{F}$ is $\mathcal{O}_S$-coherent, then $\mathcal{F}^{(n)}$ is finitely generated, and if $\mathcal{F}$ is lisse then so is $\mathcal{F}^{(n)}$. Unfortunately, the functor $\mathcal{F} \mapsto \mathcal{F}^{(n)}$ is not exact on $\mathcal{O}_S$-modules. However, we have the following result.

LEMMA 7.6.2. (i)*Let $\mathcal{F}$ be an $\mathcal{O}_S$-coherent sheaf such that its restriction to $S \setminus D$ is lisse and for every $n \geq 0$, $\mathcal{F}^{(n)}$ is lisse. Then $\mathcal{F}$ is lisse.*

(ii) *For every short exact sequence of quasicoherent $\mathcal{O}_S$-modules $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ such that $\mathcal{G}$ is $\mathcal{O}_S$-coherent, the sequence $0 \to \mathcal{E}^{(n)} \to \mathcal{F}^{(n)} \to \mathcal{G}^{(n)} \to 0$ is also exact.*

The proof of this lemma is left as an exercise to the reader.

EXAMPLE 7.6.3. Assume that $\dim S = 1$. Then $D =$ point, $\mathcal{O}_D^{(n)} = \mathbb{C}[q]/(q^{n+1})$, and $\mathcal{O}_D^{(n)}$ is just a module over this algebra.

We can also define vector fields and $\mathcal{D}$-modules for $D^{(n)}$. Note, however, that the only vector fields on S that can be restricted to $D^{(n)}$ are those tangent to D: the vector field ∂_q can not be restricted to $D^{(n)}$ as it does not preserve the relation $q^{n+1} = 0$. Thus, we can define an analogue of $\mathcal{D}_S^0$-module, but not of a $\mathcal{D}_S$-module (recall that $\mathcal{D}_S^0$ is generated by $\mathcal{O}_S$ and vector fields tangent to D, see (6.3.5)). Thus, we give the following definition:

$$\mathcal{D}^0_{D^{(n)}} = \mathcal{D}_S^0/q^{n+1}\mathcal{D}_S^0 \tag{7.6.1}$$

For example, for $\dim S = 1$, $\mathcal{D}^0_{D^{(n)}}$ is generated by $\mathcal{O}_D^{(n)} = \mathbb{C}[q]/(q^{n+1})$ and $q\partial_q$.

Since a flat connection on S with logarithmic singularities at D is the same as a lisse sheaf on S with an action of $\mathcal{D}_S^0$, it is natural to give the following definition.

DEFINITION 7.6.4. A flat connection on $D^{(n)}$ with logarithmic singularities at D ($\log D$-connnection for short) is a lisse sheaf on $D^{(n)}$ with a structure of $\mathcal{D}^0_{D^{(n)}}$-module.

We have the following obvious lemma.

LEMMA 7.6.5. (i) *Every $\log D$ flat connection on S defines a $\log D$ flat connection $D^{(n)}$ by $\mathcal{F} \mapsto \mathcal{F}^{(n)}$*

(ii) *If the connection $\mathcal{F}$ is regular—i.e., has no poles at all—then $q\partial_q$ acts by zero in $\mathcal{F}^{(0)} = \mathcal{F}/q\mathcal{F}$.*

Now let us define families of curves over $D^{(n)}$ and the bundles of coinvariants.

DEFINITION 7.6.6. A family of curves over $D^{(n)}$ is the following collection of data:

– a family C_D of stable complex curves over D

– a sheaf of algebras $\mathcal{O}^{(n)}_{C_D}$ on C_D with a structure of a flat $\mathcal{O}^{(n)}_D$-module such that $\mathcal{O}^{(n)}_{C_D}/q\mathcal{O}^{(n)}_{C_D} = \mathcal{O}_{C_D}$.

The family is called *non-singular* if the family C_D is non-singular.

In a similar way, one can define a notion of families with marked points and local parameters at these points by adding to the data above a collection of points $p_i \in C_0$ and local parameters $t_i \in \mathcal{O}^{(n)}_{p_i}$ such that $t_i(p_i) = 0 \mod q, (dt_i)_{p_i} \neq 0 \mod q$. We can also define an analogue of the $\mathcal{O}_S$-sheaf $\mathcal{O}(C_S - \vec{p}(S))$. Namely, we define $\mathcal{O}^{(n)}_D$-module $\mathcal{O}^{(n)}(C - \vec{p})$ to be the space of global sections on C_D of the sheaf $\mathcal{O}^{(n)}_C[t_i^{-1}]$.

Obviously, every family of curves over S defines a family of curves over $D^{(n)}$: it suffices to take $\mathcal{O}^{(n)}_C = \mathcal{O}_{C_S}/q^{n+1}\mathcal{O}_{C_S}$; we will call this *restriction* of the family C_S to $D^{(n)}$). It turns out that if C_D is non-singular, then this statement can be reversed.

LEMMA 7.6.7. *Locally in S, every non-singular family of curves over $D^{(n)}$ can be obtained as a restriction of an analytic family of curves over a neighborhood of D in S.* ?!

Let us give an example of a singular family over $D^{(n)}$.

EXAMPLE 7.6.8. Let $\dim S = 1$, and let C_S be a family of curves over S such that C_s is smooth for $s \neq D$, and C_D is the curve with one double point a, so that in a neighbohood of a, C_S has local coordinates t_1, t_2 and the projection is given by $q = t_1 t_2$; thus, C_0 is given by equation $t_1 t_2 = 0$.

Let us describe the corresponding family of curves over $D^{(n)}$. In this case, the curve C_D is singular—it has double point a. To describe the sheaf $\mathcal{O}^{(n)}_C$, note that its stalk at a point $b \neq a$ is given by $\mathcal{O}^{(n)}_{C,b} \simeq \mathcal{O}_{C,b} \otimes \mathcal{O}^{(n)}_D$ (note: this doesn't define the sheaf yet, as we haven't defined the gluing maps—they depend on the map $\pi\colon C_S \to S$). However, the stalk at the double point is different:

$$\mathcal{O}^{(n)}_{C,a} = \mathcal{O}(t_1, t_2)/(t_1 t_2)^{n+1}, \tag{7.6.2}$$

where $\mathcal{O}(t_1, t_2)$ is the ring of germs of analytic functions in t_1, t_2 near the origin $t_1 = t_2 = 0$.

To relate the stalk at the double point with the stalks at nearby points, let us describe $\mathcal{O}^{(n)}(U)$, where U is a punctured neighborhood of a in C_D. Since in a neighborhood of a, the curve C_D consists of two components given by equations $t_2 = 0$ and $t_1 = 0$, every small enough U can be presented as $U = U_1 \sqcup U_2$, where $U_1 = U \cap \{t_2 = 0\}, U_2 = U \cap \{t_1 = 0\}$. Thus, t_1 is a coordinate on U_1 and t_2 is a coordinate on U_2. From this it is easy to show that

$$\mathcal{O}^{(n)}(U_1) = \mathcal{O}(U_1) \otimes \mathcal{O}^{(n)}_D \simeq \mathcal{O}(U_1) \otimes (\mathbb{C}[t_2]/(t_2)^{n+1})$$

where the isomorphism is given by $f(t_1)q^k \mapsto f(t_1)t_1^k t_2^k$, and similarly for U_2. Thus:

$$\mathcal{O}^{(n)}(U) = \Big(\mathcal{O}(U_1) \otimes (\mathbb{C}[t_2]/(t_2)^{n+1})\Big) \oplus \Big(\mathcal{O}(U_2) \otimes (\mathbb{C}[t_1]/(t_1)^{n+1})\Big)$$

Now it is easy to see that for $f(t_1,t_2) \in \mathcal{O}^{(n)}_{C,a}$, its restriction to the punctured neighborhood of a is given by

$$t_1^k t_2^l \mapsto (t_1^k t_2^l) \oplus (t_1^k t_2^l) = (t_1)^{k-l} q^l \oplus (t_2)^{l-k} q^k \tag{7.6.3}$$

In particular, if $l > n$, then restriction of $t_1^k t_2^l$ to U_1 is zero, and if $k > n$, then restriction of $t_1^k t_2^l$ to U_2 is zero.

For every family C_S over S with marked points and modules $V_i \in \mathcal{O}_k^{\mathrm{int}}$ assigned to these points we have a sheaf of coinvariants $\tau(C_S)$ over S which gives rise to the sheaf $\tau^{(n)}$ over $D^{(n)}$; if C_S is a smooth family, then $\tau^{(n)}$ is lisse. It follows from Lemma 7.6.2(ii) that this module can be defined in terms of the n-th infinitesimal neighborhood of D, namely

$$\tau^{(n)} = V^{(n)}/\mathfrak{g}^{(n)}(C-\vec{p})V^{(n)}, \tag{7.6.4}$$

where $\mathfrak{g}^{(n)}(C-\vec{p}) = \mathfrak{g} \otimes \mathcal{O}^{(n)}(C-\vec{p})$, and $V^{(n)} = V \otimes \mathcal{O}^{(n)}_D$.

Therefore, it is natural to take this formula as the definition of the sheaf of coinvariants for families over $D^{(n)}$.

PROPOSITION 7.6.9. *Let $C_{D^{(n)}}$ be a family of curves with marked points over $D^{(n)}$, with local parameters at these points, and integrable $\widehat{\mathfrak{g}}$-modules assigned to these points. Let $\tau^{(n)}$ be the $\mathcal{O}^{(n)}_D$-module defined by (7.6.4). Assume that C_D is nonsingular. Then $\tau^{(n)}$ is lisse and has a natural structure of a projective $\mathcal{D}^0_{D^{(n)}}$-module such that the action of $q\partial_q$ on $\tau^{(0)} = \tau^{(n)}/q\tau^{(n)}$ is zero.*

PROOF. By Lemma 7.6.7, such a family can be obtained as a restriction of some analytic family. Now existence of the flat connection and the fact that $\tau^{(n)}$ is lisse immediately follow from Theorem 7.4.1 and Lemma 7.6.7. To prove that $q\partial_q$ acts by zero on $\tau^{(0)}$, just note that for the analytic family, we have a well-defined action of ∂_q, and thus $q\partial_q = 0 \mod q$.

It is also important to note that the structure of $\mathcal{D}^0_{D^{(n)}}$-module can be defined completely in terms of $D^{(n)}$, without extending this to a family on S. Let $\Theta^{(n)}(C-\vec{p})$ be the space of global sections (on C_D) of the sheaf of derivations of $\mathcal{O}^{(n)}(C-\vec{p})$—this is the infinitesimal analogue of the algebra of vector fields. Then we can lift any vector field θ on S which is tangent to D—in particular, the vector field $q\partial_q$—to a "vector field" $\tilde{\theta} \in \Theta^{(n)}(C-\vec{p})$. The easiest way to prove this is to use Lemma 7.6.7.

As in the analytic case (see proof of Theorem 7.4.1), define the action of θ on the bundle of coinvariants by

$$\theta(fv) = (\theta(f))v + f\sum_i \gamma_{p_i}(\tilde{\theta})(v).$$

The same arguments as in Theorem 7.4.1 show that this is indeed defines the structure of a projective $\mathcal{D}^0_{D^{(n)}}$-module on the sheaf of coinvariants. □

7.7. Coinvariants for singular curves

In this section, we give a description of the vector space $\tau(C,\vec{p},V)$ for a singular curve C. This description will be used in the next section to prove that the bundle of conformal blocks satisfies the gluing axiom and in particular has regular singularities on the boundary of the moduli space.

Let $C, \vec{p}, \vec{t}$ be stable singular curve with marked points and local parameters at these points. Choose modules $V_1, \dots, V_n \in \mathcal{O}_k^{\mathrm{int}}$ assigned to these points. We define the space of coinvariants $\tau(C, \vec{p}, V)$ (or, for brevity, $\tau(C, V)$) by the same formula as for non-singular curves (see Definition 7.3.1). For simplicity, let us only consider the case when C has only one double point; general case is completely parallel.

Denote by $C^\vee$ the *normalization* of C, i.e. the non-singular curve such that C is obtained by identifying points $a', a'' \in C$. Let us choose the local coordinates t', t'' near a', a''.

THEOREM 7.7.1. *The map*

$$V \to \bigoplus_\lambda V \otimes (L_\lambda^k \otimes DL_\lambda^k)$$

$$v_1 \otimes \ldots \otimes v_n \mapsto \bigoplus v_1 \otimes \ldots \otimes v_n \otimes 1_\lambda,$$

where DL_λ^k is defined as in Section 7.1, and $1_\lambda \in V_\lambda \otimes V_\lambda^* \subset L_\lambda^k \otimes DL_\lambda^k$ is the canonical $\mathfrak{g}$-invariant vector, induces an isomorphism of the spaces of coinvariants

$$\tau(C, V) \simeq \bigoplus_{\lambda \in P_+^k} \tau(C^\vee, V \otimes L_\lambda^k \otimes DL_\lambda^k)$$

with the modules $L_\lambda^k, DL_\lambda^k$ assigned to the points a', a'' respectively.

PROOF. The basic observation is that $\mathcal{O}(C - \vec{p}) = \{f \in \mathcal{O}(C^\vee - \vec{p}) \mid f(a') = f(a'')\}$. Therefore,

$$\mathfrak{g}(C - \vec{p}) = \{f \in \tilde{\Gamma} \mid (\gamma_{a'} \oplus \gamma_{a''})f \in \left(\widehat{\mathfrak{g}}^+ \oplus \widehat{\mathfrak{g}}^+ \oplus \Delta(\mathfrak{g})\right) \subset \widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}\} \tag{7.7.1}$$

where $\Delta(\mathfrak{g}) = \{x \oplus x\}, x \in \mathfrak{g}$, $\tilde{\Gamma} = \mathfrak{g}(C^\vee - \vec{p} - a' - a'')$.

Next, let us define the $U\widehat{\mathfrak{g}}_k \otimes U\widehat{\mathfrak{g}}_k$-module U as follows:

$$U = \operatorname{Ind}_{\tilde{U}}^{U\widehat{\mathfrak{g}}_k \otimes U\widehat{\mathfrak{g}}_k} \mathbb{C}\mathbf{1}$$

where $\tilde{U} \subset U_k\widehat{\mathfrak{g}} \otimes U_k\widehat{\mathfrak{g}}$ is the subalgebra generated by $\widehat{\mathfrak{g}}^+ \otimes 1, 1 \otimes \widehat{\mathfrak{g}}^+, x \otimes 1 + 1 \otimes x, x \in \mathfrak{g}$, which acts trivially on $\mathbb{C}$:

$$(\widehat{\mathfrak{g}}^+ \otimes 1)\mathbf{1} = (1 \otimes \widehat{\mathfrak{g}}^+)\mathbf{1} = (a \otimes 1 + 1 \otimes a)\mathbf{1} = 0. \tag{7.7.2}$$

(By Poincare-Birkhoff-Witt theorem, U is isomorphic to $U\mathfrak{g} \otimes (U\widehat{\mathfrak{g}}^-)^{\otimes 2}$ as a graded vector space.)

Since U is a $(U(\widehat{\mathfrak{g}})_k)^{\otimes 2}$-module, we can define the space of coinvariants $\tau(C^\vee, \vec{p} \cup a' \cup a'', V \otimes U)$.

LEMMA 7.7.2. *The map $v \mapsto v \otimes \mathbf{1}$ is an isomorphism $\tau(C, V) \xrightarrow{\sim} \tau(C^\vee, V \otimes U)$.*

The proof of this lemma is more or less standard: one has to check that this map is well-defined, which follows from (7.7.2); injectivity follows from the fact that U is free over $U\widehat{\mathfrak{g}}^- \otimes U\widehat{\mathfrak{g}}^-$. Proof of surjectivity is is only slightly more difficult: it suffices to prove that for every $v \in V, u \in u$ one can find $v' \in V$ such that $v \otimes u \equiv v' \otimes \mathbf{1} \mod \operatorname{Im} \ \tilde{\Gamma}$. It follows from the fact that for every $a \oplus b \in \widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}, u \in u$ there exists a function $f \in \Gamma$ such that $(\gamma_{q'} \oplus \gamma_{q''})(f)u = au$, and therefore $v \otimes (a \oplus b)u \equiv -(\gamma_{\vec{p}} f)v \otimes u$.

LEMMA 7.7.3. *Maximal integrable quotient of U is equal to $\bigoplus_{\lambda \in P_+^k} L_\lambda^k \otimes DL_\lambda^k$.*

Indeed, let us define the homomorphism of $(U(\widehat{\mathfrak{g}})_k)^{\otimes 2}$ modules $\pi : U \to \bigoplus L^k_\lambda \otimes DL^k_\lambda$ by $1 \mapsto \bigoplus 1_\lambda$ (since U is the induced module, this uniquely defines π). It is an easy exercise to show that $\bigoplus 1_\lambda$ is a cyclic vector in $\bigoplus L^k_\lambda \otimes DL^k_\lambda$ (with respect to the action of $\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}$), and therefore, the above map is surjective; thus, $\bigoplus L^k_\lambda \otimes DL^k_\lambda$ is indeed an integrable quotient of U. On the other hand, every integrable $(U(\widehat{\mathfrak{g}})_k)^{\otimes 2}$-module is of the form $\bigoplus_{\lambda,\mu \in P^k_+} N_{\lambda\mu} L^k_\lambda \boxtimes L^k_\mu$. Since U is generated by a vector 1 which is $\Delta(\mathfrak{g})$ invariant, it easily follows that any integrable quotient of U must have $N_{\lambda,\mu} \leq \delta_{\lambda,\mu^*}$. Details are left to the reader.

These two lemmas, combined with Lemma 7.3.3, give the proof of the theorem. □

7.8. Bundle of coinvariants for a singular family

In this section, we continue the study of coinvariants for singular curves. This time, we will consider a family of pointed curves C_S over a smooth base S such that C_s is stable and non-singular for $S \setminus D$, and C_s is a stable singular curve with one double point for $s \in D$, where D is a smooth divisor in S (without loss of generality we may assume that D is connected). As before, we assume that we have some integrable modules $V_1, \dots, V_n$ assigned to the marked points $p_1, \dots, p_n$. Then, by the construction of the previous sections, this data defines a vector bundle of coinvariants $\tau = \tau(C_S, \vec{p}, V)$ over $S \setminus D$.

Let us extend τ to the whole of S as an $\mathcal{O}$-module. Define the sheaf τ on S in the obvious way, as in Lemma 7.4.2. The restriction of this sheaf to $S \setminus D$ is lisse, and its fiber at a point $s \in D$ is the vector space $\tau(C_s, \vec{p}, V)$ which was discussed in the previous section. The same arguments as before show that τ_S is $\mathcal{O}_S$-coherent sheaf. The goal of this section is to prove the following theorem, which is the key step in proving the gluing axiom.

THEOREM 7.8.1. *Under the assumptions above, the sheaf τ_S is lisse.*

The remaining part of this section is devoted to the proof of this theorem. Note that by Theorem 7.4.1, the restriction of τ to $S \setminus D$ is lisse, so the only problem is analyzing the behavior of τ at D.

PROOF. The proof consists of several steps. The main idea is to use the results of the previous section, relating coinvariants for the singular fibers $C_s, s \in D$ with the coinvariants for nonsingular curve $C^\vee_s$ obtained by normalization of C_s, and extend it to an isomorphism of sheaves of coinvariants in some neighborhood of D. Unfortunately, it is impossible to do this directly: we can not extend $C^\vee$ to a family of nonsingular curves $C^\vee_S$ over S with a natural map $C^\vee_S \to C_S$. However, this becomes possible if instead of constructing a family over S we restrict ourselves to an infinitesimal neighborhood of D, as defined in Section 7.6, which is sufficient for our purposes. For simplicity, we will assume that S is a disk in the complex plane with coordinate q and $D = \{0\}$. The general case can be treated quite similarly; however, it is not even necessary to do that due to Lemma 6.3.13. We will choose coordinates t_1, t_2 in the neighborhood of the double point $a \in C_S$ such that $t_1 t_2 = q$ (this is always possible).

By Lemma 7.6.2, it suffices to prove that for every $n \geq 0$, the module $\tau^{(n)}$ over $\mathcal{O}^{(n)}_D$ defined by (7.6.4) for our family of curves is free of finite rank.

In order to prove that $\tau^{(n)}$ is free over $\mathcal{O}^{(n)}_D$, let us construct another family $C^\vee$ of curves over $D^{(n)}$. Namely, take $C^\vee_0$ to be the normalization of C_0; this is a nonsingular curve with the same marked points as C_0, plus two more marked points which we denote a', a''. The choice of local coordinates t_1, t_2 on C_S defines local coordinates t_1, t_2 in the neighborhood of $a' \in C^\vee$ (respectively, a'').

Now, let us define the sheaf $\mathcal{O}^{(n)}_{C^\vee}$ as follows. Let $U = C_0^\vee \setminus \{a', a''\} = C_0 \setminus \{a\}$. By definition, let $\mathcal{O}^{(n)}_{C^\vee}|_U = \mathcal{O}^{(n)}_C|_U$. To extend it to the points a', a'', define the stalks $\mathcal{O}^{(n)}_{a'} = \mathcal{O}(t_1) \otimes \mathcal{O}^{(n)}_D$, where $\mathcal{O}(t_1)$ is the ring of germs of analytic functions in t_1 in a neighborhood of $t_1 = 0$, and similarly for a''. Obviously, each $f \in \mathcal{O}^{(n)}_{a'}$ also defines a section of $\mathcal{O}^{(n)}_{C^\vee}|_U$ on some punctured neighborhood of a' by $t_1 \mapsto t_1, q \mapsto t_1 t_2$, and thus we can glue the sheaf $\mathcal{O}^{(n)}_{C^\vee}$ from its restriction to U and stalks at a', a''. This defines on $C^\vee$ a structure of a family of curves over $D^{(n)}$; this family is non-singular.

Now let us assign the modules $L^k_\lambda, DL^k_\lambda$ to the points a', a'' and take direct sum over all $\lambda \in P^k_+$. By Proposition 7.6.9, this defines a lisse module $\tau^{\vee(n)}$ over $\mathcal{O}^{(n)}_D$.

Proposition 7.8.2. *The map*

$$\begin{aligned} \phi\colon V^{(n)} &\to V^{(n)} \\ v &\mapsto \sum_{\lambda,i} q^{-\deg e_{\lambda,i}} v \otimes e_{\lambda,i} \otimes e^*_{\lambda,i}, \end{aligned} \tag{7.8.1}$$

where $e_{\lambda,i}$ is a homogeneous basis in L^k_λ, and $e^*_{\lambda,i}$ is the dual basis in DL^k_λ, induces an isomorphism of $\mathcal{O}^{(n)}_D$-modules $\tau^{(n)} \to \tau^{\vee(n)}$.

Proof. First of all, we have to check that this map descends to the bundle of coinvariants. To do this, note that it is immediate from the definition that we have an embedding $A : \mathcal{O}^{(n)}(C - p) \hookrightarrow \mathcal{O}^{(n)}(C^\vee - p - a' - a'')$. Near the double point this map is given by

$$\begin{aligned} \mathcal{O}^{(n)}(C - p) &\to \Big(\mathbb{C}((t_1))[[q]] \oplus \mathbb{C}((t_2))[[q]]\Big)/(q^{n+1}) \\ t_1^k t_2^l &\mapsto t_1^{k-l} q^l \oplus t_2^{l-k} q^k \end{aligned}$$

(compare with (7.6.3)). We leave it to the reader to check that in fact the image of this embedding is analytic functions.

It is also easy to show by explicit calculation that the vector

$$w_\lambda = \sum_i q^{-\deg e_{\lambda,i}} e_{\lambda,i} \otimes e^*_{\lambda,i} \in (L^k_\lambda \otimes DL^k_\lambda)^{(n)} \tag{7.8.2}$$

is invariant under the image of the embedding

$$\mathfrak{g}[[t_1, t_2]]/(t_1 t_2)^{n+1} \to \Big(\mathfrak{g}((t_1))[q] \oplus \mathfrak{g}((t_2))[q]\Big)/q^{n+1}.$$

Indeed, it suffices to show this for $xt_1^n t_2^m, x \in \mathfrak{g}$. In this case, it follows from the following sequence of identities:

$$\begin{aligned}
&(x[n-m]q^m \otimes 1 + 1 \otimes x[m-n]q^n)w_\lambda \\
&\qquad = (x[n-m]q^m \otimes 1 + 1 \otimes x[m-n]q^n)(q^{-d} \otimes 1)\sum_i e_{\lambda,i} \otimes e^*_{\lambda,i} \\
&\qquad = (q^{-d} \otimes 1)(x[n-m]q^n \otimes 1 + 1 \otimes x[m-n]q^n)\sum_i e_{\lambda,i} \otimes e^*_{\lambda,i} \\
&\qquad = (q^{-d+n} \otimes 1)(x[n-m] \otimes 1 + 1 \otimes x[n-m])1 \\
&\qquad = 0,
\end{aligned}$$

where $1 = \sum_i e_{\lambda,i} \otimes e^*_{\lambda,i}$ is considered as a vector in a certain completion of $L_\lambda^k \otimes (L_\lambda^k)^*$. Note that in the last line we replaced DL_λ^k by $(L_\lambda^k)^*$, which resulted in ?! replacing $x[m-n]$ by $x[n-m]$—see (7.1.4). We leave it to the reader to check that the fact that 1 does not lie in $L_\lambda^k \otimes (L_\lambda^k)^*$ but only in some completion does not cause any problems.

Therefore, if $f \in \mathfrak{g}^{(n)}(C_S - \vec{p}), v \in V$, then $\phi(f(v)) = A(f)\phi(v)$ and thus the map ϕ descends to the space of coinvariants; we will denote the corresponding map also by ϕ.

Now the proof of proposition is easy. Indeed, we have a morphism of $\mathcal{O}_D^{(n)}$-modules $\phi : \tau^{(n)} \to \tau^{\vee(n)}$. By Theorem 7.7.1, ϕ induces an isomorphism on the fibers at zero $\tau^{(n)}/q\tau^{(n)} \xrightarrow{\sim} \tau^{\vee(n)}/q\tau^{\vee(n)}$. Since $\tau^{\vee(n)}$ is free over $\mathcal{O}_D^{(n)}$, this immediately implies that ϕ is surjective. To prove that ϕ is injective, choose a basis $v_1, \dots, v_k$ in $\tau^{(n)}/q\tau^{(n)}$. Since $\tau^{\vee(n)}$ is free, this implies that $v_1, \dots, v_k$ are linearly independent over $\mathcal{O}_D^{(n)}$. On the other hand, it follows from the definition that the module $K = \tau^{(n)}/\langle v_1, \dots, v_k\rangle$ satisfies $qK = K$; since $q^{n+1} = 0$, this implies $K = 0$. Thus, $\tau^{(n)}$ is freely generated by $v_1, \dots, v_k$. Therefore, ϕ is an isomorphism, which completes the proof of the proposition. □

Since by Proposition 7.6.9 the sheaf $\tau^{\vee(n)}$ is lisse, this proposition implies that the same holds for $\tau^{(n)}$ and thus completes the proof of Theorem 7.8.1. □

7.9. Proof of the gluing axiom

In this section we give a proof of the gluing axiom for the WZW modular functor. Recall that this axiom describes the behaviour of the bundle of coinvariants in a neighborhood of the boundary of the moduli space; in particular, it claims that the connection has first regular singularities at the boundary, and describes the specialization of this connection.

Recall that the boundary of the moduli space consists of the stable curves with ordinary double points (see Section 6.2) and that it suffices to check the regularity condition for an open part of the boundary. Thus, we need to prove regularity and calculate specialization of the connection in τ_S, where $S, C_S, D, \dots$ are same as in the beginning of the previous section. By the construction of the previous sections, τ_S carries a natural projectively flat connection over $S \setminus D$. Also, we have shown in the previous section that τ_S is lisse, i.e., is a sheaf of sections of a vector bundle on S.

THEOREM 7.9.1. *Under the assumptions above, the connection in τ_S has logarithmic singularities at D.*

PROOF. As before, choose a local coordinate q in a neighborhood of D such that $q = 0$ is the equation of D. Recall (see (6.3.5)) that $\mathcal{D}_S^0 \subset \mathcal{D}_S$ be the subsheaf generated (as sheaf of algebras) by $\mathcal{O}_S$ and vector fields which are tangent to D.

PROPOSITION 7.9.2. *The sheaf τ has a natural structure of a $\mathcal{D}_S^0$-module.*

This proposition is a generalization of Theorem 7.4.1, and is proved in the same way. The only change is that instead of claiming that any vector field on S can be lifted to a vector field on $C_S - \vec{p}(S)$, we use the following lemma.

LEMMA 7.9.3. *Let θ be vector field on S which is tangent to D. Then locally in S, such a field can be lifted to a vector field on C_S which has poles at the marked points.*

EXAMPLE 7.9.4. Let S be a neighborhood of zero in $\mathbb{C}$, with coordinate q, $D = \{0\}$. As before, introduce coordinates t_1, t_2 near the double point in C_S such that $q = t_1 t_2$. Then in the neighborhood of the double point, the lifting of the vector field $q\partial_q$ must be of the form $\alpha t_1 \partial_{t_1} + \beta t_2 \partial_{t_2}$ for some α, β satisfying $\alpha + \beta = 1$.

This proposition, along with the fact that τ_S is lisse, immediately implies the statement of the theorem. □

EXAMPLE 7.9.5. Let S be a neighborhood of zero in $\mathbb{C}$, with coordinate q. Define the family $C_S \subset \mathbb{C}P^2 \times S$ by the equation

$$uv = qw^2, \qquad (u : v : w) \in \mathbb{C}P^2, \quad q \in S$$

with the marked points $p_1(q) = (1 : 0 : 0), p_2(q) = (0 : 1 : 0)$, and local parameters at these points $t_1 = w/u, t_2 = w/v$. The same argument as in Example 6.2.4 shows that for $q \neq 0$, the curve C_q is isomorphic to a sphere $\mathbb{P}^1$, with marked points $p_1 = 0, p_2 = \infty$ and local parameters $z, 1/z$ respectively. For $q = 0$, the fiber C_0 consists of two components, each of them isomorphic to a sphere $\mathbb{P}^1$, with coordinates $z' = u/w, z'' = v/w$ respectively, which have one common point $z' = z'' = 0$. The marked points p_1 and p_2 are the points ∞', ∞'' —infinite points of the first and the second spheres respectively, with local coordinates $t_1 = 1/z', t_2 = 1/z''$ respectively.

It is easy to see that any vector field of the form

$$\tilde{v} = \alpha u\partial_u + \beta v\partial_v + q\partial_q, \quad \alpha + \beta = 1$$

defines a vector field on C_S which is a lifting of the vector field $q\partial_q$ on S. Rewriting $\tilde{v}$ in terms of coordinates t_1, q, we get $\tilde{v} = -\alpha t_1 \partial_{t_1} + q\partial_q$, and thus $\gamma_{p_1}(\tilde{v}) = \alpha L_0$. Similarly, expansion near p_2 gives $\gamma_{p_2}(\tilde{v}) = \beta L_0$. Therefore, the action of $q\partial_q$ on coinvariants is given by $\alpha(L_0)_{p_1} + \beta(L_0)_{p_2}$.

This statement also has an infinitesimal analogue. Recall the notation $\tau^{(n)} = \tau_S/q^{n+1}\tau_S$ (see the previous section). This is a lisse $\mathcal{O}_D^{(n)}$-module. It immediately follows from Proposition 7.9.2 that $\tau^{(n)}$ has a natural action of the sheaf of algebras $\mathcal{D}^0_{D^{(n)}} = \mathcal{D}_S^0/q^{n+1}\mathcal{D}_S^0$.

Similar result also holds for the sheaf $\tau^{\vee(n)}$ described in the previous section: it follows from Proposition 7.6.9 that $\tau^{\vee(n)}$ has a natural structure of a projective $\mathcal{D}^0_{D^{(n)}}$-module. Let us twist this action, defining a new action of $q\partial_q$ by adding to the old action the constant Δ_λ, defined by (7.4.7) (cf. Example 7.4.11). We will denote this new action by $\nabla^\vee$.

Note that a lifting of the vector field $q\partial_q$ to $C^\vee_{D^{(n)}}$ can be explicilty described as follows: lift $q\partial_q$ to a derivation $\tilde{v}$ of $\mathcal{O}^{(n)}(C-\vec{p})$; as was discussed in Example 7.9.4, this lifting in a neighborhood of the double point has the form $\alpha t_1\partial_{t_1}+\beta t_2\partial_{t_2}, \alpha+\beta=1$. Define $v^\vee$ by $v^\vee=\tilde{v}$ on $C^\vee\setminus\{a',a''\}=C_0\setminus\{a\}$, and $v^\vee=\alpha t_1\partial_{t_1}+q\partial_q$ at a'; similarly, let $\tilde{v}=\beta t_2\partial_{t_2}+q\partial_q$ at a''. It is easy to check that this defines an element of $\Theta^{(n)}(C^\vee-\vec{p})$.

EXAMPLE 7.9.6. Under the assumptions of Example 7.9.5, the lifting of the vector field $q\partial_q$ is given by $v^\vee=\alpha z'\partial_{z'}+q\partial_q$ on the first component, and by $v^\vee=\beta z''\partial_{z''}+q\partial_q$ on the second one. Therefore, its action on the bundle of coinvariants is given by

$$\nabla^\vee_{q\partial_q}=q\partial_q+\alpha\big((L_0)_{p_1}-(L_0)_{a'}\big)+\beta\big((L_0)_{p_2}-(L_0)_{a''}\big)+\Delta_\lambda. \tag{7.9.1}$$

PROPOSITION 7.9.7. *The isomorphism $\phi\colon\tau^{(n)}\to\tau^{\vee(n)}$, defined by* (7.8.1)*, is an isomorphism of $\mathcal{D}^0_{D^{(n)}}$-modules.*

PROOF. It suffices to check that ϕ commutes with the action of the vector field $q\partial_q$. To prove this, it suffices to check that

$$\nabla^\vee_{q\partial_q}(v\otimes w_\lambda)=(\nabla_{q\partial_q}v)\otimes w_\lambda$$

where w_λ was defined in (7.8.2). But this is immediate from the definition of $\nabla^\vee$:

$$\begin{aligned}\nabla^\vee_{q\partial_q}(v\otimes w_\lambda)-(\nabla_{q\partial_q}v)\otimes w_\lambda&=v\otimes(q\partial_q-\alpha(L_0)_{a'}-\beta(L_0)_{a''}+\Delta_\lambda)w_\lambda\\&=v\otimes(-d+\Delta_\lambda-\alpha(L_0)_{a'}-\beta(L_0)_{a''})w_\lambda\\&=0.\end{aligned}$$

□

Now let us calculate the specialization of the connection in τ_S. Let us recall the definition of the specialization functor, slightly modifying it for our needs. As in Chapter 6, assume that (F,∇) is flat connection with first order poles at D. As before, we denote by $\mathcal{F}$ the sheaf of sections of F, and $\mathcal{F}^{(0)}=\mathcal{F}/q\mathcal{F}$. $\mathcal{F}^{(0)}$ is a sheaf on D which has a natural action of the sheaf of algebras $\mathcal{D}^{(0)}_D=\mathcal{D}^0_S/q\mathcal{D}^0_S$. It turns out that the specialization Sp_DF can be defined using only $\mathcal{F}^{(0)}$ as follows.

LEMMA 7.9.8. *Let $(G,\tilde{\nabla})$ be a vector bundle on the normal bundle ND with a monodromic $\log D$ flat connection, and let i be a homeomorphism identifying a neighborhood of D in S with a neighborhood of D in ND, as in* (6.2.8)*. Then an isomorphism of vector bundles with connections $Sp_DF\to G$ is the same as an isomorphism of $\mathcal{D}^{(0)}_S$-modules*

$$\mathcal{F}^{(0)}\to i_*\mathcal{G}^{(0)}. \tag{7.9.2}$$

As before, we leave the proof of this lemma to the reader.

Now we need to calculate the specialization of the vector bundle of coinvariants τ_S. To do so, recall first that by Lemma 6.2.5, the normal bundle to D is $ND=\{(d,v)\}, d\in D, v\in T^{(1)}_aC_d\otimes T^{(2)}_aC_d$, where C_d is the curve with one double point a, and $T^{(1)},T^{(2)}$ are the tangent spaces to the two components of C_d at a. Choice of coordinate q on S and coordinates t_1,t_2 on C_S such that $t_1t_2=q$ gives an identification of a neighborhood of D in S with a neighborhood of D in ND by

$$i\colon(d,q)\mapsto(d,q\partial_{t_1}\otimes\partial_{t_2}),$$

or, passing from vectors to covectors,

$$i\colon (d,q)\mapsto (d,\frac{dt_1\otimes dt_2}{q}). \tag{7.9.3}$$

Now, let us define a family of pointed curves over ND by $C_{d,q}=C_d^\vee$ with the parameters at a',a'' given by $t_1/q, t_2$. This defines a bundle of coinvariants $\tilde{\tau}$ on a neighborhood of D in S.

THEOREM 7.9.9. *The map*

$$\begin{aligned} \mathcal{O}_S\otimes V &\to \mathcal{O}_{ND}\otimes \sum_\lambda V\otimes L^k_\lambda\otimes DL^k_\lambda \\ f(s)v &\mapsto \sum_\lambda f(i(s))v\otimes w_\lambda \end{aligned} \tag{7.9.4}$$

where $1_\lambda\in V_\lambda\otimes V_\lambda^*\subset L^k_\lambda\otimes DL^k_\lambda$ *is the canonical* $\mathfrak{g}$*-invariant vector, gives rise to an isomorphism of* $\mathcal{D}_S^{(0)}$*-modules* $\tau_S^{(0)}\to\tilde{\tau}^{(0)}$.

PROOF. We will use as an intermediate step the sheaf $\tau^{\vee(0)}$ introduced in the previous section. By Proposition 7.9.7, the isomorphism $\phi\colon \tau^{(0)}\to\tau^{\vee(0)}$, defined by (7.8.1) is an isomorphism of $\mathcal{D}_D^{(0)}$-modules. On the other hand, let us show that the map $V\otimes L^k_\lambda\otimes DL^k_\lambda\to V\otimes L^k_\lambda\otimes DL^k_\lambda$, given by

$$v\otimes v'\otimes v''\mapsto q^{\deg v'}v\otimes v'\otimes v''$$

gives rise to an isomorphism of $\tau^{\vee(0)}$ and $\tilde{\tau}^{(0)}$ as $\mathcal{D}^0_D$-modules. Indeed, let us compare the action of the vector field $q\partial_q$ on both spaces. For $\tilde{\tau}^{(0)}$ it is given by $-(L_0)_{a'}$, and for $\tau^{\vee(0)}$, it is given by

$$\gamma_{a'}(v^\vee)+\gamma_{a'}(v^\vee)+\sum\gamma_{p_i}(v^\vee)+\Delta_\lambda.$$

It follows from Proposition 7.6.9 that the only non-zero term in this sum is Δ_λ, and therefore, (7.9.4) is indeed an isomorphism of modules.

Combining the isomorphisms $\tau^{(0)}\to\tau^{\vee(0)}\to\tilde{\tau}^{(0)}$, we get the statement of the theorem. □

Now we can prove the main result of this chapter.

THEOREM 7.9.10. *The sheaves of coinvariants* $\tau(C,\vec{p},V_i)$, $V_i\in\mathcal{O}_k^{\mathrm{int}}$, *form a modular functor with additive central charge* c.

PROOF. According to Definition 6.4.1, we need to define the gluing isomorphism and the vacuum propagation isomorphism for the spaces of coinvariants. Vacuum propagation isomorphism is given by Corollary 7.3.5; the gluing isomorphism is obtained by combining Lemma 7.9.8 and Theorem 7.9.9. Checking all the compatibility conditions for these isomorphisms is trivial. □

For technical reasons, it is more convenient to pass to the dual sheaf

$$\tau^*(C,\vec{p},V_i)=\big(\tau(C,\vec{p},DV_i)\big)^*.$$

Obviously, the previous theorem immediately implies that the sheaves $\tau^*(C,\vec{p},V_i)$ also form a modular functor with the additive central charge c. This functor will be called *Wess-Zumino-Witten modular functor.*

As a corollary, we have proved the theorem formulated in the introduction to this chapter.

COROLLARY 7.9.11. *The category $\mathcal{O}_k^{\mathrm{int}}$ has a structure of a modular tensor category, with $\mathbf{1} = L_0^k, \theta_V = e^{2\pi \mathrm{i} L_0}$, and the tensor product $\dot\otimes$ defined by*

$$\mathrm{Hom}_{\mathcal{O}_k^{\mathrm{int}}}(\mathbf{1}, V_1 \dot\otimes \dots \dot\otimes V_n) = \big(\tau(C, DV_1 \otimes \dots \otimes DV_n)\big)^*$$

where C is the "standard" n-punctured sphere, as in (6.4.3).

?! As a matter of fact, we have not yet proved the rigidity (recall that modular functor only defines weak rigidity); however, it can be shown that this category is indeed rigid.

A weaker version of this result is the following:

THEOREM 7.9.12. *Let $k \notin \mathbb{Q}$. Then the vector spaces of coinvariants $\tau(C, \vec{p}, V_{\vec\lambda}^k)$ define a genus zero modular functor. The corresponding ribbon category is the Drinfeld's category.*

PROOF. The proof is obtained by noticing that we have used integrability of L_λ^k only in two places: when checking finite-dimensionality of the spaces of coinvariants, and in the proof of Theorem 7.7.1, identifying the coinvariants for a singular curve C and its normalization $C^\vee$. On the other hand, if we restrict ourselves to genus zero curves, then the vector spaces of coinvariants are finite-dimensional by Proposition 7.3.8. It is also easy to show that the proof of Theorem 7.7.1 remains valid for $k \notin \mathbb{Q}$ if we replace $\bigoplus L_\lambda^k \otimes DL_\lambda^k$ by (infinite) sum $\bigoplus_{\lambda \in P_+} V_\lambda^k \otimes DV_\lambda^k$.

The fact that the corresponding category is exactly the Drinfeld's category follows from comparison of this modular functor with the modular functor defining Drinfeld category (see Proposition 6.5.4). Indeed, Proposition 7.3.8 shows that the corresponding vector spaces of conformal blocks can be identified, Theorem 7.4.10 shows that this identification preserves the flat connections, and Theorem 7.9.9 shows that the gluing map for these two modular functors also coincides. ☐

REMARK 7.9.13. One can note that we have most of the arguments above were quite general and didn't use much information about the coinvarints. Most of the time we were only using the action of the Virasoro algebra on integrable modules, given by the Sugawara construction. The only places were we actually used the definition of coinvariants and properties of integrable modules were the proof of finite-dimensionality of the vector spaces of coinvariants and the proof of Theorem 7.7.1, identifying the coinvariants for a singular curve C and its normalization $C^\vee$. Thus, if we could repeat these two steps in other setups—for example, replacing the category $\mathcal{O}_k^{\mathrm{int}}$ by a suitable category of Virasoro modules—we would again get a modular functor. Indeed, it is rather easy to modify these arguments to define the modular functor related to the so-called minimal models of Conformal Field Theory, in which the modules L_λ^k are replaced by irreducible unitary modules over Vir with a suitable central charge. If we try to pursue this idea as far as we can and see what is the most general situation in which we can apply the same proof, we will arrive at the notion of Rational Conformal Field Theory (or, to be more precise, the holomorphic (chiral) half of RCFT). The number of references on this subject is tremendous; some of the more suitable for mathematical audience are [**Hua**], influential but unpublished manuscript [**BFM**], and [**Gai**]. For more physical exposition and extra references, see [**FMS**].