CHAPTER 6

Moduli Spaces and Complex Modular Functor

In this chapter, we will rewrite the definition of modular functor in algebro-geometric terms, i.e., in terms of flat connections with regular singularities on the moduli spaces, instead of the topological surfaces and mapping class groups in Chapter 5. In fact, this is how the modular functor originally appeared in conformal field theory, see, e.g, [**MS1**], [**S**]. The exposition in this chapter is based on the unpublished manuscript [**BFM**]; similar ideas were also introduced in Deligne's letter to Drinfeld.

The complex version of modular functor is best formulated using the language of connections with regular singularities on the Deligne–Mumford compactification of the moduli space of complex curves. For readers' convenience, we give a short introduction to the theory of moduli spaces and connections with regular singularities.

In this chapter, "complex curve" means "complex projective curve" (thus, it is compact); unless stated otherwise, the curves are assumed to be connected and non-singular. We remind that by Riemann's theorem, every (non-singular) compact Riemann surface is projective. However, unless otherwise specified, we will consider all manifolds with analytic topology. We assume that the reader is familiar with some basic notions of algebraic geometry, such as coherent $\mathcal{O}$-modules, vector bundles, etc.; all the necessary prerequisites can be found, for example, in [**GH**].

6.1. Moduli spaces and complex Teichmüller tower

In this section, we give a definition of the Teichmüller tower of groupoids in terms of moduli spaces of complex curves. Let us first recall the relation between the moduli space and the mapping class group.

Let Σ be a compact oriented topological surface without boundary. A *complex structure* on Σ is an isomorphism class of pairs (C, φ) where C is a smooth compact complex curve and $\varphi\colon C \xrightarrow{\sim} \Sigma$ is a homeomorphism preserving orientation. Equivalently, a complex structure on a smooth surface can be defined as a polarization of the complexified tangent space, i.e., a one-dimensional complex vector sub-bundle $T^{\mathbb{C}}\Sigma \subset (T^{\mathbb{R}}\Sigma) \otimes_{\mathbb{R}} \mathbb{C}$ such that $(T^{\mathbb{R}}\Sigma) \otimes_{\mathbb{R}} \mathbb{C} = T^{\mathbb{C}} \oplus \overline{T^{\mathbb{C}}}$.

We identify two complex structures on Σ if they can be obtained one from another by an isotopy of Σ; in other words, we let $(C, \varphi) \simeq (C', \varphi')$ if there exists a commutative square

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & \Sigma \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ C' & \xrightarrow{\varphi'} & \Sigma \end{array}$$

where f is an isomorphism of complex varieties, and g is an automorphism of Σ which is homotopic to identity. The set of all complex structures on Σ up to isotopy is called the *Teichmüller space* and will be denoted by $T(\Sigma)$. For a connected surface of genus g, we will also use the notation T_g.

Denote by $\mathcal{M}_g$ the set of isomorphism classes of complex curves of genus g. It is well known that this set has a natural structure of an analytic variety. We will call $\mathcal{M}_g$ the *moduli space* of curves of genus g (to be more precise, it is a *coarse* moduli space in the terminology of Mumford—see Theorem 6.1.8). The following result immediately follows from the definitions.

PROPOSITION 6.1.1. *The moduli space $\mathcal{M}_g$ is isomorphic to T_g/Γ_g, where Γ_g is the mapping class group of a surface of genus g.*

The next result is classical, see, e.g., [**Ab**].

THEOREM 6.1.2 (Teichmüller). *The set T_g of all complex structures (up to isotopy) on a connected surface Σ of genus g has a natural structure of a complex analytic manifold such that the action of Γ_g is holomorphic. In particular, this gives a structure of an analytic variety on $\mathcal{M}_g$.*

As a real *analytic manifold, T_g is isomorphic to $\mathbb{R}^{6g-g}$ for $g > 1$.*

Note that $T_g \not\simeq \mathbb{C}^{3g-3}$ as a complex analytic manifold.

EXAMPLE 6.1.3. For Σ of genus 1, i.e., a torus

$$T_1 \simeq \mathcal{H} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} \simeq \mathbb{R}^2.$$

Then $\mathcal{M}_1 = \mathcal{H}/\mathrm{SL}_2(\mathbb{Z}) \simeq \mathbb{C}$.

The above results can be generalized to surfaces with marked points.

DEFINITION 6.1.4. A *pointed curve* is a complex curve C with an ordered set of marked points $y_1, \dots, y_n \in C$ and with a non-zero tangent vector v_i given at each point.

Note that choosing a non-zero tangent vector is equivalent to choosing a non-zero cotangent vector: it can be defined by $\langle v_i, v_i^* \rangle = 1, v_i \in T_{y_i}C, v_i^* \in T^*_{y_i}C$.

One defines isomorphism of pointed curves in an obvious way. Let us denote

$$\begin{aligned} \mathcal{M}_{g,n} = &\text{the set of isomorphism classes of pointed curves} \\ &\text{of genus } g \text{ with } n \text{ marked points.} \end{aligned} \tag{6.1.1}$$

As before, we will call $\mathcal{M}_{g,n}$ the *moduli space of pointed curves.*

REMARK 6.1.5. This moduli space is different from the moduli space considered in [**Kn**]. The latter space, which we will denote $\mathcal{M}_g^n$, is defined as the set of isomorphism classes of curves of genus g with n marked points, but without tangent vectors. However, they are closely related: for example, if g, n are such that there are no non-trivial automorphisms of a n-pointed genus g curve, then $\mathcal{M}_{g,n}$ is a $(\mathbb{C}^\times)^n$ bundle over $\mathcal{M}_g^n$, so all the results of [**Kn**] can be easily reformulated for $\mathcal{M}_{g,n}$. One can also define more general moduli spaces $\mathcal{M}_{g,r}^n$ in an obvious way; they will not be used in our work.

Let us define the *Teichmüller space* $T_{g,n}$ to be the set of all complex structures on a surface Σ of genus g with n marked points and tangent vectors up to an isotopy which fixes the marked points and vectors. This space has a natural structure of an analytic manifold. Then the previous results can be generalized as follows:

THEOREM 6.1.6. (i) *The Teichmüller space $T_{g,n}$ is contractible.*

(ii) *Let $\Gamma'_{g,n} \subset \Gamma_{g,n}$ be the group of automorphisms of an extended topological surface of genus g with n boundary components which act trivially on the set of boundary components. Then this group acts holomorphically on $T_{g,n}$, and the stabilizer of every point is finite.*

(iii) *As a complex variety, $\mathcal{M}_{g,n} \simeq T_{g,n}/\Gamma'_{g,n}$. In particular, $\mathcal{M}_{g,n}$ is connected.*

REMARK 6.1.7. In fact, it is shown in [**DM**], [**Kn**], that $\mathcal{M}_{g,n}$ is an irreducible quasiprojective algebraic variety over $\mathbb{C}$—this is a difficult theorem.

If the action of $\Gamma'_{g,n}$ on the Teichmüller space $T_{g,n}$ were free, then $\pi_1(\mathcal{M}_{g,n})$ would be equal to $\Gamma'_{g,n}$. Unfortunately, the action of $\Gamma'_{g,n}$ is not free: the stabilizer of a point coincides with the group of automorphisms of the corresponding complex curve. Therefore, in general $\pi_1(\mathcal{M}_{g,n}) \neq \Gamma'_{g,n}$, as can be seen already for $g = 1$: in this case, $\pi_1(\mathcal{M}_{1,0}) = \{1\}$, while $\Gamma_{0,1} \simeq \mathrm{SL}_2(\mathbb{Z})$.

Now, let us discuss in what sense $\mathcal{M}_{g,n}$ is the moduli space of curves. Let us recall (see, e.g., [**Ha**]) that a family of curves C over a smooth variety U by definition is a variety C_U with a proper flat morphism $\pi\colon C_U \to U$, such that $\pi^{-1}(t) = C_t$ is a compact complex curve (unless specified otherwise, we will assume that the fibers are connected). Note that $\pi^{-1}(t)$ can be singular even if C_U is smooth, as shown by the example of the surface $xy = tu$ in $\mathbb{P}^4$. Similarly, a family of pointed curves is a family C_U together with n non-intersecting sections $p_i\colon U \to C_U$ and a non-vanishing vertical vector field v_i on $p_i(U)$ (vertical means that $\pi_*(v_i) = 0$).

THEOREM 6.1.8. *$\mathcal{M}_{g,n}$ is the* coarse moduli space *of curves in the sense of* [**MFK**]: *for every family of pointed curves C_U over U, the induced map $U \to \mathcal{M}_{g,n}$, $t \mapsto [C_t]$, is analytic.* (*Here $[C]$ denotes the isomorphism class of a curve C.*)

Unfortunately, it is not true that the construction above gives a bijection

$$\{\text{families of curves over } U\} \xrightarrow{\sim} \{\text{maps } U \to \mathcal{M}_{g,n}\};$$

in other words, $\mathcal{M}_{g,n}$ is not the fine moduli space. The reason for the failure is that $\mathcal{M}_{g,n}$ carries no information about the automorphisms of a curve.

EXERCISE 6.1.9. Let C be a pointed curve, and σ—a non-trivial automorphism of C. Construct a family of curves C_t over $\mathbb{C}^\times$ such that $C_t \simeq C$ for any t, but this family is not isomorphic to the direct product $C \times \mathbb{C}^\times$.

It turns out that this was the only problem: if we assume that the curves have no non-trivial automorphisms, then $\mathcal{M}_{g,n}$ is the fine moduli space.

THEOREM 6.1.10. *Assume that $g > 0, n > 0$ or $G = 0, n > 1$. Then:*

1. *For every complex curve C of genus g with n marked point, the group of automorphisms is trivial.*
2. *The action of the group $\Gamma'_{g,n}$ on the corresponding Teichmüller space is free, so $\mathcal{M}_{g,n} = T_{g,n}/\Gamma'_{g,n}$ is smooth.*
3. *$\mathcal{M}_{g,n}$ is the fine moduli space: for every variety S, the functors*

$$S \mapsto \textit{families of curves of genus } g \textit{ with } n \textit{ marked points over } S$$

and

$$S \mapsto \mathrm{Mor}(S, \mathcal{M}_{g,n})$$

are canonically isomorphic. In other words, every family of curves on S can be obtained as a pull-back of a universal family over $\mathcal{M}_{g,n}$ for a unique map $\psi\colon S \to \mathcal{M}_{g,n}$.

It turns out that one can also define a suitable "fine moduli space" $M_{g,n}$ under less restrictive assumptions that (g,n) is stable, i.e.

$$(g,n) \neq (0,0),(0,1),(1,0). \tag{6.1.2}$$

In this case, the group of automorphisms of every curve $C \in \mathcal{M}_{g,n}$ is finite. It turns out that under this assumption, it is possible to account for these automorphisms and define a "fine" moduli space, if we allow the moduli space to be not a variety, but a *stack*, as defined in [**DM**], [**Ar**]. Intuitively, this means that every point of $\mathcal{M}_{g,n}$ has some additional structure, which encodes the group of automorphisms of the corresponding curve. Unfortunately, an accurate exposition of the theory of algebraic stacks goes far beyond the scope of this book; we can only refer the reader to the Appendix to [**Vi**] for an introduction to this language. Another approach, which applies if g,n are such that the generic curve $C \in \mathcal{M}_{g,n}$ has no automorphisms, is to consider $\mathcal{M}_{g,n}$ as an *orbifold*, or *V-variety* (see [**Sat**]). Finally, the third possibility, used in [**TUY**], is to consider "local universal families of curves", which can be viewed as local charts of the algebraic stack. For our purposes, we can use any of these approaches: all of them will yield the same results, and each has its own advantages and disadvantages. We chose to use the language of algebraic stacks.

We will denote by $M_{g,n}$ the *moduli stack* of pointed curves of genus g with n marked points; as we said, we will not explain what it is, referring the reader to [**DM**] instead. Nevertheless, we can say what are points, vector bundles, etc., on $M_{g,n}$. Namely: for a complex manifold S, a morphism $S \to M_{g,n}$ is by definition the same as a family of pointed complex curves of genus g with n marked points over S. In particular, this implies that the set of (closed) points of $M_{g,n}$ is $\mathcal{M}_{g,n}$. Similarly, a vector bundle E on $M_{g,n}$ is the same as a collection of vector bundles ϕ^*E for every morphism $\phi : S \to M_{g,n}$, such that this collection is functorial in S. Local systems, flat connections, etc., can be defined in a similar way. We can also define a divisor $D \subset M_{g,n}$ as a compatible collection of divisors $\phi^*D \subset S$ for every étale (i.e., finite unramified covering) $\phi : S \to M_{g,n}$. Finally, we define the fundamental group of $M_{g,n}$ by

$$\pi_1(M_{g,n}, C) = \{(C_t)_{0\le t\le 1}, \varphi_0 : C_0 \xrightarrow{\sim} C, \varphi_1 : C_1 \xrightarrow{\sim} C\}/\text{homotopy}.$$

Here C_t is a C^∞ family of complex curves, i.e., a C^∞ real manifold Σ with a map $\pi\colon \Sigma \to [0,1]$ such that $d\pi \neq 0$, and for every $t \in \mathbb{R}$, $C_t = \pi^{-1}(t)$ is a smooth compact oriented surface, and with a family of complex structures μ_t in C_t such that μ_t is a C^∞ function of t (this should be modified in an obvious way for pointed curves). Later we will show that in fact, $\pi_1(M_{g,n}) = \Gamma'_{g,n}$.

Of course, we are just hiding the real problem: why so defined $M_{g,n}$ is a reasonable geometric object, i.e., why the standard results about, say, sheaves on varieties apply to $M_{g,n}$? This is indeed a difficult question, and the best we can do here is to refer to [**DM**]. Their results show that as far as we are concerned, $M_{g,n}$ can be treated in the same way as a non-singular variety: all the standard results from algebraic geometry we will be using apply to $M_{g,n}$.

As was mentioned above, for $g > 0, n > 0$ or $g = 0, n > 1$, $\mathcal{M}_{g,n}$ itself is the fine moduli space, so in this case we have $\mathcal{M}_{g,n} = M_{g,n}$. In general, this is not true.

From now on, we will use $M_{g,n}$ as the moduli space, and all geometric constructions will be understood in the stack sense. A reader who is not too experienced in this language can just think of $M_{g,n}$ as a smooth manifold.

Sometimes, it is convenient to define a slightly different space. Let A be a finite set. Denote $\mathcal{M}_{g,A} = \{(C, f)\}$, where C is a genus g complex curve with n **unordered** marked points and non-zero tangent vectors, and f is a bijection ?! $A(C) \xrightarrow{\sim} A$, where $n = |A|$, and $A(C)$ is the set of marked points of C. In other words, $\mathcal{M}_{g,A}$ is the moduli space of curves of genus g with n marked points labeled by elements of A, and with non-zero tangent vectors at these points. Obviously, for $A = \{1, \dots, n\}$, this coincides with the definition of $\mathcal{M}_{g,n}$. One defines the stack $M_{g,A}$ in a similar way.

We will also consider the moduli space $\mathcal{M}_{*,n}$ of not necessarily connected n-pointed curves, and the space $\mathcal{M} = \sqcup_{n \geq 0} \mathcal{M}_{*,n}$. One easily sees that $\mathcal{M}_{*,n}$ is an unramified finite cover over the space

$$\sqcup(\mathcal{M}_{g_1,n_1} \times \cdots \times \mathcal{M}_{g_k,n_k}),$$

where the disjoint union is taken over all finite sequences $(g_1, n_1), \dots, (g_k, n_k)$ such that $\sum n_i = n$, up to permutation. Thus, we can easily define the stacks $M_{*,n}, M_{*,A}$. The last stack will be frequently used later, so we state its definition explicitly:

$$\begin{aligned} M_{*,A} = &\text{moduli stack of stable smooth possibly disconnected curves} \\ &\text{with unordered marked points and non-zero tangent vectors} \\ &\text{and a bijection (marked points)} \xrightarrow{\sim} A \end{aligned} \tag{6.1.3}$$

Recall that we have defined the notion of a tower of groupoids, which is just a groupoid Γ with a functor $A\colon \Gamma \to \mathcal{S}ets$ and with the functors of disjoint union and gluing (see Definition 5.6.1). In particular, we have defined the Teichmüller tower $\mathcal{T}eich$, in which the objects are (topological) surfaces with boundary, morphisms are homeomorphisms of surfaces, and gluing is the gluing of two boundary components (see Definition 5.1.7, Section 5.6). The formula $\pi_1(M_{g,n}) = \Gamma'_{g,n}$, which is an obvious corollary of Theorem 6.1.6 if the action of $\Gamma'_{g,n}$ is free, suggests that the same tower can be defined in terms of the moduli spaces $M_{g,n}$.

Recall that for a topological space M, its *Poincaré groupoid* (also called the *fundamental groupoid*) is defined as the groupoid with objects: points of M, and morphisms: homotopy classes of paths in M connecting two points.

DEFINITION 6.1.11. The *complex Teichmüller tower of groupoids* $\mathcal{T}eich^{\mathbb{C}}$ is the fundamental groupoid of the stack M, i.e.,

$$\operatorname{Ob} \mathcal{T}eich^{\mathbb{C}} = \text{pointed complex curves}$$

and

$$\operatorname{Mor}(C', C'') = \{(C_t)_{0 \leq t \leq 1}, \varphi_0 \colon C_0 \xrightarrow{\sim} C', \varphi_1 \colon C_1 \xrightarrow{\sim} C''\}/\text{homotopy},$$

where, as before, C_t is a C^∞ family of pointed curves. We also define the functor $A\colon \mathcal{T}eich^{\mathbb{C}} \to \mathcal{S}ets$ by $A(C, y_i, v_i) = \{y_i\}$, and the disjoint union and empty set in an obvious way.

In a similar way, for a finite set A we define the groupoid $\mathcal{T}eich^{\mathbb{C}}_A$ as the fundamental groupoid of the stack $M_{*,A}$.

Note that in particular, for every curve C we have a canonical map $\operatorname{Aut} C \to \operatorname{Mor}(C, C)$, which assigns to $\sigma \in \operatorname{Aut} C$ the data $C_t = C \times [0,1], \varphi_0 = \mathrm{id}, \varphi_1 = \sigma$, which explains why we introduced φ_0, φ_1 in the definition.

To complete the definition, we also have to define the gluing functor. This will be done in the next section.

Now we can compare this complex Teichmüller groupoid with the groupoid $\mathcal{T}eich$ defined in the previous chapter in terms of topological surfaces with boundary. Note, however, that since we have imposed the stability condition (6.1.2), it only makes sense to compare $\mathcal{T}eich^{\mathbb{C}}$ with the subgroupoid $\mathcal{T}eich^{stab} \subset \mathcal{T}eich$, formed by topological surfaces all connected components of which satisfy the stability condition (6.1.2).

THEOREM 6.1.12. *The towers of groupoids $\mathcal{T}eich^{stab}$ and $\mathcal{T}eich^{\mathbb{C}}$ are equivalent. In particular, $\pi_1(M_{g,n}) = \Gamma'_{g,n}$.*

PROOF. The proof essentially repeats the proof of the fact that for a simply-connected T, one has $\pi_1(T/\Gamma) = \Gamma$.

First we construct a functor $\mathcal{T}eich^{stab} \to \mathcal{T}eich^{\mathbb{C}}$ as follows. Let Σ be an object of $\mathcal{T}eich^{stab}$, i.e., an extended surface. We will use Definition 5.1.10 of extended surface; thus, Σ is a topological surface with marked points and non-zero tangent vectors at these points. Fix a complex structure μ on the surface Σ. Let C_μ be the complex curve obtained from Σ with the complex structure μ. It is a pointed curve, with the same marked points and tangent vectors as Σ (recall that a complex structure defines an $\mathbb{R}$-linear isomorphism of the real tangent space $T_p^{\mathbb{R}}\Sigma$ and the complex tangent space $T_p^{\mathbb{C}}C$; for example, for $\Sigma = \mathbb{R}^2$ and the standard complex structure, we get $\partial_x \mapsto \partial_z, \partial_y \mapsto \mathrm{i}\partial_z$). This construction depends on the choice of μ. By Theorem 6.1.6, the set $T(\Sigma)$ of all complex structures on Σ is contractible. Therefore, every two complex structures μ_0, μ_1 can be connected by a unique path μ_t in $T(\Sigma)$. This gives a canonical family of curves C_{μ_t} connecting C_{μ_0} with C_{μ_1}, or a canonical morphism $C_{\mu_0} \to C_{\mu_1}$ in $\mathcal{T}eich^{\mathbb{C}}$. Thus, we have assigned to a topological surface C a collection of objects $C_\mu \in \mathcal{T}eich^{\mathbb{C}}$, canonically isomorphic to each other. As was discussed before (see Definition 1.1.11, Lemma 1.1.12), such a collection can be viewed as an object of $\mathcal{T}eich^{\mathbb{C}}$.

This defines the functor $\mathcal{T}eich^{stab} \to \mathcal{T}eich^{\mathbb{C}}$ on the objects of $\mathcal{T}eich^{stab}$. To define it on morphisms, we note that any homeomorphism of extended surfaces $f\colon \Sigma \xrightarrow{\sim} \Sigma'$ gives an identification of the Teichmüller spaces $f_*\colon T(\Sigma) \xrightarrow{\sim} T(\Sigma')$. Thus, for any $\mu \in T(\Sigma), \mu' \in T(\Sigma')$ there is a unique path connecting $f_*\mu$ with μ' in $T(\Sigma')$, and thus, a unique path connecting C_μ with $C_{\mu'}$ in the moduli stack M.

The inverse functor $\mathcal{T}eich^{\mathbb{C}} \to \mathcal{T}eich^{stab}$ is constructed as follows. On objects, it is just the forgetful functor, which assigns to a complex curve C the underlying topological surface. To define it on morphisms, let $C_t, t \in [0,1]$ be a family of curves. As before, let us forget the complex structure and view it as a family of extended surfaces. Then each of the surfaces Σ_t is homeomorphic to Σ_0, and the homeomorphism is unique if we additionally require that it depends continuously on t (this follows from the discreteness of the mapping class group). This gives a family of homeomorphisms $\varphi_t\colon \Sigma_0 \xrightarrow{\sim} \Sigma_t$. In particular, this defines $\varphi_1\colon \Sigma_0 \to \Sigma_1$.

It is easy to check that the above two functors are inverse to each other and are compatible with gluing (see the next section). $\square$

REMARK 6.1.13. Sometimes we will use an alternative definition of pointed curve. Recall that extended surface can be defined in any of the following three ways: 1) as a surface with boundary and a point on each boundary component; 2) as a surface with boundary and a parametrization of every boundary component; 3) as a surface without boundary but with marked points and non-zero tangent vectors. All these definitions give rise to equivalent groupoids (see Proposition 5.1.8).

Similarly, in the complex situation we can use the following definition of pointed curve: a pointed curve is a complex curve with marked points y_i and a local parameter z_i near each of these points. The corresponding moduli space (which is infinite-dimensional) will be denoted $\mathcal{M}_{g,n}^{(\infty)}$; similarly, one defines $\mathcal{M}^{(\infty)}$, and the groupoid $\mathcal{T}eich^{\mathbb{C}(\infty)}$. One has an obvious forgetting functor $\mathcal{T}eich^{\mathbb{C}(\infty)} \to \mathcal{T}eich^{\mathbb{C}}\colon (C, y_i, z_i) \mapsto (C, y_i, v_i)$, where the vector v_i is defined by $\langle v_i, dz_i\rangle = 1$. Since the set $\{f = z + \sum_{n>1} a_n z^n \mid f \text{ converges in a neighborhood of } 0\}$ is contractible, this functor is an equivalence. Therefore, we can use either definition of the Teichmüller groupoid.

Finally, there exists yet one more definition: a pointed curve is a topological surface Σ with a boundary, with a complex structure μ and with parametrizations $\pi_i\colon (\partial\Sigma)_i \to S^1$ of the boundary components which are analytic with respect to the complex structure μ. We leave it to the reader to check that this definition is equivalent to the two previous ones.

6.2. Compactification of the moduli space and gluing

In this section, we will define the gluing functor for the complex Teichmüller groupoid. A straightforward approach would be to cut from a curve small disks around the marked points, and glue the boundary circles together (this was first suggested by Vafa, see [**V1**]). However, there is a much better way of defining the gluing, which uses the so-called Deligne–Mumford compactification of the moduli space.

Following [**DM**], let us call a possibly singular complex curve C *stable* if its only singularities are ordinary double points, and its group of automorphisms is finite. (We recall that $p \in C$ is called an *ordinary double point* if locally C is isomorphic to the coordinate cross $xy = 0$ in $\mathbb{C}^2$ with $p = (0,0)$. Such a singularity is also called a *node*, or a *quadratic singularity*.) For such a curve C, one can define its genus g as the genus of the smooth curve obtained by deforming away all double points; in local coordinates, this can be described by replacing the equation $xy = 0$ by $xy = \varepsilon$ (for readers familiar with algebraic geometry, we note that so defined genus coincides with the "arithmetic genus" of C). Thus, we can consider the set $\overline{\mathcal{M}}_g$ of all isomorphism classes of connected stable curves of genus g. Similarly, one can define stable pointed curves (the marked points must be non-singular, i.e., they can not coincide with the double points), and define the space $\overline{\mathcal{M}}_{g,n}$ of isomorphism classes of such curves. This moduli space (or, rather, the analogous space $\overline{\mathcal{M}}_g^n$—see Remark 6.1.5) was considered by Deligne and Mumford [**DM**] and Knudsen [**Kn**]. Note that for non-singular curves, the stability condition automatically follows from the condition (6.1.2) which we imposed in the previous section.

As before, we can also define the corresponding stack $\overline{M}_{g,n}$ by $\mathrm{Mor}(S, \overline{M}_{g,n}) = \{\text{families of (possibly singular) stable pointed curves over } S\}$.

THEOREM 6.2.1 ([**DM**], [**Kn**]). (i) *$\overline{M}_{g,n}$ is a stack in the sense of* [**DM**].
(ii) *$\overline{M}_{g,n}$ is connected.*

(iii) *The complement* $D = \overline{M}_{g,n} \setminus M_{g,n}$ *is a divisor with normal crossings in* $\overline{M}_{g,n}$ *in the stack sense.* (*Often* D *is also called the* boundary, *or* points at infinity, *of the moduli stack* $M_{g,n}$.)

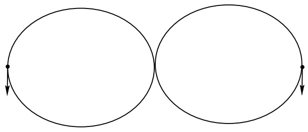
Let us recall that a *divisor with normal crossings* $D \subset M$, where M is an analytic manifold, is a union of a finite number of smooth codimension 1 subvarieties D_i, such that if p lies in the intersection of k components, then one can introduce local coordinates $z_1, \dots, z_n$ ($n = \dim M$) near p such that D is given by the equation $z_1 \cdots z_k = 0$. This definition should be properly modified for stacks.

As far as we are concerned, part (i) of the theorem could read as follows: "most standard results about smooth manifolds apply to $\overline{M}_{g,n}$".

Remark 6.2.2. As in Remark 6.1.5, we note that in [**DM**], [**Kn**], a different space is considered—they consider the space $\overline{\mathcal{M}}_g^n$, which is a compactification of the moduli space $\mathcal{M}_g^n$. It is shown in [**Kn**] that $\overline{\mathcal{M}}_g^n$ is a projective variety; in particular, it is compact. Our space $\overline{\mathcal{M}}_{g,n}$ is not compact for obvious reasons: the tangent vectors lie in a punctured affine space, which is not compact. However, other results from [**DM**], [**Kn**],—most importantly, the fact that $D = \overline{M}_{g,n} \setminus M_{g,n}$ is a divisor with normal crossings—can be easily generalized to our situation. One could further extend $\overline{\mathcal{M}}_{g,n}$, replacing $T_p^\times(C)$ by a projective space. However, this is not necessary for our purposes.

The stack $\overline{M}_{g,n}$ is called the *Deligne–Mumford compactification* of the moduli space (warning: it is not compact unless $n = 0$—see the remark above).

Example 6.2.3. Let $g = 0, n = 2$. Then one easily sees that $M_{0,2} = \mathcal{M}_{0,2} = \mathbb{C}^\times$, and $\overline{\mathcal{M}}_{0,2} = \overline{M}_{0,2} = \mathbb{C}$. The "infinite point" is the singular curve shown below.



Note that every divisor with normal crossings is naturally stratified: its codimension k (in D) stratum D^k consists of points which lie in the intersection of $k+1$ components. In particular, the set of non-singular points of D is given by

$$D^0 = \sqcup_i D_i^0, \tag{6.2.1}$$

where $D_i^0 = D_i \setminus (\cup_{j \neq i} D_j)$, D_i being the components of D.

For stacks, this definition should be suitably modified. It can be shown that for $\overline{M}_{g,n}$ the stratification of D is given by

$$D^k = \{\text{curves with exactly } k+1 \text{ double points}\}. \tag{6.2.2}$$

In particular, the open stratum $D^0 \subset D$ consists of the curves with exactly one double point. Every such curve is obtained by identifying two distinct points of a stable non-singular curve $C^\vee$ (*normalization* of C). In other words, if we denote by $M^2_{*,A}$ the set of stable possibly not connected curves with $|A| + 2$ marked points, out of which all but two are labeled by elements of A and have non-zero tangent vector, and the remaining two are not labeled or ordered and have no tangent vector assigned, then we have a natural isomorphism $S\colon M^2_{*,A} \xrightarrow{\sim} D^0$. This shows that the components of D are in bijection with the components of $M^2_{*,A}$, which are easy to describe. For example, for $g = 0$, we get the following result: the irreducible components of $D \subset \overline{M}_{0,A}$ are given by

$$\textit{Irr } D \leftrightarrow \{(A', A'') \mid A', A'' \subset A, A = A' \sqcup A'', |A'|, |A''| \geq 1\} \tag{6.2.3}$$

(here (A', A'') is an unordered pair). The corresponding component of D is defined by

$$D^0_{A',A''} = S(M^1_{0,A'} \times M^1_{0,A''});$$

in other words, these are curves which can be obtained by identifying a point on $C^{(1)} \in M_{0,A'}$ with a point on $C^{(2)} \in M_{0,A''}$, as in Example 6.2.3.

Let us show how such a curve can be obtained as a limit of a family of non-singular curves.

EXAMPLE 6.2.4. Let $z_1, \dots, z_n \in \mathbb{C}$, $v_1, \dots, v_n \in \mathbb{C}^\times$ be such that $z_i \neq z_j$. Denote

$$\begin{aligned}(\mathbb{P}^1; z_1, \dots, z_n; v_1, \dots, v_n) = {}& \text{the projective line } \mathbb{P}^1\\ & \text{with the standard coordinate } z \in \mathbb{C} \cup \infty,\\ & \text{with marked points } z = z_i \text{ and tangent vectors } v_i \partial_z.\end{aligned} \tag{6.2.4}$$

Obviously, any curve $C \in M_{0,n}$ can be written in such form.

Now, choose $z'_1 \dots, z'_k, a, z''_1, \dots, z''_m$ such that $z'_i \neq z'_j, z'_i \neq a, z''_i \neq z''_j$. Choose $q \in \mathbb{C}^\times$ small enough and define the curve C_q by

$$C_t = (\mathbb{P}^1; z'_1, \dots, z'_k, a + qz''_1, \dots, a + qz''_m; v'_1, \dots, v'_k, qv''_1, \dots, qv''_m).$$

Then we claim that the limit $C_0 = \lim_{q \to 0} C_q$ exists in $\overline{M}_{0,k+m}$, and is given by the singular curve obtained by identifying the point $a \in C^{(1)}$ with $\infty \in C^{(2)}$, where

$$\begin{aligned}C^{(1)} &= (\mathbb{P}^1; z'_1, \dots, z'_k; v'_1, \dots, v'_k),\\ C^{(2)} &= (\mathbb{P}^1; z''_1, \dots, z''_m; v''_1, \dots, v''_m).\end{aligned}$$

Speaking informally, one can say that as $q \to 0$, the curve C_q looks to a bare eye as $C^{(1)}$ and m points $a + qz''_i$ all collapsed at a; looking at the point a with a microscope, one can separate these m points and see that their relative position is described by $C^{(2)}$.

SKETCH OF PROOF. For simplicity, we will disregard the tangent vectors and will take $a = 0$. First of all, recall that the topology in $\overline{M}_{g,n}$ is defined so that every map $S \to M$ is continuous. In particular, if one can construct an analytic family $C_q, q \in U$, of curves over a disk U in the q-plane, then $\lim_{q \to 0} C_q = C_0$.

Let us construct such a family with C_0 defined above. Let Σ be the surface in $\mathbb{P}^2 \times U$ given by the equation

$$uv = qw^2, \quad (u : v : w) \in \mathbb{P}^2, q \in U \tag{6.2.5}$$

and the marked points given by $\{((z'_i)^2 : q : z'_i), (q(z''_i)^2 : 1 : z''_i)\}_{i=1}^n$. Obviously, this is a smooth family of pointed curves over U, with an obvious projection to U; define $\pi^{-1}(q) = \tilde{C}_q$. We claim that for $q \neq 0$, the curve $\tilde{C}_q \simeq C_q$. Explicitly, the isomorphism $C_q \to \tilde{C}_q$ is given by $(z : s) \mapsto (u : v : w) = (z^2 : qs^2 : zs)$. Similarly, for $q = 0$, $\tilde{C}_0$ can be identified with C_0 by $\psi_1 : C^{(1)} \to \tilde{C}_0, \psi_2 : C^{(2)} \to \tilde{C}_0$ given by $\psi_1 \colon (z : s) \mapsto (z : 0 : s), \psi_2 \colon (w : s) \mapsto (0 : s : w)$. This completes the proof. We leave it to the reader to check that the above construction in fact also gives correct tangent vectors. □

Let us denote by $N(D)$ the normal bundle to D in $\overline{M}_{g,n}$: for $C \in D, N_C(D) = T_C\overline{M}_{g,n}/T_CD$; and let $N^\times(D)$ be the complement to the zero section: $N_C^\times(D) = N_C(D) \setminus \{\text{zero section}\}$.

LEMMA 6.2.5. *If C is a stable singular curve with only one double point a, then N_CD is one-dimensional, and can be canonically identified with $T_a^{(1)}C \otimes T_a^{(2)}C$, where $T_a^{(1)}C$, $T_a^{(2)}C$ are the tangent spaces to the two components of C at a.*

SKETCH OF PROOF. By definition, the space N_CD is the set of equivalence classes of one-parameter families of curves C_U, defined over a disk U in the complex plane, such that $C_0 = C$ and C_q is non-singular for $q \neq 0$. A typical example of such a family is given by (6.2.5).

Let $a \in C_0 \subset C_U$ be the double point. Then it can be shown that one can always introduce local coordinates x_1, x_2 on Σ near a such that $x_1x_2 = q$; when restricted to C_0, these coordinates become the local coordinates on the two components of C_0. Now, define the map $N_CD \to T_a^{(1)}C \otimes T_a^{(2)}C$ by $\partial_q \mapsto \partial_{x_1} \otimes \partial_{x_2}$. We leave it to the reader to check that this map does not depend on the choice of local coordinates t_1, t_2. □

?! Informally, the family C_q corresponding to the vector $v \in N_C^\times D$ can be presented as "thickening" of the double point, as shown in the figure below.

THERE WILL BE A FIGURE HERE

FIGURE 6.1. Family of smooth curves converging to a singular curve.

More generally, if $C \in \overline{M}_{g,n}$ is a curve with k double points $a_1, \dots, a_k$ (equivalently, C lies in the intersection of k components of D: $C \in D_1 \cap \dots \cap D_k$), then $N_CD = T_C\overline{M}_{g,n}/\cap T_CD_i$ is k-dimensional, and we have a canonical isomorphism

$$N_CD \simeq \bigoplus_{i=1}^{k} T_{a_i}^{(1)}C \otimes T_{a_i}^{(2)}C. \tag{6.2.6}$$

Using this lemma, we can now define the gluing functor for the complex Teichmüller groupoid. This is done in two steps.

First, let A be a finite set, $\alpha, \beta \in A$—an unordered pair. Then we define the "clutching" map

$$\begin{aligned} S_{\alpha\beta}\colon M_{*,A} &\to N(D^0), \quad D^0 \subset \overline{M}_{*,A\setminus\{\alpha,\beta\}}, \\ C^\vee &\mapsto (C, v), \end{aligned} \tag{6.2.7}$$

where $C \in D^0$ is the singular curve obtained by identifying the marked points α, β of $C^\vee$, and $v = v_\alpha \otimes v_\beta \in T_\alpha C^\vee \otimes T_\beta C^\vee \simeq N_C(D)$. The map S is a $\mathbb{C}^\times$-bundle over $N(D^0)$. We will also denote by $S_{\alpha,\beta}$ the corresponding functor between fundamental groupoids:

$$S_{\alpha,\beta}\colon \mathcal{T}eich_A^{\mathbb{C}} \to \pi_1(N^\times D),$$

where $\pi_1(X)$ denotes the fundamental groupoid of X.

The second step is to pass from $N^\times D \subset \overline{M}$ to M. Choose some tubular neighborhood N_ε of D in $N(D)$, and a C^∞ embedding

$$i\colon N_\varepsilon \to \overline{M} \tag{6.2.8}$$

such that i is identity on the normal bundles (note that the normal bundle to D in N_ε is canonically identified with $N(D)$). Such a map exists; moreover, it can be shown that the set of all such maps is contractible. Restricting this map to $N_\varepsilon^\times(D) = N_\varepsilon \setminus D$, we get a well-defined functor between the fundamental groupoids:

$$i\colon \pi_1(N_\varepsilon^\times(D)) \to \pi_1(M) = \mathcal{T}eich^{\mathbb{C}}.$$

Since the embedding $N_\varepsilon^\times(D) \to N^\times D$ is a homotopy equivalence, $\pi_1(N_\varepsilon^\times(D)) \simeq \pi_1(N^\times D)$. Thus, we can view i as a functor

$$i\colon \pi_1(N^\times D) \to \mathcal{T}eich^{\mathbb{C}}. \tag{6.2.9}$$

Now, let us define the gluing functor for the complex Teichmüller groupoid as the composition

$$F_{\alpha,\beta}\colon \mathcal{T}eich_A^{\mathbb{C}} \xrightarrow{S_{\alpha,\beta}} \pi_1(N^\times D) \xrightarrow{i} \mathcal{T}eich^{\mathbb{C}}_{A\setminus\{\alpha,\beta\}}. \tag{6.2.10}$$

Note that it is defined only for those curves $C \in M_{*,A}$ for which $S_{\alpha,\beta}(C)$ is stable.

Example 6.2.6. Let us describe the gluing map for genus zero. Let $A' = \{\infty', 1', \dots, k', a\}$, $A'' = \{\infty'', 1'', \dots, m''\}$. Then the gluing map

$$F_{a,\infty''}\colon M_{0,A'} \times M_{0,A''} \to M_{0,B}$$

where $B = (A' \sqcup A'') \setminus \{a, \infty''\} = \{\infty', 1', \dots, k', 1'', \dots, m''\}$, can be described explicitly as follows. Choose for any $C^{(1)} \in M_{0,A'}$ a presentation in the form

$$C^{(1)} = (\mathbb{P}^1; \infty, z_1', \dots, z_k', a; v_\infty, v_1', \dots, v_k', t)$$

as in (6.2.4), where the tangent vector at ∞ is given by $v_\infty = -\partial_{1/z}$. (More formally, choose a section of the projection $X_{k+1} \to M_{0,A'}$, where $X_{k+1} = (\mathbb{C}^{k+1} \setminus \text{diagonals}) \times (\mathbb{C}^\times)^{k+1}$.) Do the same for $M_{0,A''}$. Then a simple generalization of the arguments of Example 6.2.4 shows that the gluing functor $F_{a,\infty''}$ is given by

$$\begin{aligned} C^{(1)} \sqcup C^{(2)} \mapsto (\mathbb{P}^1; &\infty, z_1', \dots, z_k', a + tz_1'', a + tz_m''; \\ &v_\infty', v_1', \dots, v_k', tv_1'', \dots, tv_m''). \end{aligned} \tag{6.2.11}$$

This map is well defined only for small enough t, and depends on the choice of presentation of $C^{(2)}$ in the form (6.2.4); however, the induced functor of fundamental groupoids is well defined up to a unique isomorphism.

The gluing operation satisfies the associativity property formulated in Definition 5.6.1, i.e., for distinct $\alpha, \beta, \gamma, \delta \in A$, the functors $F_{\alpha,\beta}F_{\gamma,\delta}, F_{\gamma,\delta}F_{\alpha,\beta}\colon \mathcal{T}eich_A^{\mathbb{C}} \to \mathcal{T}eich_{A'}^{\mathbb{C}}$, where $A' = A \setminus \{\alpha, \beta, \gamma, \delta\}$, are canonically isomorphic. The proof of this fact can be obtained from noting that each of them is isomorphic to the composition

$$\pi_1(M_{*,A}) \to \pi_1(N^\times(D^1)) \to \pi_1(M_{*,A'}) = \mathcal{T}eich_{A'}^{\mathbb{C}},$$

where D^1 is the strata of the boundary $D = \overline{M}_{*,A'} \setminus M_{*,A'}$ consisitng of curves with two double points, and the first arrow is given by identifying the points $\alpha \leftrightarrow \beta, \gamma \leftrightarrow \delta$ of C, thus producing a curve with two double points, and taking the normal vector to be $(v_\alpha \otimes v_\beta) \otimes (v_\gamma \otimes v_\delta)$ (see (6.2.6)). The second map is defined as in (6.2.9). The details are left to the reader.

It is easy to check that the gluing operation is also compatible with the disjoint union and empty set. Thus, the groupoid $\mathcal{T}eich^{\mathbb{C}}$ is a tower of groupoids in the sense of Definition 5.6.1. It is also easy to verify that the equivalence $\mathcal{T}eich^{stab} \to \mathcal{T}eich^{\mathbb{C}}$, constructed in Theorem 6.1.12, identifies this gluing operation with the gluing in $\mathcal{T}eich$. Thus, $\mathcal{T}eich^{stab} \xrightarrow{\sim} \mathcal{T}eich^{\mathbb{C}}$ as towers of groupoids.

The construction of gluing above requires that all the curves we use (including the singular ones) be stable; otherwise, the moduli spaces of curves are not stacks in the sense of [**DM**], which makes life much more difficult. In particular, we can not define the gluing $M_{0,1} \times M_{g,n} \to M_{g,n-1}$ because $M_{0,1}$ is not a DM-stack. Note, however, that in the topological approach the groupoid $\mathcal{T}eich_{0,1}$ is trivial (i.e., equivalent to the group with one element), and the operation of gluing $\mathcal{T}eich \times \mathcal{T}eich_{0,1}$ coincides with the operation of erasing a marked point (or patching a hole, depending on what definition of an extended surface was used). This operation is also well-defined as a functor $\mathcal{T}eich^{\mathbb{C}}_{A} \to \mathcal{T}eich^{\mathbb{C}}_{A\setminus\alpha}$ in the complex Teichmüller groupoid.

6.3. Connections with regular singularities

In this section, we briefly give the main definitions and results regarding flat connections with regular singularities. This will be used in the next section to define the modular functor in terms of connections on the moduli spaces of curves. Most of these results are due to Deligne and can be found in [**De1**] or in the review [**Ma**]. We assume that the reader is familiar with basic geometric notions such as vector bundles, sheaves, and $\mathcal{O}$-modules (as usual, we denote by $\mathcal{O}$ the structure sheaf, i.e., the sheaf of germs of analytic functions on M). As before, the word "manifold" stands for complex analytic manifold, and vector bundles are holomorphic vector bundles, etc. The notation $s \in \mathcal{F}$ means that s is a local section of the sheaf $\mathcal{F}$.

Let M be a manifold. By definition, a *local system* on M is a representation of the Poincaré groupoid of M. It is well-known that this is the same as a locally constant sheaf of vector spaces on M.

A convenient way of constructing local systems on a manifold is given by vector bundles with flat connections. Recall that a *connection* in a vector bundle E over M is a morphism of sheaves

$$\nabla\colon \mathcal{E} \to \mathcal{E} \otimes \Omega^1, \tag{6.3.1}$$

such that

$$\nabla(sf) = (\nabla s)f + s \otimes df, \qquad s \in \mathcal{E}, f \in \mathcal{O}_M,$$

where $\mathcal{E}$ is the sheaf of sections of E, and Ω^n is the sheaf of differential forms of degree n,

We can extend ∇ to a map from $\mathcal{E} \otimes \Omega^n$ to $\mathcal{E} \otimes \Omega^{n+1}$, $n = 0, 1, \dots$. The connection ∇ is called *flat* if the resulting $\mathcal{E} \otimes \Omega^{\bullet}$ is a complex, i.e., if $\nabla^2 = 0$.

For any vector field X on M, (6.3.1) gives a linear morphism

$$\nabla_X\colon \mathcal{E} \to \mathcal{E} \tag{6.3.2}$$

such that

$$\nabla_X(sf) = (\nabla_X s)f + s\,X(f).$$

Then ∇ is flat iff $X \mapsto \nabla_X$ is a Lie algebra homomorphism, i.e., $[\nabla_X, \nabla_Y] = \nabla_{X,Y}$. In local coordinates x_i, $X_i = \partial/\partial x_i$, $\nabla_i = \nabla_{X_i}$, this means $[\nabla_i, \nabla_j] = 0$. In other words, a flat connection is the same as an action of the sheaf Θ_M of vector fields on sections of E, compatible with action of $\mathcal{O}$.

We can say that a flat connection is a consistent system of partial differential equations. Any flat connection gives rise to a locally constant sheaf—the sheaf of solutions to this system of differential equations; its sections are $\{s \in \mathcal{E} \mid \nabla_X s = 0 \text{ for all } X\}$; usually they are referred to as "flat sections".

If M is a C^∞ real manifold, then it is known that the converse is also true: any local system can be obtained from a vector bundle with a flat connection. More formally, one can say that in this case the categories of local systems and of vector bundles with flat connections are equivalent. The same holds if M is an analytic complex manifold, and we consider holomorphic vector bundles with holomorphic flat connections.

Recalling the definition (6.2.10) of gluing for the complex Teichmüller groupoid, we see that in order to describe it in terms of flat connections we need somehow to extend our flat connections on $M_{g,n}$ to the boundary $D = \overline{M} \setminus M$. In the simplest example when M is one-dimensional and D is a point, it is well known that though one can not define a value of a flat section at D, one has a well-defined notion of asymptotics provided that our system of differential equations has regular singularities (see, e.g., [**CL**]). Thus, it is natural to expect that in order to define the gluing axiom, one has to introduce local systems with regular singularities.

Definition 6.3.1. Let $\overline{M}$ be a complex analytic manifold, $D \subset \overline{M}$ be a divisor with normal crossings, and $M = \overline{M} \setminus D$. Let E be a holomorphic vector bundle on $\overline{M}$ with a holomorphic flat connection defined over M. This flat connection is said to have *logarithmic singularities* at D (*log* D connection for short) if in a neighborhood of every $p \in D$ the bundle E admits a trivialization such that the connection has the form

$$\begin{aligned} \nabla_i &= \frac{\partial}{\partial z_i} + \frac{A_i(z)}{z_i}, && 1 \le i \le k, \\ \nabla_i &= \frac{\partial}{\partial z_i} + A_i(z), && k+1 \le i \le n, \end{aligned} \tag{6.3.3}$$

where z_i are local coordinates near p chosen so that the divisor D is given by the equation $z_1 \cdots z_k = 0$, and $A_i(z)$ are regular matrix-valued functions in a neighborhood of p.

The following lemma describes (locally) the local system of flat sections of a *log* D connection.

Lemma 6.3.2. *In the notation of Definition* 6.3.1*, let U be a small ball around p, and $U^0 = U \setminus (U \cap D)$. Then*:

(i) $\pi_1(U^0) = \mathbb{Z}^k$*, with the generators* $\gamma_i\colon z_i \mapsto e^{\mathrm{i}\varphi} z_i$, $0 \le \varphi \le 2\pi$, $1 \le i \le k$.

(ii) *Let* $i \le k$*. Then the conjugacy class of the matrix* $A_i(z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_n)$ *does not depend on* $z_1, \dots, z_n$*. In particular, the eigenvalues* $\lambda_i^a, a = 1, \dots, \dim E$ *of* $A_i|_{z_i=0}$ *do not depend on* z_j.

(iii) *Let us assume that the connection* ∇ *satisfies the following non-integrality condition*:

$$\lambda_i^a - \lambda_{i'}^{a'} \notin (\mathbb{Z} \setminus \{0\}) \quad \text{for any } i, i' = 1, \dots, k, \quad a, a' = 1, \dots, \dim E. \tag{6.3.4}$$

(*Note that multiple eigenvalues are allowed.*) *Then the corresponding representation of* $\pi_1(U^0)$ *is given by* $\gamma_i \mapsto e^{-2\pi \mathrm{i} A_i(z_i=0)}$.

The proof of this lemma is not too difficult and essentially follows from the one-dimensional case. We refer the reader to [**Ma**], [**De1**] for details.

Note that part (iii) of the lemma may fail if we do not impose the non-integrality condition.

EXAMPLE 6.3.3. Let $\overline{M} = \mathbb{C}, D = \{0\}$. Let E be the trivial 2-dimensional vector bundle over $\mathbb{C}$ with the connection given by $\nabla = d - \frac{A(z)}{z}dz$, with

$$A(z) = \begin{pmatrix} 0 & 0 \\ z & 1 \end{pmatrix}.$$

Show that this connection has nontrivial monodromy even though $e^{2\pi i A(0)} = 1$.

One defines morphisms between bundles with connections in an obvious way. However, it is convenient also to introduce a more general notion of a morphism as follows. Let E, F be two holomorphic vector bundles over $\overline{M}$ with flat connections defined over M. Let $\mathcal{E}[D]$ be the sheaf of **meromorphic** sections of E which are holomorphic outside of D. Assume that the connections preserve $\mathcal{E}[D], \mathcal{F}[D]$ (this holds automatically for $log\, D$ connections). We define *meromorphic morphisms* between E and F to be the morphisms of sheaves $\mathcal{E}[D] \to \mathcal{F}[D]$ which commute with the connection (note that this is more general than the usual definition of a morphism between two vector bundles). We will say that (E, ∇_E) is *meromorphically equivalent* to (F, ∇_F) if there exists an invertible meromorphic morphism $E \to F$.

EXERCISE 6.3.4. Let $\overline{M} = \mathbb{C}, D = \{0\}$, and let $\nabla^s = d + s\frac{dz}{z}$ be the connection in the trivial one-dimensional vector bundle. Show that ∇^s is meromorphically equivalent to ∇^t iff $s - t \in \mathbb{Z}$.

Let E be a vector bundle on $\overline{M}$ with a flat connection ∇ defined on M.

DEFINITION 6.3.5. The connection ∇ has *regular singularities at D* if (E, ∇) is meromorphically equivalent to a bundle with a $log\, D$ connection (see Definition 6.3.1).

EXERCISE 6.3.6. Show that if $\dim M = 1$, then ∇ has regular singularities iff ∇ is a $log\, D$ connection. (For $\dim M > 1$, this is not true.)

For brevity, we will refer to the pair (E, ∇) in the definition as a "connection on $\overline{M}$ with regular singularities at D". The category of such connections with respect to meromorphic morphisms will be denoted by $\mathcal{RS}(\overline{M}, M)$ (or just $\mathcal{RS}(M)$ when there is no ambiguity). Note that meromorphic morphisms do not change monodromy, and thus we have a well-defined functor $\mathcal{RS}(\overline{M}, M) \to$ {local systems on M}.

REMARK 6.3.7. If we considered algebraic theory rather than analytic one, then any vector bundle on M admits a unique meromorphic continuation to $\overline{M}$, so the category of sheaves on $\overline{M}$ up to meromorphic equivalence is the same as the category of sheaves on M. Moreover, in this case it was proved by Deligne that the notion of connection with regular singularities on M can be defined purely in terms of M, without using $\overline{M}$ at all. In analytic situation, it is not true.

We quote here without proofs several important results of Deligne about connections with regular singularities. Proofs and details can be found in [**De1**] or in [**Ma**].

THEOREM 6.3.8. *Let $(E, \nabla) \in \mathcal{RS}(\overline{M}, M)$. For every $z \in \mathbb{C}/\mathbb{Z}$, choose a representative $\tau(z) \in \mathbb{C}$ (τ needs not to be continuous). Then there is a unique vector bundle $\tilde{E}$ with a flat $\log D$ connection $\tilde{\nabla}$ such that $(\tilde{E}, \tilde{\nabla})$ is meromorphically equivalent to (E, ∇), and all eigenvalues $\tilde{\lambda}_i^a$ (see Lemma 6.3.2) lie in the image of τ.*

COROLLARY 6.3.9. *Every flat connection with regular singularities is meromorphically equivalent to a $\log D$ connection which satisfies the non-integrality property (6.3.4).*

THEOREM 6.3.10. *In the notation of Definition 6.3.1, let D^0 be the smooth part of the divisor D (cf. (6.2.1)). Let E be a vector bundle on $\overline{M}$, and ∇ be a flat connection with regular singularities at D^0. Then ∇ has regular singularities at D.*

In other words, it suffices to check the regularity condition only for the open part of D. (Note: the proof of this theorem in [**De1**] contains a mistake, which Deligne later corrected.)

THEOREM 6.3.11. *In the notation of Definition 6.3.1, any holomorphic vector bundle on M with a flat connection can be extended to a vector bundle on $\overline{M}$ with a connection which has regular singularities at D. This extension is unique up to a meromorphic isomorphism.*

COROLLARY 6.3.12 (The Riemann–Hilbert correspondence). *The natural functor*

$$\mathcal{RS}(\overline{M}, M) \to \text{local systems on } M$$

is an equivalence.

In practical applications, it is convenient to use the following criterion of regularity, which is easy to prove.

LEMMA 6.3.13. *Let E, ∇ be as in Definition 6.3.5. Then ∇ has regular singularities iff for every holomorphic map $u\colon U \to \overline{M}$, where U is a disk, and $u^{-1}(D) = \{0\}$, the induced connection $u^*\nabla$ on U has regular singularities at 0.*

In fact, due to Theorem 6.3.10, it suffices to check this condition for $u(0) \in D^0$.

It will be convenient to rewrite the notion of flat connection with regular singularities in terms of $\mathcal{D}$-modules. As was noted before, a connection ∇ in a vector bundle E is flat if it defines for every open $U \subset M$ an action of the Lie algebra $\Theta(U)$ of vector fields on U on the space of sections $\mathcal{E}(U)$, which is compatible with multiplication by functions. Such an action is the same as an action on $\mathcal{E}$ of the sheaf of associative algebras $\mathcal{D}$ of differential operators on M. Thus, if E is a vector bundle with a flat connection, then the sheaf $\mathcal{E}$ of its sections is naturally a module over the sheaf $\mathcal{D}$ of differential operators, or a *$\mathcal{D}$-module* for short. Conversely, it is easy to see that if $\mathcal{E}$ is a $\mathcal{D}$-module which is locally free of finite rank as $\mathcal{O}$-module, then $\mathcal{E}$ is a the sheaf of sections of some vector bundle with a flat connection. It is known (but not easy) that it suffices to require that $\mathcal{E}$ be *coherent*, i.e., locally finitely generated over $\mathcal{O}$—for $\mathcal{D}$-modules, this automatically implies that $\mathcal{E}$ is locally free of finite rank. we refer the reader to [**Ber**], [**Bor**], [**Bjo**] for the proof of this and other facts about $\mathcal{D}$-modules.

In a similar way, it is easy to show that a connection with logarithmic singularities at D is the same as a sheaf of modules over the sheaf

$$\mathcal{D}^0_{\overline{M}} = \{\partial \in \mathcal{D}_{\overline{M}} \mid \partial I \subset I\} \tag{6.3.5}$$

where I is the sheaf of functions on $\overline{M}$ vanishing at D. The sheaf $\mathcal{D}^0_{\overline{M}}$ is generated as a sheaf of algebras by $\mathcal{O}$ and vector fields tangent to D. For example, for $\overline{M} = \mathbb{C}, D = \{0\}$, the sheaf $\mathcal{D}^0_{\overline{M}}$ is generated by $\mathcal{O}_{\mathbb{C}}$ and the vector field $q\partial_q$.

Now, let us show how connections with regular singularities allow us to pass to the boundary of the moduli space. Before doing so, we need to introduce one more notion.

DEFINITION 6.3.14. Let X be a $\mathbb{C}^\times$ bundle over M. A monodromic flat connection on X is a pair (E, ∇), where E is a $\mathbb{C}^\times$-equivariant vector bundle on X, and ∇ a connection which commutes with the action of $\mathbb{C}^\times$.

In local coordinates, such a connection can always be written as $\nabla = d + \sum A_i(x)dx_i + A(x)\frac{du}{u}$, where x_i are coordinates in M, and u is coordinate along the fibers of X.

LEMMA 6.3.15. (i) *Let $D \subset \overline{M}$ be a smooth divisor. Then there exists a natural specialization functor*

$$Sp_D \colon \mathcal{RS}(\overline{M}, M) \to \mathcal{RS}(ND, N^\times D)$$

such that $Sp_D(\nabla)$ is monodromic along the fibers of the projection $ND \to D$ and has the same monodromy around D as ∇.

(ii) *Let D be a divisor with normal crossings: $D = \cup D_i$. Fix one of the components D_i and let $D_i^0 = D_i \setminus (\cup_{j\neq i} D_j)$. Then we have a natural specialization functor*

$$Sp_{D_i} \colon \mathcal{RS}(\overline{M}, M) \to \mathcal{RS}(ND_i, N^\times D_i^0)$$

with the same properties as above.

PROOF. (i) The easiest way to define this functor is to use the terminology of $\mathcal{D}$-modules. By Corollary 6.3.9, we can assume that ∇ has logarithmic singularities and satisfies the non-integrality property (6.3.4). First of all, note that one can describe the structure sheaf of $\mathcal{O}_{ND}$ in terms of the restriction of the structure sheaf $\mathcal{O}_M$ to D. Namely, the latter sheaf is naturally filtered by the powers of the ideal I: $\mathcal{O}_M = I^0 \supset I \supset I^2 \supset \dots$. We claim that $\mathcal{O}_{ND} = \widehat{\bigoplus}_{n\geq 0} I^n/I^{n+1}$ is the completion of the associated graded algebra (we need completion to get all analytic functions, not just polynomial). Similarly, the sheaf of differential operators on ND which preserve I is nothing but the (completion of) associated graded sheaf for $\overline{M}$: $\mathcal{D}^0_{ND} = \widehat{\bigoplus}_{n\geq 0} I^n \mathcal{D}^0_{\overline{M}}/I^{n+1}\mathcal{D}^0_{\overline{M}}$

As was mentioned above, a flat connection with first order poles at D is the same as a $\mathcal{O}_M$-coherent $\mathcal{D}^0_{\overline{M}}$-module $\mathcal{E}$. Such a module is also naturally filtered, and action of $\mathcal{D}^0_{\overline{M}}$ preserves this filtration. Now define

$$Sp_D(\mathcal{E}) = \widehat{\bigoplus_{n\geq 0}} I^n\mathcal{E}/(I^{n+1}\mathcal{E}).$$

This is naturally a $\mathcal{D}^0_{ND}$ module, and thus a sheaf of sections of a vector bundle on ND with a flat connection which has first order poles at D.

Here is a more explicit construction. Choose coordinates $z_1, \dots, z_n$ in a neighborhood of the point $p \in D$ such that D is given by the equation $z_1 = 0$. This also gives coordinates $t, z_2, \dots, z_n$ in ND, where $t(a, v) = \langle v, dz_1\rangle, a \in D, v \in T_a M$.

Choose a trivialization of E near p; then ∇ is given by (6.3.3), with $k=1$. Define the connection $Sp\nabla = Sp_D(\nabla)$ in $N^\times D$ by

$$\begin{aligned}(Sp\nabla)_t &= \partial_t + A_1(0, z_2, \dots)/t,\\ (Sp\nabla)_{z_i} &= \partial_{z_i} + A_i(0, z_2, \dots), \quad i = 2, \dots, n.\end{aligned}$$

One easily sees that this connection is flat, invariant with respect to the action of $\mathbb{C}^\times$ by dilations on ND, and does not depend on the choice of coordinates up to a unique isomorphism. By Lemma 6.3.2, $Sp\nabla$ has the same monodromy as ∇. This definition uses a choice of a local coordinate system and an extension of the vector bundle to D. However, it can be shown that this construction coincides with the previous one (described by passing from filtered modules to associated graded ones) and thus $Sp\nabla$ does not depend on these choices.

In terms of the corresponding local systems, the specialization functor is defined as $Sp_D = i^*$, where i is is an identification of ND with a neighborhood of D in $\overline{M}$, as in (6.2.8).

Equivalently, the same flat connection can be defined by specifying its flat sections. Let f be a flat section of the original local system on M, i.e., $f(z)$ is a solution of the system $\nabla_i f = 0$. Let us restrict this solution to the curve $z(t) = p+tv, t \in \mathbb{R}_{>0}, (p, v) \in N^\times(D)$. By the classical theory of ODE's with regular singularities (see, e.g., [**CL**]), there exists a vector $g(p, v)$ such that $f(p+tv) = F(t)g(p, v)$, where $F(t)$ is the fundamental (matrix) solution; usually, g is called asymptotics of f along this curve. Then $g(p, v)$ is a flat section of the connection $Sp_D(\nabla)$.

To prove (ii), we need to check that $Sp_{D_i}(\nabla)$ has regular singularities at $D_i \cap D_j$, which can be done explicitly. □

REMARKS 6.3.16. (i) The specialization functor can be easily described in terms of the functor of nearby cycles for $\mathcal{D}$-modules (see [**KasS**].

(ii) Note that the specialization functor is defined even if the eigenvalues of $A_i(z_i = 0)$ differ by a non-zero integer. However, in this case this functor is not so easy to describe: one first needs to replace the flat connection by a meromorphically equivalent one which satisfies the non-integrality condition. Such a connection exists by Corollary 6.3.9, but explicitly constructing it can be difficult.

Finally, it is natural to consider the following question. Suppose we have a divisor with normal crossings, and D_1, D_2 are two of the components. Is it true that Sp_{D_1} and Sp_{D_2} commute?

In order for this question to make sense, one must first define the composition $Sp_{D_1} Sp_{D_2}$. For simplicity, let us assume that $\dim \overline{M} = 2$, $D_1 \cap D_2 = \{p\}$. Let $N_1 = N(D_1)$; it contains the divisor $D_{12} = N_p(D_1)$. Let N_{12} be the normal bundle to D_{12} inside N_1.

LEMMA 6.3.17. (i) *There exists a canonical homeomorphism $N_{12} \simeq T_p$, where for brevity we denoted $T_p = T_p\overline{M}$. Thus, one can define the specialization functor $Sp_{12} : \mathcal{RS}(\overline{M}, M) \to \mathcal{RS}(T_p, T_p^\times)$, where $T_p^\times = T_p \setminus (T_pD_1 \cup T_pD_2)$, as the composition*

$$\mathcal{RS}(\overline{M}, M) \to \mathcal{RS}(N_1, N_1^\times) \to \mathcal{RS}(N_{12}, N_{12}^\times) \simeq \mathcal{RS}(T_p, T_p^\times).$$

(ii) *The functors Sp_{12}, Sp_{21}, defined as in part* (i), *are canonically isomorphic.*

The proof of this lemma is not difficult and is left as an exercise.

6.4. Complex analytic modular functor

In this section, we give a definition of modular functor in terms of flat connections on the moduli spaces.

Let $\mathcal{C}$ be a semisimple abelian category over $\mathbb{C}$, and $R \in \text{ind}-\mathcal{C}^{\boxtimes 2}$ be a symmetric object, as in Section 2.4. Recalling the definition of a $\mathcal{C}$-extended modular functor (Definition 5.1.12) and the results of the previous section, we can rewrite the definition of modular functor as follows.

DEFINITION 6.4.1. A *complex $\mathcal{C}$-extended modular functor* is the following collection of data:

(i) For every finite set A and $W \in \mathcal{C}^{\boxtimes A}$, a finite-dimensional vector bundle over $\overline{M}_{*,A}$ with a flat connection with regular singularities. This bundle is called the *bundle of conformal blocks*; its fiber at a point $C \in \overline{M}_{*,A}$ will be denoted by $\langle W \rangle_C$.

(ii) Isomorphisms $\langle X \rangle_{C'} \otimes \langle Y \rangle_{C''} \simeq \langle X \boxtimes Y \rangle_{C' \sqcup C''}$.

(iii) **Gluing isomorphisms.** Let A be a finite set, $\alpha, \beta \in A$—an unordered pair, $A' = A \setminus \{\alpha, \beta\}, W \in \mathcal{C}^{\boxtimes A'}$. For every such collection, we require an isomorphism of vector bundles with connections on $M_{*,A}$:

$$G_{\alpha,\beta} \colon \langle W \boxtimes R \rangle \xrightarrow{\sim} S^*_{\alpha,\beta} Sp_D \langle W \rangle, \tag{6.4.1}$$

where $S_{\alpha,\beta} \colon M_{*,A} \to N(D^0), D^0 \subset M_{*,A'}$, is the "clutching" (6.2.7), and R is placed at positions with indices α, β. (Since $S_{\alpha,\beta}$ is a $\mathbb{C}^\times$-bundle, the definition of $S^*_{\alpha,\beta}$ causes no problems.)

(iv) **Vacuum propagation.** We have a distinguished element $\mathbf{1} \in \operatorname{Ob} \mathcal{C}$, and for every $\alpha \in A$, we require an isomorphism of vector bundles with connections on $M_{*,A}$:

$$G_\alpha \colon \langle W \boxtimes \mathbf{1} \rangle \xrightarrow{\sim} S^*_\alpha \langle W \rangle, \tag{6.4.2}$$

where $S_\alpha \colon M_{*,A} \to M_{*,A \setminus \alpha}$ is the operator of erasing the point α.

These data have to satisfy the following properties:

Functoriality: $W \mapsto \langle W \rangle_C$ is functorial in W, and the isomorphisms (ii)-(iv) are functorial isomorphisms.

Equivariance: $W \mapsto \langle W \rangle_C$ is equivariant with respect to the action of the symmetric group S_A.

Compatibility: the isomorphisms (ii)-(iv) are compatible with each other and with the commutativity, associativity, and unit morphisms in $\mathcal{V}ec_f$ (cf. Definition 4.2.1).

Normalization: $\langle \mathbf{1}, \mathbf{1} \rangle_{\mathbb{P}^1} = \mathbb{C}$.

We can now formulate the main result of this section.

THEOREM 6.4.2. *The notions of a* (*topological*) *$\mathcal{C}$-extended modular functor and a complex $\mathcal{C}$-extended modular functor are equivalent.*

PROOF. By Lemma 5.6.13, a modular functor is the same as a functor $\mathcal{T}eich \to \mathcal{F}un(\mathcal{C})$. We leave it to the reader to check that the same definition can be rewritten in terms of the groupoid $\mathcal{T}eich^{stab}$ (that is, without using surfaces of type $(g, n) = (0,0), (0,1), (1,0)$) if, in addition to the gluing of such surfaces, we also add an operation of erasing a marked point. This operation makes up for the operation of gluing a sphere with one hole, i.e., a disk, in $\mathcal{T}eich$.

Using the equivalence $\mathcal{T}eich^{stab} \simeq \mathcal{T}eich^{\mathbb{C}}$, we see that modular functor is the same as a functor $\mathcal{T}eich^{\mathbb{C}} \to \mathcal{F}un(\mathcal{C})$. Thus, for every fixed $W \in \mathcal{C}^{\boxtimes A}$, we get a local system on $M_{*,A}$. By the results of the previous section, every local system can be presented by a unique up to meromorphic equivalence holomorphic vector bundle on $\overline{M}_{*,A}$ with a flat connection on $M_{*,A}$ which has regular singularities at $D = \overline{M}_{*,A} \setminus M_{*,A}$. Then all the properties, except for the gluing axiom, are obvious reformulations of the axioms of modular functor.

To show that the definition of gluing given above coincides with the one given for topological modular functor, recall that the gluing functor $F_{\alpha,\beta}\colon \mathcal{T}eich^{\mathbb{C}}_A \to \mathcal{T}eich^{\mathbb{C}}_{A'}$ was defined using the composition

$$M_{*,A} \xrightarrow{S_{\alpha,\beta}} N^{\times}(D) \to M_{*,A'},$$

see (6.2.10). In the language of local systems, the gluing isomorphism should identify the vector spaces $\langle W \boxtimes R\rangle_C \simeq \langle W\rangle_{C'}$ in such a way that it agrees with morphisms in $\mathcal{T}eich_A$. This is equivalent to saying that it must be an isomorphism of local system on $M_{*,A}$, $\langle W \boxtimes R\rangle \simeq F^*_{\alpha,\beta}\langle W\rangle$. Replacing local systems by connections with regular singularities, we note that the identification $i\colon N^{\times}(D) \to M_{*,A'}$ was defined so that i^* is exactly the specialization functor. This leads to the definition of the gluing isomorphism given above.

We leave ti to the reader to check the equivalence of normalization axioms in topological and complex-analytic settings. $\square$

Remark 6.4.3. Let us restrict the gluing functor to

$$S_{\alpha,\beta}\colon M_{g_1,A} \times M_{g_2,B} \to N^{\times}(D^0), \quad D \subset M_{g_1+g_2,C},$$

where $\alpha \in A, \beta \in B$, and $C = (A \sqcup B) \setminus \{\alpha, \beta\}$. Then the gluing axiom says that the bundle of conformal blocks $\langle W_A \boxtimes W_B\rangle$ on $M_{g_1+g_2,C}$ factors into tensor product of bundles $\langle W_A, R^{(1)}\rangle$ on $M_{g_1,A}$ and $\langle R^{(2)}, W_B\rangle$ on $M_{g_2,B}$ as we approach the corresponding component of the boundary in $\overline{M}_{g_1+g_2,C}$.[1] This is known as the *factorization property* of the bundle of conformal blocks and was first introduced in [**FS**].

Corollary 6.4.4. *Every modular tensor category $\mathcal{C}$ over $\mathbb{C}$ with $p^+/p^- = 1$ gives rise to a complex $\mathcal{C}$-extended modular functor such that*

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_1 \otimes \cdots \otimes W_n) = \langle W_1, \dots, W_n\rangle_C$$

where C is the standard n-punctured sphere

$$C = (\mathbb{P}^1; z_1, \dots, z_n; v_1, \dots, v_n) \tag{6.4.3}$$

with $z_1 < \cdots < z_n \in \mathbb{R}, v_i > 0$. The Dehn twist θ_{W_i} corresponds to the monodromy around the loop $v_i \mapsto e^{i\varphi}v_i, 0 \le \varphi \le 2\pi$, and the braiding $\sigma_{W_i,W_{i+1}}$ corresponds to the holonomy around the path b_i shown in Figure 6.2

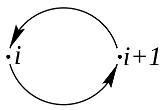


Figure 6.2. Braiding for the complex modular functor.

[1] We are using the same notation as in Definition 5.1.12.

Conversely, if $\mathcal{C}$ is a semisimple abelian category over $\mathbb{C}$, then every complex $\mathcal{C}$-extended modular functor gives rise to a structure of weakly ribbon category on $\mathcal{C}$ such that the above properties hold; if this category is rigid, then it is also modular with $p^+/p^- = 1$.

This corollary is nothing but the reformulation of Theorem 5.5.1 in the language of complex modular functor.

REMARK 6.4.5. Equivalently, one can describe the relation between conformal blocks and the spaces of homomorphisms in $\mathcal{C}$ as follows:

$$\mathrm{Hom}_{\mathcal{C}}(W_\infty^*, W_1 \otimes \cdots \otimes W_n) = \langle W_\infty, W_1, \dots, W_n \rangle_C$$

where $C = (\mathbb{P}^1; \infty, z_1, \dots, z_n; v_\infty, v_1, \dots, v_n)$, with $z_1 < \cdots < z_n, v_1 > 0, \dots, v_n > 0$ and the tangent vector at ∞ given by $v_\infty = -\partial_{1/z}$. Indeed, this curve can be reduced to the $(n+1)$-punctured standard sphere (6.4.3) by the change of variables $z \mapsto -1/z$.

6.5. Example: Drinfeld's category

In this section, we study one example of modular functor in genus zero, associated with a simple Lie algebra $\mathfrak{g}$. This modular functor is defined in terms of the Knizhnik–Zamolodchikov equations; the corresponding tensor category is the Drinfeld's category $\mathcal{D}$ defined in Chapter 1.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, and let $\mathcal{C} = \mathcal{R}ep_f\mathfrak{g}$ be the category of finite dimensional $\mathfrak{g}$-modules. Let $R = \bigoplus_{\lambda \in P_+} V_\lambda \boxtimes V_\lambda^*$, where V_λ is the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight λ, and $*$ is the usual duality for $\mathcal{R}ep_f\mathfrak{g}$. Fix $\varkappa \in \mathbb{C} \setminus \mathbb{Q}$. We will construct a (complex) modular functor in genus 0 for $\mathcal{C}$.

First, note that the moduli space $M_{0,n}$ of pointed curves of genus zero is given by $M_{0,n} = X_n/\mathrm{PSL}_2(\mathbb{C})$, where

$$X_n = \{z_1, \dots, z_n \in \mathbb{P}^1, v_i \in T_{z_i}^\times \mathbb{P}^1 \mid z_i \neq z_j\}. \tag{6.5.1}$$

This, in particular, implies that $M_{0,n}$ is smooth for $n > 0$; for $n = 0$, $M_{0,0} = \{pt\}$ and this case will not be considered. We will start by constructing the bundle of conformal blocks on the open part

$$X_n^0 = \{(\mathbf{z}, \mathbf{v}) \in X_n \mid z_i \neq \infty\} = \{z_1, \dots, z_n \in \mathbb{C} \mid z_i \neq z_j\} \times (\mathbb{C}^\times)^n. \tag{6.5.2}$$

For $(\mathbf{z}, \mathbf{v}) \in X_n^0$, the corresponding curve is $\mathbb{P}^1$ with marked points $z_1, \dots, z_n$ and tangent vectors v_i (since $T_z\mathbb{C} = \mathbb{C}$).

Let $W_1, \dots, W_n$ be the representations of $\mathfrak{g}$ assigned to these points. Define the bundle of conformal blocks to be the trivial vector bundle over X_n^0 with fiber $(W_1 \otimes \cdots \otimes W_n)^{\mathfrak{g}}$ and with the *Knizhnik–Zamolodchikov connection* (cf. (KZ_n)):

$$\begin{aligned} \nabla_{z_j} &= \frac{\partial}{\partial z_j} - \frac{1}{\varkappa} \sum_{\substack{1 \le k \le n \\ k \neq j}} \frac{\Omega_{jk}}{z_j - z_k}, \qquad 1 \le j \le n, \\ \nabla_{v_j} &= \frac{\partial}{\partial v_j} - \frac{1}{2\varkappa} \frac{D_j}{v_j}. \end{aligned} \tag{6.5.3}$$

Here D is the Casimir element of $\mathfrak{g}$ defined by (1.4.4); all other notation is as in (KZ_n). This connection, which is originally defined in the bundle with fiber

$W_1 \otimes \cdots \otimes W_n$, commutes with the action of $\mathfrak{g}$ and therefore can be restricted to the sub-bundle of invariants $(W_1 \otimes \cdots \otimes W_n)^{\mathfrak{g}} \subset W_1 \otimes \cdots \otimes W_n$.

LEMMA 6.5.1. (i) *The KZ connection* (6.5.3) *on* X_n^0 *is flat and* S_n*-equivariant.*

(ii) *This connection can be uniquely extended to a* $\mathrm{PSL}_2(\mathbb{C})$*-invariant connection on* X_n.

PROOF. The first statement easily follows from the flatness of the usual KZ connection—see Lemma 1.4.7. The second can be checked explicitly; it suffices to check that under a change of variables $w = (az+b)/(cz+d)$, the KZ equations in terms of w have the same form as in terms of z. Details can be found, for example, in [**EFK**]. Note that the second statement fails for the KZ connection in the form of Chapter 1: to ensure projective invariance, one needs to add the equation in v_i and restrict to $\mathfrak{g}$-invariants. $\square$

Thus, the connection (6.5.3) defines a flat connection on the moduli space $M_{0,n}$, which will also be called the KZ connection.

REMARK 6.5.2. It is more convenient to describe the same connection in a slightly different way. Namely, the map

$$(z_1, \dots, z_n; v_1, \dots, v_n) \mapsto (\mathbb{P}^1; \infty, z_1, \dots, z_n; v_\infty, v_1, \dots, v_n),$$

where, as before, $v_\infty = -\partial_{1/z}$, gives an identification $M_{0,n+1} = X_n^0/\mathbb{C}$, where $\mathbb{C}$ acts on X_n^0 by $z_i \mapsto z_i + a$. Let us fix $W_\infty, W_1, \dots, W_n$ and consider the connection (6.5.3) in $W_1 \otimes \cdots \otimes W_n$. This connection induces a connection in $(W_\infty \otimes W_1 \otimes \cdots \otimes W_n)^{\mathfrak{g}}$, which is obviously translation invariant and thus defines a connection on $M_{0,n+1}$. One easily checks that this connection coincides with the KZ connection defined above.

THEOREM 6.5.3. *The KZ connection, considered as a connection on the trivial vector bundle with fiber* $(W_1 \otimes \cdots \otimes W_n)^{\mathfrak{g}}$ *over the compactification* $\overline{M}_{0,n}$*, has first order poles and satisfies the gluing axiom, with* $R = \bigoplus V_\lambda \boxtimes V_\lambda^*$ *and the gluing isomorphism given by*

(6.5.4)

$$\bigoplus_\lambda (W_1 \otimes \ldots \otimes W_k \otimes V_\lambda)^{\mathfrak{g}} \otimes (V_\lambda^* \otimes W_{k+1} \otimes \ldots \otimes W_n)^{\mathfrak{g}} \xrightarrow{\sim} (W_1 \otimes \ldots \otimes W_n)^{\mathfrak{g}}$$

$$(w_1 \otimes \ldots \otimes w_k \otimes v) \otimes (v^* \otimes w_{k+1} \otimes \ldots \otimes w_n) \mapsto (v, v^*) w_1 \otimes \ldots \otimes w_n.$$

PROOF. By Theorem 6.3.10, it suffices to check the regularity condition for the open strata of $D = \overline{M}_{0,n} \setminus M_{0,n}$, i.e., for the curves with one double point. Thus, it suffices to check the regularity and the gluing axiom for the gluing functor $F_{a,\infty''}\colon M_{0,k+2} \times M_{0,m+1} \to M_{0,k+m+1}$ considered in Example 6.2.6. Using the notation from that example, we represent $M_{0,k+2} = X_{k+1}^0/\mathbb{C}$, $M_{0,m+1} = X_m^0/\mathbb{C}$ as in Remark 6.5.2. Under this identification, the gluing functor is given by

$$\begin{aligned} &F_{a,\infty''}\colon X_{k+1}^0 \times X_m^0 \to X_{k+m}^0, \\ &(z', a; v', t) \times (z''; v'') \mapsto (z_1', \dots, z_k', a + t z_1'', \dots, a + t z_m''; \\ &\qquad\qquad v_1', \dots, v_k', t v_1'', t v_1'', \dots, t v_m''). \end{aligned} \tag{6.5.5}$$

By definition, the KZ connection on $M_{0,A'}$, $M_{0,A''}$ is given by (6.5.3) in the variables $z_1', \dots, z_k'$, $a = z_{k+1}'$ and $z_1'', \dots, z_m''$, respectively, while on $M_{0,B}$ it is given by (6.5.3) in the variables $z_1, \dots, z_{k+m}$.

Then to prove the theorem it suffices to check that

$$\begin{aligned}Sp_{t=0}F^*_{a,\infty''}\langle W'_\infty, W'_1,\dots,W'_k,W''_1,\dots,W''_m\rangle\\ = \bigoplus_i \langle W'_\infty, W'_1,\dots,W'_k,V_i\rangle\otimes\langle V_i^*,W''_1,\dots,W''_m\rangle\end{aligned}$$

as vector bundles with connections. To obtain the left hand side, we need to substitute in (6.5.3)

$$\begin{aligned}
z_i &= z'_i, & v_i &= v'_i, & i &\le k;\\
z_{k+i} &= a+tz''_i, & v_{k+i} &= tv'_i, & i &\le m,
\end{aligned}$$

and then specialize to $t=0$.

Explicit calculation shows that this substitution gives the following connection in terms of the variables z', z'', a, v', v'', t:

$$\begin{aligned}
\nabla_{z'_i} &= \frac{\partial}{\partial z'_i} - \frac{1}{\varkappa}\left(\sum_{j\neq i}\frac{\Omega_{ij}}{z'_i-z'_j} + \sum_q \frac{\Omega_{ik+q}}{z'_i-(a+tz''_q)}\right),\\
\nabla_{v'_i} &= \frac{\partial}{\partial v'_i} - \frac{1}{2\varkappa}\frac{D_i}{v'_i},\\
\nabla_{z''_p} &= \frac{\partial}{\partial z''_p} - \frac{t}{\varkappa}\left(\sum_j \frac{\Omega_{jk+p}}{(a+tz''_p)-z'_j} + \sum_{q\neq p}\frac{\Omega_{k+pk+q}}{t(z''_p-z''_q)}\right),\\
\nabla_{v''_p} &= \frac{\partial}{\partial v''_p} - \frac{1}{2\varkappa}\frac{D_{k+p}}{v''_p},\\
\nabla_a &= \frac{\partial}{\partial a} - \frac{1}{\varkappa}\sum_j\sum_q \frac{\Omega_{jk+q}}{(a+tz''_q)-z'_j},\\
\nabla_t &= \frac{\partial}{\partial t} - \frac{1}{\varkappa}\sum_p\left(\sum_i \frac{z''_p\Omega_{ik+p}}{(a+tz''_p)-z'_i} + \sum_{q\neq p}\frac{z''_p\Omega_{k+pk+q}}{t(z''_p-z''_q)} + \frac{D_{k+p}}{2t}\right),
\end{aligned}$$

where the indices $i,j=1,\dots,k$ and $p,q=1,\dots,m$.

It is obvious that this is a connection with regular singularities at $t=0$. Let us specialize it to $t=0$ as described in Lemma 6.3.15; we will check the non-integrality condition later. This gives:

$$\begin{aligned}
\nabla_{z'_i} &= \frac{\partial}{\partial z'_i} - \frac{1}{\varkappa}\left(\sum_{j\neq i}\frac{\Omega_{ij}}{z'_i - z'_j} + \frac{\Omega_{i,(2)}}{z'_i - a}\right),\\
\nabla_{v'_i} &= \frac{\partial}{\partial v'_i} - \frac{1}{2\varkappa}\frac{D_i}{v'_i},\\
\nabla_{z''_p} &= \frac{\partial}{\partial z''_p} - \frac{1}{\varkappa}\left(\sum_{q\neq p}\frac{\Omega_{k+pk+q}}{z''_p - z''_q}\right),\\
\nabla_{v''_p} &= \frac{\partial}{\partial v''_p} - \frac{1}{2\varkappa}\frac{D_{k+p}}{v''_p},\\
\nabla_a &= \frac{\partial}{\partial a} - \frac{1}{\varkappa}\sum_j \frac{\Omega_{j,(2)}}{a - z'_j},\\
\nabla_t &= \frac{\partial}{\partial t} - \frac{1}{2\varkappa t}\sum_{p,q}\Omega_{k+p,k+q},
\end{aligned}$$

where we denoted $\Omega_{i,(2)} = \sum_p \Omega_{i,k+p} = \Omega_{W'_i, W''_1\otimes\cdots\otimes W''_m}$.

Let us identify

$$\begin{aligned}
&(W'_\infty \otimes W'_1 \otimes \cdots \otimes W'_k \otimes W''_1 \otimes \cdots \otimes W''_m)^{\mathfrak{g}}\\
&\qquad = \bigoplus_\lambda (W'_\infty \otimes W'_1 \otimes \cdots \otimes W'_k \otimes V_\lambda)^{\mathfrak{g}} \otimes (V^*_\lambda \otimes W''_1 \otimes \cdots \otimes W''_m)^{\mathfrak{g}}.
\end{aligned}$$

Under this identification, one has:

$$\begin{aligned}
\Omega_{j,(2)} &= \Omega_{W'_j, V_\lambda},\\
\sum_{p,q}\Omega_{k+p,k+q} &= D_{V^*_\lambda} = D_{V_\lambda}.
\end{aligned}$$

Thus, we see that the specialization exactly coincides with the product of KZ connections on $X^0_{k+1} \times X^0_m$.

Finally, in order to justify that our calculation of the specialization functor is valid, we have to check that our connection satisfies the non-integrality property (6.3.4). This follows from the fact that in our case, the operator $A(t=0)$ is given by $\frac{1}{2\varkappa}D_{V_i}$. Thus, its eigenvalues are of the form $\langle\lambda, \lambda + 2\rho\rangle/2\varkappa$. Since $\langle\lambda, \lambda+2\rho\rangle \in \mathbb{Q}$, and $\varkappa \notin \mathbb{Q}$, these eigenvalues can not differ by a non-zero integer (this is the only place where we use the condition $\varkappa \notin \mathbb{Q}$!).

This completes the proof of the gluing axiom. □

Therefore, we see that the KZ connection given above does define a genus zero complex modular functor. Thus, it defines a structure of a weakly rigid tensor category on $\mathcal{R}ep_f\mathfrak{g}$.

PROPOSITION 6.5.4. *The weakly rigid tensor structure on $\mathcal{R}ep_f(\mathfrak{g})$ defined as above coincides with the Drinfeld category structure defined in Section* 1.4.

The proof of this proposition immediately follows from the definition of Drinfeld's category. This, in particular, shows that this category is rigid.

The reader might notice that the proof of the gluing axiom given above is essentially the same proof we used in Chapter 1 to prove the associativity axiom for the Drinfeld's category—only now we have the language of connections with regular singularities and specialization functors in our disposition, which allows us to make all the arguments absolutely rigorous.

6.6. Twisted $\mathcal{D}$-modules

So far, we have only discussed modular functors in which the bundle of confor-mqal blocks carries a natural flat connection ("modular functors with zero central charge"). Comparing this with the discussion of Section 5.7, we see that these modular functors correspond to categories with $p^+/p^- = 1$. However, most interesting examples — for example, the category of representations of quantum groups at roots of unity — do not satisfy this property. As was discussed in Section 5.7, the way to incorporate modular categories with $p^+/p^- \neq 1$ is to define modular functor with central charge. In topological language, it was defined as a projective representation of the tower of mapping class groups, or, more precisely, as a representation of a suitable central extension of this tower with the central element acting by the fixed constant K (multiplicative central charge).

In order to give an analogous description of the modular functor with central charge, we need to introduce the appropriate formalizm — namely, the notion of twisted $\mathcal{D}$-modules. This is done in this section; in the next section, we will use this formalizm to define modular functor with central charge.

As before, the simplest way to describe such modular functors is to replace the requirement that the bundle carry a flat connection by a *projectively flat* connection, i.e., a connection such that $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is an operator of multiplication by a function (depending on X, Y). Equivalently, we can say that the sheaf of sections carries a projective action of the algebra of vector fields, or that the sheaf of sections is a projective $\mathcal{D}$-module.

However, we want to describe the failure of the connection to be flat more precisely, by describing the corresponding central extension of the Lie algebra of vector fields. It can be done as follows.

First, let us say that an $\mathcal{O}_S$-module is a sheaf of Lie algebras if we have a Lie algebra structure on local sections; the Lie bracket does not have to be $\mathcal{O}_S$-linear. For example, the sheaf Θ_S of vector fields on S is a sheaf of Lie algebras. A map of Lie algebra sheaves is just a sheaf morphism which preserves both the Lie bracket and $\mathcal{O}_S$-module structure.

DEFINITION 6.6.1. A central extension of Θ_S is a sheaf of Lie algebras $\mathcal{A}$ on S along with given maps of sheaves of Lie algebras giving a short exact sequence

$$0 \to \mathcal{O}_S \xrightarrow{\psi} \mathcal{A} \xrightarrow{\varepsilon} \Theta_S \to 0 \tag{6.6.1}$$

(here $\mathcal{O}_S$ is considered as sheaf of Lie algebras with zero bracket) such that:

1. $\psi(1)$ is central in $\mathcal{A}$.
2. For $a, b \in \mathcal{A}, f \in O_S$, we have $[a, fb] = f[a,b] + (\varepsilon(a)f)b$.

A module $\mathcal{F}$ over $\mathcal{A}$ is a quasicoherent $\mathcal{O}$-module with the action of $\mathcal{A}$ (as a Lie algebra) on $\mathcal{F}$ which agrees with the $\mathcal{O}$-module structure: $\psi(f)s = fs,\ f \in \mathcal{O}, s \in \mathcal{F}$.

REMARK 6.6.2. This is a special case of a more general notion of *Atiyah algebra*, in which $\mathcal{O}_S$ is replaced by an arbitrary sheaf of associative algebras over $\mathcal{O}$, see [**BS**] for details.

One can easily see that locally in S, we can choose a lifting, i.e. a morphism of $\mathcal{O}_S$-modules $a\colon \Theta \to \mathcal{A}$; then the bracket can be written as $[a(X), a(Y)] = a([X,Y]) + c(X,Y)$, where $c(X,Y) \in \mathcal{O}$ is a 2-cocycle on Θ_S.

EXAMPLES 6.6.3.

1. Let $\mathcal{A}_{\mathcal{O}} = \mathcal{O}_S \oplus \Theta_S$ (direct sum as $\mathcal{O}_S$-modules), with the bracket given by $[X+f, Y+g] = [X,Y] + X(g) - Y(f)$, where $X, Y \in \Theta$, $f, g \in \mathcal{O}$, and $[X,Y]$ is the usual bracket of vector fields. This plays the role of a trivial central extension.
2. Let L be a line bundle on S, $\mathcal{L}$— sheaf of sections of L. Define $\mathcal{A}_{\mathcal{L}}$ as the algebra of first order differential operators in L. If we choose a local trivialization of L, then sections of $\mathcal{A}$ have the form $\partial = X + f, X \in \Theta_S, f \in \mathcal{O}_S$. In other words, choice of trivialization $\mathcal{L}|_U \xrightarrow{\sim} \mathcal{O}|_U$ defines an isomorphism $\mathcal{A}_{\mathcal{L}}|_U \xrightarrow{\sim} \mathcal{A}_{\mathcal{O}}|_U$.
3. Let $\mathcal{A}$ be a central extension of Θ, $k \in \mathbb{C}^\times$. Then we can define the central extension $\mathcal{A}^k$; as a sheaf of Lie algebras, it coincides with $\mathcal{A}$, but the embedding $\mathcal{O} \to \mathcal{A}^k$ given by ψ/k, where ψ is the embedding $\mathcal{O} \to \mathcal{A}$. Equivalently, if we locally choose a lifting $\Theta \to \mathcal{A}$ so that the extension $\mathcal{A}$ is given by a 2-cocycle $c(X,Y)$, then $\mathcal{A}^k$ is given by the 2-cocycle $kc(X,Y)$, which also shows that it is well-defined for $k=0$. One can easily check that for integer k, one has $\mathcal{A}_{\mathcal{L}^k} = (\mathcal{A}_{\mathcal{L}})^k$. Using this, we will define for any $k \in \mathbb{C}$ the "sheaf of first order differential operators in $\mathcal{L}^k$" by
$$\mathcal{A}_{\mathcal{L}^k} = (\mathcal{A}_{\mathcal{L}})^k.$$

Now one can easily see that every projectively flat connection ∇ in a vector bundle E defines a central extension $\mathcal{A}$ of Θ such that ∇ defines a true action of $\mathcal{A}$ by first order differential operators in E. In other words, failure of a projectively flat connection to be flat can be described by a central extension $\mathcal{A}$ of the Lie algebra of vector fields.

EXERCISE 6.6.4. Let L be a line bundle on S. Show that $\mathcal{A}_{\mathcal{L}}$ is isomorphic to the Lie algebra of vector fields on the total space of L which commute with the action of $\mathbb{C}^\times$ on L by dilations; locally, such vector fields have the form $\partial = X + f(s)u\partial_u, X \in \Theta_S, f \in \mathcal{O}_S$, where u is coordinate along the fibers of L. Using this, show that an action of $\mathcal{A}_{\mathcal{L}^k}$ on a vector bundle E on S is the same as a monodromic flat connection on the pullback π^*E of E to $L^\times = L \setminus \{\text{zero section}\}$ such that the monodromy of this connection around the zero section is equal to $e^{-2\pi \mathrm{i} k}$.

As with the usual flat connections, we can also use the language of $\mathcal{D}$-modules. The appropriate generalization of the notion of $\mathcal{D}$-module is the notion of twisted $\mathcal{D}$-module.

DEFINITION 6.6.5. A *twisted sheaf of differential operators* on S is a sheaf of associative algebras $\mathcal{U}$ on S and an embedding $\mathcal{O}_S \hookrightarrow \mathcal{U}$ such that locally, the pair $(\mathcal{U}, \mathcal{O} \hookrightarrow \mathcal{U})$ is isomorphic to $(\mathcal{D}, \mathcal{O} \hookrightarrow \mathcal{D})$. A *twisted $\mathcal{D}$-module* is a sheaf of modules over a twisted sheaf of differential operators.

It turns out that the notions of twisted sheaves of differential operators and of central extensions of Θ are equivalent. Namely, for a twisted sheaf $\mathcal{U}$ of d.o., we can define the subsheaf $\mathcal{U}_1$ of differential operators of first order. A reader can easily check that $\mathcal{U}_1$ is closed under the Lie bracket $[a,b] = ab - ba$ and the action of $\mathcal{U}_1$

on $\mathcal{O}$ by $\partial(f) = [\partial, f]$ defines an isomorphism $\mathcal{U}_1/\mathcal{O} \xrightarrow{\sim} \Theta$, and thus $\mathcal{U}_1$ is a central extension of Θ. Conversely, if $\mathcal{A}$ is a central extension of Θ, then define

$$\mathcal{U}(\mathcal{A}) = U(\mathcal{A})/(f - \psi(f)), \qquad f \in \mathcal{O},$$

where $U(\mathcal{A})$ is the universal enveloping algebra of $\mathcal{A}$ (as an $\mathcal{O}$-module, it is isomorphic to $\mathcal{O} \oplus \mathcal{A} \oplus S^2(\mathcal{A}) \oplus \dots$).

LEMMA 6.6.6. *The functors $\mathcal{U} \mapsto \mathcal{U}_1$, $\mathcal{A} \mapsto \mathcal{U}(\mathcal{A})$ are inverse to each other and thus give an equivalence of categories of twisted algebras of differential operators and central extensions of Θ. In particular, twisted $\mathcal{D}$-modules are the same as modules over central extensions of Θ.*

We refer the reader to [**BS**] for the proof (easy) and more details.

EXAMPLE 6.6.7. For a line bundle L, let $\mathcal{D}_L$ be the sheaf of differential operators in L. This is a twisted sheaf of d.o., which corresponds to the central extension $\mathcal{A}_{\mathcal{L}}$.

In a similar way, we can define the "sheaf $\mathcal{D}_{\mathcal{L}^k}$ of differential operators in L^k" as the twisted sheaf of d.o. corresponding to the central extension $\mathcal{A}_{\mathcal{L}^k}$ (see Example 6.6.3):

$$\mathcal{D}_{\mathcal{L}^k} = \mathcal{U}(\mathcal{A}_{\mathcal{L}^k}). \tag{6.6.2}$$

EXERCISE 6.6.8. Let us choose local trivializations of L: $\varphi_\alpha \colon \mathcal{L}|_{U_\alpha} \xrightarrow{\sim} \mathcal{O}|_{U_\alpha}$ (where $\{U_\alpha\}$ is an open cover of S), and let $f_{\alpha\beta} = \varphi_\beta\varphi_\alpha^{-1} \in \mathcal{O}(U_\alpha \cap U_\beta)$ be the corresponding transition functions. Show that this defines isomorphisms $\varphi_\alpha \colon \mathcal{D}_{L^k}|_{U_\alpha} \xrightarrow{\sim} \mathcal{D}|_{U_\alpha}$, and the transition functions $\varphi_\beta\varphi_\alpha^{-1} \colon \mathcal{D}(U_\alpha \cap U_\beta) \to \mathcal{D}(U_\alpha \cap U_\beta)$, when restricted to vector fields, are given by

$$\varphi_\beta\varphi_\alpha^{-1} \colon v \mapsto v + k\frac{v(f_{\alpha\beta})}{f_{\alpha\beta}}.$$

Note that the right-hand side is not a vector field but a first order differential operator.

EXERCISE 6.6.9. Show that if L admits a flat connection, then $\mathcal{D}_L \simeq \mathcal{D}$, $\mathcal{A}_{\mathcal{L}} \simeq A_{\mathcal{O}}$.

As for usual $\mathcal{D}$-modules, we can define the category $RS_{L^c}(\overline{M}, M)$ (where L is a vector bundle over $\overline{M}$) as the category of vector bundles over $\overline{M}$ (up to meromorphic equivalence) with an action of $\mathcal{D}_{L^c}$ on the sheaf of sections such that E admits a trivialization such that the action of vector fields has first order poles. Indeed, the regularity condition is local, and locally a twisted sheaf of differential operators is isomorphic to the usual sheaf $\mathcal{D}$ of differential operators.

Similarly, we can define the specialization functor

$$Sp_D \colon RS_{L^c}(\overline{M}, M) \to RS_{L^c}(ND, N^\times D).$$

Note that restriction of L to the fiber $N_dD \simeq \mathbb{C}^\times$ is necessarily trivial, so that there is no need to change the definition of monodromic connection.

6.7. Modular functor with central charge

In order to apply the technique of twisted D-modules to modular functors, we must choose a line bundle L on each of the moduli spaces $M_{g,n}$ in a consistent way. We will use the so-called *determinant line bundle* introduced by Grothendieck (see [**KM**] for details). As before, in order to define a line bundle on $M_{g,n}$ we need to define a line bundle on S for every family of curves C_S over S.

Before doing so, let us introduce some notation. Let L be a finite-dimensional vector space; we define one dimensional vector space $\det L$ as the highest exterior power of L:

$$\det L = \Lambda^{\dim L} L.$$

More generally, for a finite complex of finite-dimensional vector spaces $L^\bullet = \ldots \to L_{i-1} \to L_i \to L_{i+1} \to \ldots$, we denote $\det L^\bullet = \bigotimes (\det L_i)^{(-1)^i}$, where, for a one-dimensional vector space X, we let $X^1 = X$, $X^{-1} = X^*$.

EXERCISE 6.7.1. Show that there is a canonical isomorphism

$$\det L^\bullet = \bigotimes (\det H^i(L^\bullet))^{(-1)^i}.$$

This definition can be generalized to vector bundles over a smooth base S: if E is a vector bundle of dimension n over S, then we define line bundle $\det L$ by $\det L = \Lambda^n E$. Again, this can be trivially generalized to complexes of vector bundles, and we have the following proposition.

LEMMA 6.7.2. *Let $E^\bullet, F^\bullet$ be finite complexes of vector bundles over S, and let $f\colon E^\bullet \to F^\bullet$ be a morphism of complexes of vector bundles which is a quasi-isomorphism, i.e., it induces isomorphism of the cohomology sheaves $\mathcal{H}^i f\colon \mathcal{H}^i(\mathcal{E}^\bullet) \xrightarrow{\sim} \mathcal{H}^i(\mathcal{F}^\bullet)$. Then f defines an isomorphism of the line bundles* $\det E^\bullet \simeq \det F^\bullet$.

This lemma allows one to define $\det \mathcal{E}$ for arbitrary coherent $\mathcal{O}$-module $\mathcal{E}$, generalizing the case when $\mathcal{E}$ is the sheaf of sections of a vector bundle E. Indeed, every coherent $\mathcal{O}$-module admits a resolution by vector bundles: we can find a complex of vector bundles $F^\bullet$ such that $\mathcal{H}^0(\mathcal{F}^\bullet) = \mathcal{E}, \mathcal{H}^i(\mathcal{F}^\bullet) = 0$ for $i \neq 0$. By definition, let $\det \mathcal{E} = \det F^\bullet$. Lemma 6.7.2 shows that this is independent of the choice of resolution. The same argument shows that we can define $\det \mathcal{E}^\bullet$ for a complex of coherent $\mathcal{O}$-modules $\mathcal{E}^\bullet$, and that it only depends on the quasi-isomorphism class of $\mathcal{E}^\bullet$; in other words, det is well defined on the derived category of coherent $\mathcal{O}$-modules (see [**KM**] for details).

REMARK 6.7.3. In [**KM**], the determinant line bundle is defined as a pair, consisting of a line bundle a and "parity", i.e. an element of $\mathbb{Z}/2\mathbb{Z}$. Parity is important for tracking correct signs in isomorphisms like $\det(F \oplus G) \simeq \det F \otimes \det G$. However, for us these signs are not important, and therefore we omit parity.

DEFINITION 6.7.4. Let C_S be a family of pointed curves over S. We define the corresponding *determinant line bundle* Q_S by

$$Q = (\det R\pi_* \mathcal{O}_{C_S})^{-1} = \bigotimes (\det R^i \pi_* \mathcal{O}_{C_S})^{(-1)^{i+1}}, \tag{6.7.1}$$

where π is the projection $C_S \to S$.

Note that this definition does not use the marked points. Also, this definition is valid even if the family C_S is singular: in this case, $R^i \pi_* \mathcal{O}_{C_S}$ needs not be a vector bundle, but is always a coherent $\mathcal{O}_S$-module, and thus $\det R^i\pi_*\mathcal{O}_{C_S}$ is a line bundle.

For readers who prefer to avoid using the notion of higher direct images, it suffices to say that the fiber of this line bundle at point $s \in S$ is

$$Q_s = \bigotimes(\det H^i(C_s, \mathcal{O}_{C_s}))^{(-1)^{i+1}}.$$

For a smooth family, this definition can be simplified. Let us assume that C_S is a non-singular family with connected fiber. Then $H^0(C_s, \mathcal{O}) = \mathbb{C}$, and $H^i(C_s, \mathcal{O}) = 0$ for $i > 1$ (since $\mathcal{O}$ is coherent). Thus, in this case $Q = \det R^1\pi_*(\mathcal{O}_{C_S})$, so its fiber at point s is given by

$$Q_s = \det(H^1(C_s, \mathcal{O}_{C_s})). \tag{6.7.2}$$

These spaces are well known (see, e.g., [**GH**]): for a compact complex curve of genus g, $H^1(C, \mathcal{O}_C)$ is a vector space of complex dimension g. In particular, for $g = 0, H^1(C, \mathcal{O}_C) = 0$ and thus, the determinant line bundle is trivial.

One easily sees that definition of the determinant bundle Q_S is functorial in S, and thus, we have a well-defined line bundle Q_M over the moduli stack $M_{g,n}$. The same definition also works for singular curves, and thus Q_M is well defined over the completion $\overline{M}_{g,n}$.

Finally, let us discuss the behavior of the determinant bundle with respect to gluing. Let A be a finite set, $\alpha, \beta \in A$—an unordered pair, $A' = A \setminus \{\alpha, \beta\}$. Such a pair defines a "clutching" map $S_{\alpha,\beta}\colon M_{*,A} \to N(D^0)$, where D^0 is the corresponding component of the boundary in $\overline{M}_{*,A'}$ (see (6.2.7)).

PROPOSITION 6.7.5. *Let $S_{\alpha,\beta}$ be as above, Q–the determinant line bundle on $M_{*,A'}$. Let us also denote by Q the corresponding line bundle on $N(D^0)$. Then $S^*_{\alpha,\beta}(Q)$ is canonically isomorphic to the determinant line bundle over $M_{*,A}$.*

We omit the proof of this proposition, referring the reader to [**BFM**].

Now, we are ready to formulate the definition of the modular functor with central charge, which is parallel to Definition 6.4.1, but with replacement of vector bundles with flat connection by vector bundles with the action of the central extension $\mathcal{A}_{Q^c}$ of the Lie algebra of vector fields, or, equivalently, with the action of $\mathcal{D}_{Q^c}$. As before, let $\mathcal{C}$ be an abelian category over the field $\mathbb{C}$, and R – a symmetric object in ind$-\mathcal{C}^{\boxtimes 2}$.

DEFINITION 6.7.6. A *complex $\mathcal{C}$-extended modular functor with* (*additive*) *central charge* $a \in \mathbb{C}$ is the following collection of data:

(i) For every finite set A and $W \in \mathcal{C}^{\boxtimes A}$, a finite-dimensional vector bundle over $\overline{M}_{*,A}$ with an action of $\mathcal{D}_{Q^a}$. This bundle is called the *bundle of conformal blocks*; its fiber at a point $C \in \overline{M}_{*,A}$ will be denoted by $\langle W\rangle_C$.

(ii) Isomorphisms $\langle X\rangle_{C'} \otimes \langle Y\rangle_{C''} \simeq \langle X \boxtimes Y\rangle_{C' \sqcup C''}$.

(iii) **Gluing isomorphisms.** Let A be a finite set, $\alpha, \beta \in A$—an unordered pair, $A' = A \setminus \{\alpha, \beta\}, W \in \mathcal{C}^{\boxtimes A'}$. For every such collection, we require an isomorphism of $\mathcal{D}_{Q^a}$-modules on $M_{*,A}$:

$$G_{\alpha,\beta}\colon \langle W \boxtimes R\rangle \xrightarrow{\sim} S^*_{\alpha,\beta} Sp_D\langle W\rangle, \tag{6.7.3}$$

where $S_{\alpha,\beta}\colon M_{*,A} \to N(D^0), D^0 \subset M_{*,A'}$, is the "clutching" (6.2.7), and R is placed at positions with indices α, β. (Since $S_{\alpha,\beta}$ is a $\mathbb{C}^\times$-bundle, the definition of $S^*_{\alpha,\beta}$ causes no problems.)

(iv) **Vacuum propagation.** We have a distinguished element $\mathbf{1} \in \operatorname{Ob}\mathcal{C}$, and for every $\alpha \in A$, we require an isomorphism of $\mathcal{D}_{Q^a}$-modules on $M_{*,A}$:

$$G_\alpha \colon \langle W \boxtimes \mathbf{1} \rangle \xrightarrow{\sim} S_\alpha^* \langle W \rangle, \tag{6.7.4}$$

where $S_\alpha \colon M_{*,A} \to M_{*,A\setminus\alpha}$ is the operator of erasing the point α.

These data have to satisfy the following properties:

Functoriality: $W \mapsto \langle W \rangle_C$ is functorial in W, and the isomorphisms (ii)-(iv) are functorial isomorphisms.

Compatibility: the isomorphisms (ii)-(iv) are compatible with each other and with the commutativity, associativity, and unit morphisms in $\mathcal{V}ec_f$ (cf. Definition 4.2.1).

Normalization: $\langle \mathbf{1}, \mathbf{1} \rangle_{\mathbb{P}^1} = \mathbb{C}$.

Note that the requirement that $\langle W \rangle$ be a $\mathcal{D}_{Q^a}$ is equivalent to saying that the pull-back of $\langle W \rangle$ to the total space of the line bundle Q has a monodromic flat connection with the monodromy $e^{-2\pi a}$ around the zero section (see Exercise 6.6.4).

Let us now relate it to the topological formulation of the modular functor with central charge. To do it, let us recall the definition of the central charge for the modular functor in the topological setup (see Section 5.7).

It was defined using the central extension of the usual tower of mapping class groups by the groupoid $T_\Sigma = T_{H^1(\Sigma,\mathbb{R})}$, where Σ is a closed oriented topological surface of genus g, and for a symplectic real vector space V, T_V is the Poincare (fundamental) groupoid $T_V = \pi_1(\Lambda_V)$ of the set Λ_V of all Lagrangian subspaces in V. Recall also that for every $L \in T_V, \operatorname{Hom}_{T_V}(L, L) \simeq \mathbb{Z}$.

It is well known (see, e.g., [**GH**]) that for a connected compact complex curve the natural map of sheaves $\mathbb{R} \to \mathcal{O}$ induces an isomorphism of the cohomology spaces $H^1(C,\mathbb{R}) \simeq H^1(C,\mathcal{O})$, where both sides are considered as vector spaces over $\mathbb{R}$. In other words, a choice of a complex structure on a topological surface Σ defines a complex structure on the $2g$-dimensional real vector space $V = H^1(\Sigma,\mathbb{R})$.

THEOREM 6.7.7. *Let C be a complex curve. Then one has a canonical equivalence of groupoids*

$$T_\Sigma = \pi_1(Q_C^{\otimes 2} \setminus \{0\})$$

where, as before, $Q_C = \det H^1(C, \mathcal{O}_C)$.

PROOF. Let us note that the complex structure on $V = H^1(C,\mathbb{R})$ defined by the identification $V \simeq H^1(C,\mathcal{O})$, agrees with the symplectic structure in V as follows:

$$\langle \cdot, i_C \cdot \rangle \text{ is symmetric positive definite} \tag{6.7.5}$$

where $\langle \cdot,\cdot \rangle$ is the symplectic form, and $i_C \in \operatorname{End}_{\mathbb{R}}(V)$ is the operator of multiplication by $\mathrm{i} = \sqrt{-1}$ in the complex structure defined by C.

Let V be an arbitrary symplectic real vector space. Denote by H_V the set of all complex structures on V satisfying the condition above. It is usually called *Siegel upper half plane of* V. We quote without proof the following standard result, which can be found, for example, in [**GH**].

THEOREM 6.7.8. *H_V is a contractible space.*

EXERCISE 6.7.9. Show that for $V = \mathbb{R}^2$ with the standard symplectic form, H_V can be identified with the upper half-plane of $\mathbb{C}$.

Let us consider the line bundle λ over H_V, whose fiber at point $h \in H_V$ is $\lambda_h = \det_{\mathbb{C}} V$; here V is considered as g-dimensional vector space over $\mathbb{C}$ with the complex structure given by h. Consider the total space of $\lambda^{\otimes 2} \setminus \{\text{zero section}\}$.

PROPOSITION 6.7.10. *For every $h \in H_V$, one has canonical equivalence of groupoids*

$$\pi_1(\lambda^{\otimes 2} \setminus \{\textit{zero section}\}) \simeq \pi_1(\lambda_h^{\otimes 2} \setminus \{0\}) \simeq T_V. \tag{6.7.6}$$

Obviously, this proposition immediately implies the statement of the theorem.

Let us first construct equivalence $\pi_1(\lambda^{\otimes 2} \setminus \{\text{zero section}\}) \simeq \pi_1(\lambda_h^{\otimes 2} \setminus \{0\})$. This equivalence follows from the fact that H_V is contractible, and therefore, embedding $\lambda_h^{\otimes 2} \setminus \{0\} \hookrightarrow \lambda^{\otimes 2} \setminus \{\text{zero section}\}$ is a homotopy equivalence.

Next, let us construct equivalence $\pi_1(\lambda_h^{\otimes 2} \setminus \{0\}) \simeq T_V$. To do so, let us rewrite the definition of T_V as follows. Let L be the tautological vector bundle of dimension g over Λ_V: it is a sub-bundle of the trivial bundle $V \times \Lambda_V$ such that its fiber at point $L \in T(V)$ is the subspace $L \subset V$. Consider the one-dimensional vector bundle $\lambda_{\mathbb{R}} = \det_{\mathbb{R}} L = \Lambda_{\mathbb{R}}^g L$ (to avoid confusion, we use subscript $\mathbb{R}$ for determinant of vector spaces over $\mathbb{R}$). Then $\lambda_{\mathbb{R}}^{\times} / \pm 1$ is a bundle with fiber $\mathbb{R}_+$ over Λ_V; thus, $\lambda_{\mathbb{R}}^{\times} / \pm 1 \to \Lambda_V$ is a homotopy equivalence, and $T_V = \pi_1(\lambda_{\mathbb{R}}^{\times} / \pm 1)$.

Now let $h \in H_V$ be a complex structure on V, $\lambda_h = \det V = \Lambda^g V$, where V is considered as a vector space over $\mathbb{C}$ using the complex structure h. Thus, we have a well-defined map $\lambda_{\mathbb{R}} = \Lambda_{\mathbb{R}}^g L \to \lambda_h = \Lambda^g V$; the left-hand side is one dimensional real vector space, the right hand side is one-dimensional complex vector space. This gives rise to a map $\lambda_{\mathbb{R}}^{\times} / \pm 1 \to \lambda^{\otimes 2} \setminus \{0\}$.

This completes the proof of the proposition. On the other hand, the proposition immediately implies the statement of the theorem. $\square$

EXAMPLE 6.7.11. Let $g = 1$, so Σ is a torus (or, equivalently, C is an elliptic curve). Then $V = \mathbb{R}^2$ with the canonical symplectic form, so Λ_V is the set of all real one-dimensional subspaces in $\mathbb{R}^2$, thus $\Lambda_V \simeq S^1$. On the other hand, in this case $\lambda_h = \det V = V$ (as a complex vector space). Thus, in this case the canonical map $\pi_1(\Lambda_V) \to \pi_1(\lambda_h^{\otimes 2} \setminus \{0\})$ is obvious.

As an immediate corollary of Theorem 6.7.7, we see get the following result. As before, let $\mathcal{C}$ be an abelian category over $\mathcal{C}$.

THEOREM 6.7.12. *The notions of $\mathcal{C}$-extended topological MF with multiplicative central charge K and of $\mathcal{C}$-extended complex MF with additive central charge a are equivalent if $K = e^{\pi \mathrm{i} a}$.*

COROLLARY 6.7.13. *Any MTC category over $\mathbb{C}$ gives rise to a $\mathcal{C}$-extended complex MF with additive central charge a such that*

$$e^{\pi i a} = p^+/p^-. \tag{6.7.7}$$

Conversely, every $\mathcal{C}$-extended complex MF with additive central charge a defines a weakly ribbon structure on $\mathcal{C}$; if this category is rigid, then it is modular, and p^+/p^- is given by the formula above.

Note that for modular functors coming from rational conformal field theory, the additive central charge is given by

$$a = c/2,$$

where c is the Virasoro central charge of the theory (we will illustrate it in the next chapter, where the Wess-Zumino-Witten model is considered). Combining this with the corollary above, we see that in this case one has

$$K = \frac{p^+}{p^-} = e^{\pi \mathrm{i} c/2}$$

(cf. Remark 3.1.20).