CHAPTER 5

Modular Functor

Given a modular tensor category $\mathcal{C}$, in the previous chapter we constructed a 3-dimensional Topological Quantum Field Theory (3D TQFT). Moreover, this 3D TQFT was based on an extended notion of a manifold (a usual manifold with additional data). In this chapter, we will show that the notion of a modular tensor category (MTC) is essentially equivalent to some geometric construction in dimension 2. The right notion here is that of a modular functor, which was introduced by Segal (see [**S**]). Our exposition mostly follows the papers [**S, MS1, MS2, T**] and folklore of mathematical physicists.

5.1. Modular functor

DEFINITION 5.1.1. A (*topological*) *d-dimensional modular functor* (MF for short) is the following collection of data:

(i) A vector space $\tau(N)$ assigned to any oriented compact d-manifold N without boundary.

(ii) An isomorphism $f_*\colon \tau(N_1) \xrightarrow{\sim} \tau(N_2)$ of vector spaces assigned to every homeomorphism $f\colon N_1 \xrightarrow{\sim} N_2$, which depends only on the isotopy class of f.

(iii) Isomorphisms $\tau(\emptyset) \xrightarrow{\sim} k$, $\tau(N_1 \sqcup N_2) \xrightarrow{\sim} \tau(N_1) \otimes \tau(N_2)$, where k is the base field.

These data have to satisfy the following axioms:

Multiplicativity: $(fg)_* = f_* g_*$, $\mathrm{id}_* = \mathrm{id}$.

Functoriality: the isomorphisms (iii) are functorial.

Compatibility: the isomorphisms of part (iii) are compatible with the canonical isomorphisms $N \sqcup \emptyset = N$, $N_1 \sqcup N_2 = N_2 \sqcup N_1$, $(N_1 \sqcup N_2) \sqcup N_3 = N_1 \sqcup (N_2 \sqcup N_3)$.

Normalization: We have an isomorphism $\tau(S^d) = k$, where S^d is the d-dimensional sphere.

Detailed statement of the functoriality and compatibility axioms can be found in Remark 4.2.2, where the same conditions appear in the definition of TQFT.

REMARK 5.1.2. Any $(d{+}1)$D TQFT (see Definition 4.2.1) gives a d-dimensional MF, because the axioms of a MF, except for the requirement that f_* depends only on the isotopy class of f, are contained in the axioms of a TQFT, and this last condition is satisfied by Theorem 4.2.3.

This modular functor is *unitary*: in addition to the data above, there are functorial isomorphisms $\tau(\overline{\Sigma}) \xrightarrow{\sim} \tau(\Sigma)^*$, where $\overline{\Sigma}$ is the manifold Σ with opposite orientation, which are compatible with the isomorphisms of part (iii).

DEFINITION 5.1.3. (i) We define a category Γ with:

Objects: d-manifolds.

Morphisms: $\mathrm{Mor}_\Gamma(N_1, N_2)$ = isotopy classes of orientation-preserving homeomorphisms $N_1 \xrightarrow{\sim} N_2$.

This is a symmetric tensor category with the "tensor product" given by disjoint union, and the unit given by $\emptyset$. (Note that this category is not additive: one can not add homeomorphisms!)

(ii) For a manifold N, its *mapping class group* $\Gamma(N)$ is the group of isotopy classes of homeomorphisms $N \xrightarrow{\sim} N$. In other words, $\Gamma(N) := \mathrm{Mor}_\Gamma(N, N)$.

The category Γ is a *groupoid*, i.e., a category in which every morphism is invertible. One easily sees that d-dimensional modular functor is the same as a representation of the groupoid Γ, i.e., a tensor functor $\Gamma \to \mathcal{V}ec_f(k)$. This explains the origin of the term "modular functor".

In particular, by 5.1.1(ii), every MF defines a representation of the mapping class group $\Gamma(N)$ of any d-manifold N on the vector space $\tau(N)$.

From now on, let us assume that $d = 2$. Then every connected compact oriented surface is determined up to homeomorphism by its genus g, and defining a modular functor is equivalent to defining for every $g \geq 0$ a representation of the mapping class group Γ_g. We quote here some classical results regarding the mapping class groups.

THEOREM 5.1.4 (Dehn). *Let Σ be a compact oriented surface, and let c be a simple closed curve on Σ. Define the* Dehn twist $t_c \in \Gamma(\Sigma)$ *by Figure* 5.1.[1] *Then the elements t_c generate the mapping class group $\Gamma(\Sigma)$.*

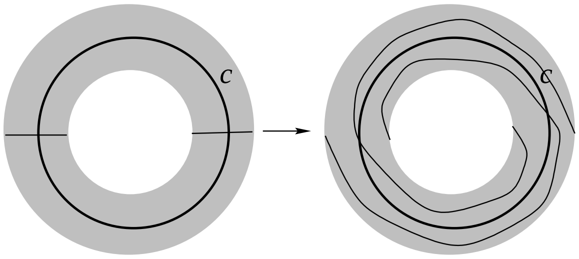


FIGURE 5.1. Dehn twist.

This theorem was later refined by Lickorish [**Li**], who suggested a finite set of Dehn twists generating $\Gamma(\Sigma)$. Finally, an approach allowing one to describe the generators and relations in $\Gamma(\Sigma)$ was given in [**HT**]. For surfaces of genus g with 0 or 1 boundary components (or marked points), the ideas of [**HT**] were fully developed in [**Waj**], where a complete set of generators and relations for $\Gamma_g \equiv \Gamma_{g,0}$ and $\Gamma_{g,1}$ is written.

EXAMPLE 5.1.5. Let $g = 1$, i.e., let Σ be a two-dimensional torus. Then, by Theorem 4.1.3, $\Gamma_1 \simeq \mathrm{SL}_2(\mathbb{Z})$, which can be described as the group with generators

[1] Here we put some auxiliary lines on the surface to demonstrate the action of the homeomorphisms. These lines are for illustration purposes only. Note that c is not required to be oriented.

s, t and relations $(st)^3 = s^2, s^4 = 1$ (which implies $s^2t = ts^2$). It can also be generated by the elements

$$t_a = t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad t_b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

which correspond to Dehn twists around the meridian and the parallel of the torus.

It turns out that for $d = 2$ the notion of modular functor can be generalized by allowing surfaces with "holes", i.e., with boundary.

DEFINITION 5.1.6. An *extended surface* is a compact oriented surface Σ, possibly with boundary, together with an orientation-preserving parameterization $\pi_i\colon (\partial\Sigma)_i \xrightarrow{\sim} S^1$ of every boundary circle. Here $(\partial\Sigma)_i$ is considered with the orientation induced from Σ, and $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with the counterclockwise orientation.

By a *genus* of an extended surface, we will mean the genus of the closed surface $cl(\Sigma)$ obtained by "patching the holes of Σ", i.e., gluing a disk to every boundary circle.

A *homeomorphism* of extended surfaces $f\colon \Sigma \xrightarrow{\sim} \Sigma'$ is an orientation-preserving homeomorphism which also preserves parameterizations.

Finally, for an extended surface $(\Sigma, \pi_i\colon (\partial\Sigma)_i \xrightarrow{\sim} S^1)$ we define the operation of *orientation reversal* by $(\overline{\Sigma}, -\overline{\pi_i})$ (**note the minus sign!**).

The notion of isotopy of homeomorphisms is trivially generalized to this case, as well as the notion of disjoint union. Thus, we can define the extended groupoid $\mathcal{T}eich$ similarly to Definition 5.1.3(i).

DEFINITION 5.1.7. (i) The (*extended*) *Teichmüller groupoid* $\mathcal{T}eich$ is the category with objects extended surfaces, and morphisms isotopy classes of homeomorphisms of extended surfaces (see Definition 5.1.6).

(ii) For any extended surface Σ, its *mapping class group* $\Gamma(\Sigma)$ is the group of all isotopy classes of homeomorphisms $\Sigma \xrightarrow{\sim} \Sigma$. (Sometimes the name "mapping class group" is used for the smaller group $\Gamma'(\Sigma)$ of all isotopy classes of homeomorphisms $\Sigma \xrightarrow{\sim} \Sigma$ which act trivially on the set of connected components of the boundary.) If Σ is a surface of genus g with n boundary components, we will denote $\Gamma(\Sigma) \equiv \Gamma_{g,n}$.

Again, it can be shown that $\Gamma'(\Sigma)$ is generated by Dehn twists (a complete set of relations for $\Gamma'_{g,n}$ is given in [**Ge1**], [**Luo**], [**Ge2**]), and $\Gamma_{g,n}$ is generated by Dehn twists and the "braiding operation" shown in Figure 5.2.[2]

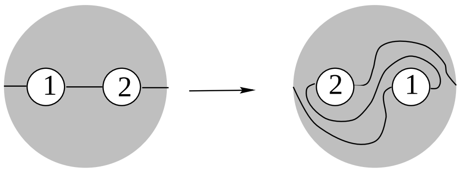


FIGURE 5.2. Braiding.

It will be useful in the future to give an alternative definition of an extended surface. We give below two such definitions. Both of them are equivalent to Definition 5.1.6 in the following sense:

[2]See the footnote on page 94.

PROPOSITION 5.1.8. *The extended groupoids $\mathcal{T}eich$, defined by Definitions* 5.1.6, 5.1.9 *and* 5.1.10, *are equivalent as categories, and this equivalence preserves the operation of orientation reversal.*

DEFINITION 5.1.9. An *extended surface* is an oriented compact surface with boundary and with a specified point p_i on every component of the boundary.

A *homeomorphism* of extended surfaces is an orientation-preserving homeomorphism $\Sigma \to \Sigma'$ which maps marked points to marked points.

Orientation reversal is defined in the obvious way, by reversing the orientation of Σ while leaving the points p_i unchanged.

DEFINITION 5.1.10. An *extended surface* is an oriented compact surface Σ without boundary, with marked points z_i, and with non-zero tangent vectors v_i attached to each marked point.

A *homeomorphism* of extended surfaces is an orientation-preserving homeomorphism $\Sigma \to \Sigma'$ which maps marked points to marked points, and marked tangent vectors to marked tangent vectors.

Orientation reversal is defined by $\overline{(\Sigma, z_i, v_i)} = (\overline{\Sigma}, z_i, -v_i)$.

This definition is analogous to Definition 4.4.1.

PROOF OF PROPOSITION 5.1.8. To establish the equivalence of Definitions 5.1.6 and 5.1.9, note that a parameterization of a boundary circle gives a distinguished point $p_i = \pi_i^{-1}(\mathrm{i})$. Since the set of all homeomorphisms $S^1 \xrightarrow{\sim} S^1$ preserving orientation and the distinguished point $\mathrm{i} \in S^1$ is contractible, this is an equivalence of categories. Similarly, to establish the equivalence of Definitions 5.1.6 and 5.1.10, note that given Σ as in Definition 5.1.6, we can glue to Σ n copies of the standard disk $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ (with reversed orientation), using the identifications of the boundary circles of Σ with S^1. This gives a new surface $cl(\Sigma)$ without boundary, with marked points images of $0 \in D$, and tangent vectors images of the unit vector going along the real axis in D. As before, it is easy to check that this gives an equivalence of categories. $\square$

EXAMPLES 5.1.11. (i) Let Σ be a two-dimensional torus "with one puncture": $\partial\Sigma \simeq S^1$ and Σ has genus 1. Then the mapping class group $\Gamma_{1,1} = \Gamma(\Sigma)$ is generated by the elements s, t with the relations $(st)^3 = s^2, s^2$ is central (compare with Example 5.1.5). Moreover, s^4 is the inverse of the Dehn twist around the puncture. The easiest way to check this is to use the realization of the torus with one puncture as the quotient $\mathbb{R}^2/\mathbb{Z}^2$ with a non-zero tangent vector at the origin.

(ii) Let $\Sigma_n = \mathbb{R}^2$, with n marked points on the x-axis and with the tangent vector v_i going along this axis in positive direction (all such surfaces are canonically isomorphic). This surface is not compact, so it does not formally satisfy our definition, but let us ignore this. Then the group $\Gamma(\Sigma)$ is isomorphic to the group FB_n of all framed braids with n strands. This group is a semidirect product of the usual braid group B_n and $\mathbb{Z}^n$ (see Definition 1.2.1). In general, there is indeed a relationship between the group $\Gamma(\Sigma)$, where Σ is an extended surface with n holes, and the framed braid group $FB_n(cl(\Sigma))$, where $cl(\Sigma)$ is the closed surface obtained by patching the holes of Σ. This relationship is studied in detail in [**B2**].

The most important difference between extended surfaces and usual surfaces is that extended surfaces can be glued (or sewed) together along the boundary circles. Therefore, if we additionally require a modular functor to behave nicely under this operation, we could define $\tau(\Sigma)$ by gluing Σ from simpler pieces. This motivates the following definition.

DEFINITION 5.1.12. Let $\mathcal{C}$ be an abelian category over a field k, and let R be a symmetric object in ind$-\mathcal{C}^{\boxtimes 2}$ (see Section 2.4). Then a *$\mathcal{C}$-extended modular functor* is the following collection of data:

(i) To every extended surface Σ is assigned a polylinear functor $\tau(\Sigma)\colon \mathcal{C}^{\boxtimes \pi_0(\partial\Sigma)} \to \mathcal{V}ec_f$, where $\pi_0(\partial\Sigma)$ is the set of boundary components (or punctures, depending on the point of view) of Σ. In other words, for every choice of objects $W_a \in \mathcal{C}$ attached to every boundary component of Σ (so, a runs through the set of connected components of $\partial\Sigma$) is assigned a finite-dimensional vector space $\tau(\Sigma; \{W_a\})$, and this assignment is functorial in W_a.

(ii) To every homeomorphism $f\colon \Sigma \xrightarrow{\sim} \Sigma'$ is assigned a functorial isomorphism $f_*\colon \tau(\Sigma) \xrightarrow{\sim} \tau(\Sigma')$.

(iii) Functorial isomorphisms $\tau(\emptyset) \xrightarrow{\sim} k$, $\tau(N_1 \sqcup N_2) \xrightarrow{\sim} \tau(N_1) \otimes \tau(N_2)$.

(iv) **Gluing isomorphism:** Let $c \subset \Sigma$ be a closed curve without self-intersections and p be a marked point on c. Cutting Σ along c, we obtain a new surface Σ' (which may be connected or not). Σ' has a natural structure of an extended surface in the sense of Definition 5.1.9 which has the same boundary components as Σ plus two more components c_1, c_2, which come from the circle c (with marked points p_1, p_2 coming from p).

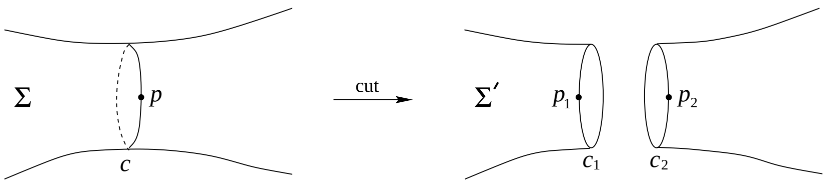


FIGURE 5.3. Cutting of a surface.

Then we are given a functorial isomorphism

$$\tau(\Sigma'; \{W_a\}, R^{(1)}, R^{(2)}) \xrightarrow{\sim} \tau(\Sigma; \{W_a\}), \tag{5.1.1}$$

where we use the notation of Section 2.4.

The above data have to satisfy the following axioms:

Multiplicativity: $(fg)_* = f_* g_*$, $\mathrm{id}_* = \mathrm{id}$.
Functoriality: all isomorphisms in parts (iii), (iv) above are functorial in Σ.
Compatibility: all isomorphisms in parts (iii), (iv) above are compatible with each other.
Normalization: $\tau(S^2) = k$.

As before, we leave it to the reader to write the explicit statements of the functoriality and compatibility axioms, taking as an example the definitions in Section 4.2. From now on, we will always work with extended modular functors (unless otherwise specified).

DEFINITION 5.1.13. A $\mathcal{C}$-extended MF is called *non-degenerate* if for every object $V \in \operatorname{Ob}\mathcal{C}$ there exists an extended surface Σ and $\{W_a\} \subset \operatorname{Ob}\mathcal{C}$ such that $\tau(\Sigma; V, \{W_a\}) \neq 0$.

The main goal of this chapter is to show that *for a given **semisimple** abelian category $\mathcal{C}$ defining a non-degenerate $\mathcal{C}$-extended MF is essentially equivalent to defining a structure of a modular tensor category on $\mathcal{C}$*, with the object $R = \bigoplus V_i \boxtimes V_i^*$, where $\{V_i\}$ are representatives of the equivalence classes of simple objects in $\mathcal{C}$. The precise statements are given in Theorems 5.4.1 and 5.5.1.

Finally, let us introduce the notion of a unitary MF.

DEFINITION 5.1.14. An extended modular functor is called *unitary*, if in addition to the data above, we are also given functorial isomorphisms $\tau(\overline{\Sigma}) \xrightarrow{\sim} \tau(\Sigma)^*$, where $\overline{\Sigma}$ is the manifold Σ with opposite orientation. These isomorphisms must be compatible with the isomorphisms f_* and the isomorphisms of part (iii) of Definition 5.1.12 in the natural way. Also, we require the following compatibility of the unitary structure with the gluing isomorphism. Let $\langle , \rangle_\Sigma : \tau(\Sigma) \otimes \tau(\overline{\Sigma}) \to k$ be the pairing induced by the isomorphism $\tau(\Sigma) \simeq \tau(\overline{\Sigma})^*$. Let Σ, Σ' be as in part (iv) of Definition 5.1.12, and for $f \in \tau(\Sigma), g \in \tau(\overline{\Sigma})$, write $f = \sum f_i$, $g = \sum g_i$ with $f_i \in \tau(\Sigma'; A_i, B_i)$, $g_i \in \tau(\overline{\Sigma}'; B_i, A_i)$, using (5.1.1). Then:

$$\langle f, g \rangle_\Sigma = \sum a_i \langle f_i, g_i \rangle_{\Sigma'} \tag{5.1.2}$$

?! for some non-zero constants a_i which do not depend on Σ.

5.2. The Lego game

Let us denote by $S_{0,n}$ "the standard sphere with n holes":

$$S_{0,n} = \mathbb{CP}^1 \setminus \{D_1, \dots, D_n\}, \quad D_j = \{z \mid |z - z_j| < \varepsilon\}, \quad z_1 < \dots < z_n, \tag{5.2.1}$$

where $\varepsilon > 0$ is small enough so that the disks D_j do not intersect, and let us mark on each boundary circle a point $p_j = z_j - \varepsilon \mathrm{i}$. This endows $S_{0,n}$ with the structure of an extended surface which is independent of the choice of z_j, ε (i.e., surfaces obtained for different choices of z_j, ε are canonically homeomorphic). Note that the set of boundary components of the standard sphere is naturally indexed by numbers $1, \dots, n$; we will use bold numbers for denoting these boundary components: $\pi_0(\partial S_{0,n}) = \{\mathbf{1}, \dots, \mathbf{n}\}$.

Obviously, every extended surface Σ can be obtained by gluing together standard spheres. Therefore, using the gluing axiom we can define the vector space $\tau(\Sigma)$ once we know $\tau(S_{0,n})$. However, the same surface Σ can be obtained by gluing the standard spheres in many ways, and in order for $\tau(\Sigma)$ to be correctly defined we need to construct canonical isomorphisms between the resulting vector spaces. This leads to the following problem.

DEFINITION 5.2.1. Let Σ be an extended surface. A *parameterization* of Σ is the following collection of data, considered up to isotopy:

(i) A finite set $C = \{c_1, \dots\}$ of simple non-intersecting closed curves (cuts) on Σ, with one point marked on every cut (the cuts do not have to be ordered).

(ii) A collection of homeomorphisms $\psi_a : \Sigma_a \xrightarrow{\sim} S_{0,n_a}$, where Σ_a are the connected components of $\Sigma \setminus C$.

We denote the set of all parameterizations of Σ by $M(\Sigma)$.

Our goal is to construct some number of edges ("moves") and 2-cells ("relations among moves") which would turn $M(\Sigma)$ into a connected and simply-connected 2-complex. This problem was first considered by Moore and Seiberg [**MS1**], who conjectured a set of moves and relations. However, their paper contains certain gaps making it not rigorous even by the physicists standards. An accurate proof was recently found independently by the authors [**BK**], and by [**FG**]. Our exposition follows the paper [**BK**] with minor changes.

Define the homeomorphisms

$$z\colon S_{0,n} \xrightarrow{\sim} S_{0,n},\qquad b\colon S_{0,3} \xrightarrow{\sim} S_{0,3} \tag{5.2.2}$$

as follows: z is rotation of the sphere which preserves the real axis and induces a cyclic permutation of the holes $\mathbf{1} \mapsto \mathbf{2} \mapsto \cdots \mapsto \mathbf{n} \mapsto \mathbf{1}$, and b is the braiding of the 2nd and 3rd punctures, as shown in Figure 5.2.

Also, for $k, l \geq 0$, denote by $S_{0,k+1} \sqcup_{k+1,1} S_{0,l+1}$ the surface obtained by identifying the $(k+1)$-st hole of $S_{0,k+1}$ with the first hole of $S_{0,l+1}$, and define the map

$$\alpha_{k,l}\colon S_{0,k+1} \sqcup_{k+1,1} S_{0,l+1} \to S_{0,k+l} \tag{5.2.3}$$

by the condition that it maps the first hole of $S_{0,k+1}$ to the first hole of $S_{0,k+l}$ and preserves the real axis (these properties define $\alpha_{k,l}$ uniquely up to isotopy).

Now, let us define the following edges ("simple moves") in $M(\Sigma)$. To avoid confusion, we will write $E\colon M_1 \rightsquigarrow M_2$ if the edge E connects parameterizations M_1, M_2.

Z-move (rotation): If $M = (C, \{\psi_a\}) \in M(\Sigma)$ and Σ_i is one of the connected components of $\Sigma \setminus C$, then we define an edge

$$Z \equiv Z_i\colon M \rightsquigarrow (C, \{\psi_a, z \circ \psi_i\}_{a \neq i}).$$

B-move (braiding): If $M = (C, \{\psi_a\}) \in M(\Sigma)$ and Σ_i is a connected component of $\Sigma \setminus C$ which has three holes, then we define an edge

$$B \equiv B_i\colon M \rightsquigarrow (C, \{\psi_a, b \circ \psi_i\}_{a \neq i}).$$

F-move (fusion): If $M = (C, \{\psi_a\}) \in M(\Sigma)$ and $c \in C$ separates two different components Σ_i, Σ_j, with $k+1$ and $l+1$ holes respectively, and $\psi_i(c) = \mathbf{k+1}, \psi_j(c) = \mathbf{1}$, then we define an edge

$$F \equiv F_c\colon M \rightsquigarrow (C \setminus \{c\}, \{\psi_a, \alpha_{kl} \circ (\psi_i \sqcup \psi_j)\}_{a \neq i,j}).$$

Before describing the relations, it is convenient to introduce some notation. First of all, let us place on each of the standard spheres $S_{0,n}$ the graph m_0 as shown in Figure 5.4 (for $n = 4$). This graph has one internal vertex, marked by a star; all other vertices are 1-valent and coincide with the marked points on the boundary components of $S_{0,n}$. The graph has a distinguished edge—the one which connects the vertex $*$ with the boundary component $\mathbf{1}$; in the figure, this edge is marked by an arrow. Also, this graph has a natural cyclic order on the set of all edges, given by $\mathbf{1} < \cdots < \mathbf{n} < \mathbf{1}$. Whenever we draw such a graph in the plane, we will always do it in such a way that this order coincides with the clockwise order.

Every parameterization M of a given surface Σ gives rise to a graph $m = \bigcup \psi_a^{-1}(m_0)$ on Σ, which we call the *marking graph* of M. It is easy to show that a parameterization is uniquely determined by C and m; therefore, these graphs give a way to visualize the parameterizations. In some cases, we will draw such graphs on Σ to illustrate a certain sequence of moves. However, in many cases it suffices just to draw the corresponding graphs on the plane, and then the moves can be reconstructed uniquely.

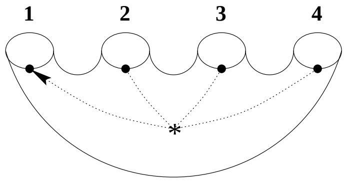


FIGURE 5.4. A standard sphere (with 4 holes).

EXERCISE 5.2.2. Show that the moves Z, B, F connect the parameterizations corresponding to the marking graphs shown in Figures 5.5, 5.6 and 5.7 below.

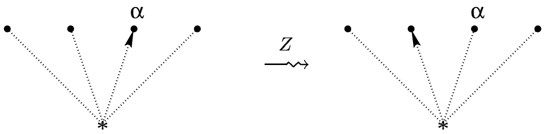


FIGURE 5.5. Z-move ("rotation").

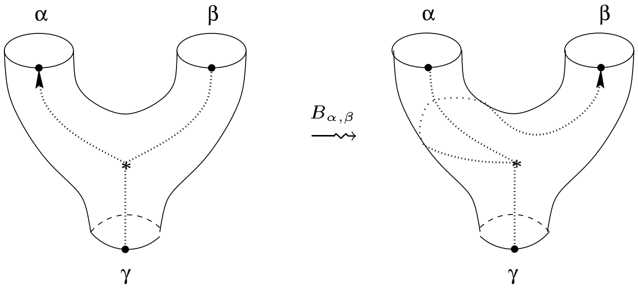


FIGURE 5.6. B-move ("braiding").

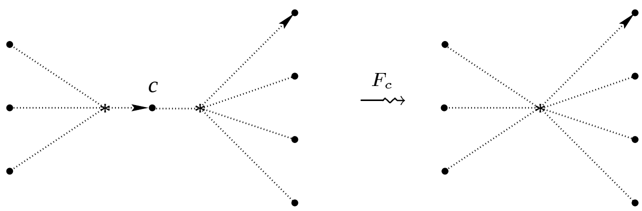


FIGURE 5.7. F-move ("fusion" or "cut removal").

Next, one often needs compositions of the form $Z^a F_c(Z_i^m \sqcup Z_j^n)$, where c is a cut separating components Σ_i and Σ_j (compare with the definition of the F-move). We will call any such composition a *generalized F-move*; for brevity, we will frequently denote it just by F_c. The Rotation axiom formulated below implies that such a composition is uniquely determined by the original parameterization M and by the choice of the distinguished edge for the resulting parameterization $F_c(M)$. Moreover, the Symmetry of F axiom along with the commutativity of disjoint union, also formulated below, imply that if we switch the roles of Σ_1 and Σ_2, then we get the same generalized F-move. Thus, the generalized F-move is completely determined by the marking graph of M and by the choice of the distinguished edge for the resulting marking graph of $F_c(M)$.

Finally, let $M \in M(\Sigma)$ and let Σ_i be one of the components of Σ. As discussed before, the parameterization ψ_i defines an order on the set of boundary components of Σ_i. Let us assume that we have a presentation of $\pi_0(\partial\Sigma_i)$ as a disjoint union, $\pi_0(\partial\Sigma_i) = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$, where the order is given by $I_1 < I_2 < I_3 < I_4$ (some of the I_k may be empty). Then we define the *generalized braiding move* B_{I_2,I_3} to be the product of simple moves shown in Figure 5.8 below (note that we are using generalized F-moves, see above). It is easy to show that this figure uniquely defines the cuts c_1, c_2, c_3 and thus, the generalized braiding move B.

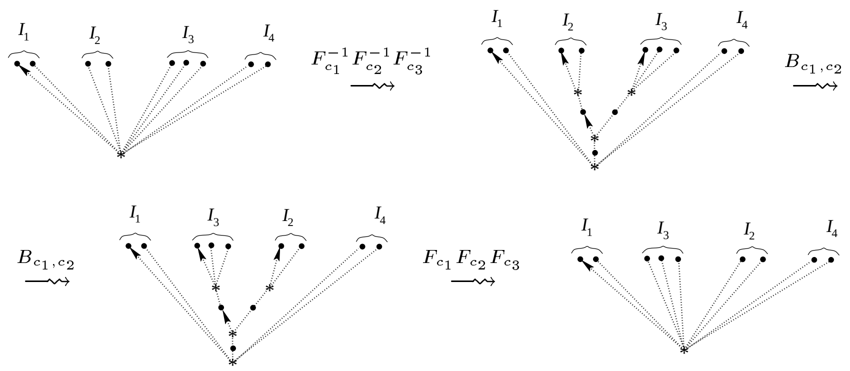


FIGURE 5.8. Generalized braiding move.

Now let us impose some relations among these moves:

MF1: Rotation axiom: If Σ_i is a component with n holes, then $Z_i^n = \mathrm{id}$.

MF2: Symmetry of F: If c, Σ_i, Σ_j are as in the definition of the F-move, then $Z^{k-1}F_c = F_c(Z_i^{-1} \sqcup Z_j)$.

MF3: Associativity of F: If Σ is a connected surface of genus zero, and $M = (C, m) \in M(\Sigma)$ is a parameterization with two cuts, $C = \{c_1, c_2\}$, then

$$F_{c_1}F_{c_2}(M) = F_{c_2}F_{c_1}(M) \tag{5.2.4}$$

(here F denotes generalized F-moves).

MF4: Commutativity of disjoint union: If E_1, E_2 are simple moves that involve non-intersecting sets of components, then $E_1E_2 = E_2E_1$.

MF5: Cylinder axiom: Let $S_{0,2}$ be a cylinder with boundary components α_0, α_1 and with the standard parameterization $M_0 = (\emptyset, \mathrm{id})$. Let Σ be an extended surface, $M \in M(\Sigma)$ be a parameterization, and α be a boundary component of Σ. Then, for every move $E\colon M \rightsquigarrow M'$ we require that the following square be commutative:

$$\begin{array}{ccc} M \sqcup_{\alpha,\alpha_1} M_0 & \xrightarrow{E \sqcup_{\alpha,\alpha_1} \mathrm{id}} & M' \sqcup_{\alpha,\alpha_1} M_0 \\ {\scriptstyle F_\alpha}\downarrow & & \downarrow{\scriptstyle F_\alpha} \\ M & \xrightarrow[E]{} & M' \end{array} , \tag{5.2.5}$$

see Figure 5.9 below.

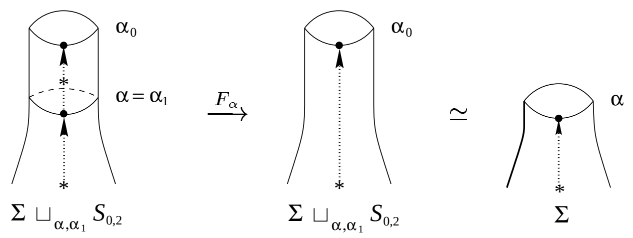


FIGURE 5.9. Cylinder Axiom.

MF6: Braiding axiom: Let Σ_i be a connected component of $\Sigma \setminus C$ which has 4 holes. Denote the boundary components $\psi_i^{-1}(\mathbf{1}), \dots, \psi_i^{-1}(\mathbf{4})$ of Σ_i by $\alpha, \dots, \delta$, respectively. Then:

$$B_{\alpha,\beta\gamma} = B_{\alpha,\gamma} B_{\alpha,\beta}, \tag{5.2.6}$$

$$B_{\alpha\beta,\gamma} = B_{\alpha,\gamma} B_{\beta,\gamma}. \tag{5.2.7}$$

For an illustration of Eq. (5.2.6), see Figure 5.10. Note that all braidings involved are generalized braidings as defined above.

MF7: Dehn twist axiom: Let Σ_i be a connected component of $\Sigma \setminus C$ which has 2 holes: $\alpha = \psi_i^{-1}(\mathbf{1}), \beta = \psi_i^{-1}(\mathbf{2})$. Then

$$Z_i B_{\alpha,\beta} = B_{\beta,\alpha} Z_i \tag{5.2.8}$$

(as before, B denotes the generalized braidings). This axiom is equivalent to the identity $T_\alpha = T_\beta$, where T_α is the Dehn twist defined in Example 5.2.4 below (see Figure 5.11).

THEOREM 5.2.3. *Let Σ be an extended surface of genus zero. Denote by $\mathcal{M}(\Sigma)$ the 2-complex with a set of vertices $M(\Sigma)$, edges given by the B-, Z-, and F-moves*

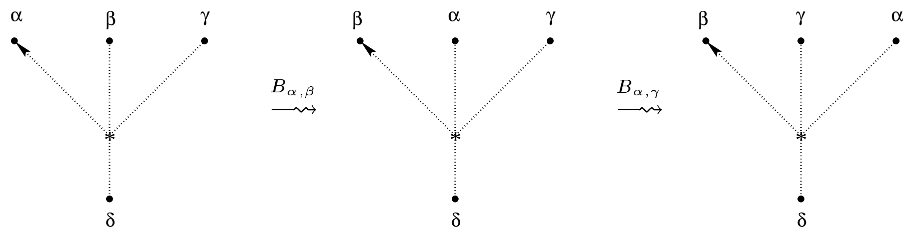


FIGURE 5.10. Braiding axiom (5.2.6).

defined above, and 2-cells given by relations MF1–MF7*. Then $\mathcal{M}(\Sigma)$ is connected and simply-connected.*

As mentioned above, this theorem was first proved (in a different form) in [**MS1**]; our exposition follows [**BK**].

EXAMPLE 5.2.4. Let Σ be an extended surface, $\psi\colon \Sigma \xrightarrow{\sim} S_{0,n}$ be a homeomorphism, and let α be one of the boundary components. Then one can connect the parameterization $(\emptyset, \psi)$ with $(\emptyset, t_\alpha \circ \psi)$, where $t_\alpha \in \Gamma(S_{0,n})$ is the Dehn twist around α (see Figure 5.1), by the following sequence of moves:

$$T_\alpha = F_c B_{\alpha,c} F_c^{-1},$$

where c is a small closed curve around the hole α (see Figure 5.11).

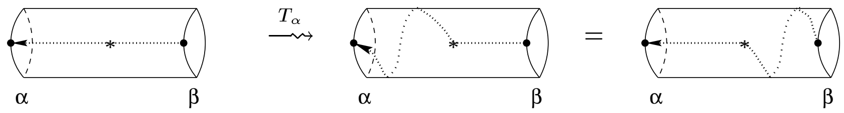


FIGURE 5.11. Dehn twist ($T_\alpha = T_\beta$).

EXERCISE 5.2.5. Let $S_{0,3}$ be the standard sphere with 3 holes, with the marking as shown in the left hand side of Figure 5.6. Deduce from the axioms MF1–MF7 the following relation in $\mathcal{M}(S_{0,3})$:

$$T_\gamma = B_{\beta,\alpha} B_{\alpha,\beta} T_\alpha T_\beta. \tag{5.2.9}$$

Hint: this is analogous to Step 7 in the proof of Theorem 5.3.8.

Now, let us consider extended surfaces of positive genus. In this case, we need to add to the complex $\mathcal{M}(\Sigma)$ one more simple move and several more relations.

S-move: Let $S_{1,1}$ be a "standard" torus with one boundary component and one cut, and with the parameterization M corresponding to the graph in the left hand side of Figure 5.12. Then we add the edge $S\colon M \rightsquigarrow M'$ where the parameterization M' corresponds to the graph shown in the right hand side of Figure 5.12.

More generally, let Σ_a be a component of an extended surface and ψ be a homeomorphism $\psi\colon \Sigma_a \xrightarrow{\sim} S_{1,1}$. Then we add the move $S\colon \psi^{-1}(M) \rightsquigarrow \psi^{-1}(M')$.

REMARK 5.2.6. If Σ is a surface of genus one with one hole, we can identify the set of all parameterizations with one cut on Σ with the set of all homeomorphisms $\psi\colon \Sigma \xrightarrow{\sim} S_{1,1}$. Then the S-move connects the marking ψ with $s \circ \psi$, where $s \in \Gamma(S_{1,1})$ is as in Example 5.1.11(i).

Now, let us formulate the new relations. In addition to relations MF1–MF7, let us also impose the following ones:

MF8: Relations for $g = 1, n = 1$: Let Σ be a marked torus with one hole α, isomorphic to the one shown in the left hand side of Figure 5.13. For any parameterization $M = (\{c\}, \psi)$ with one cut, we let T act on M as the edge

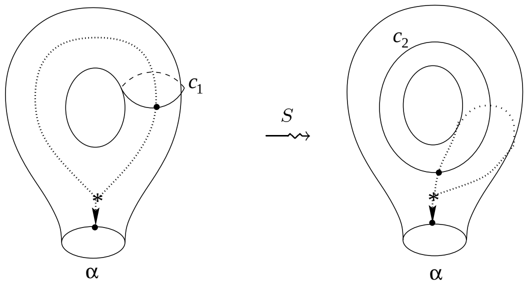


FIGURE 5.12. S-move.

Dehn twist T_c around c (see Example 5.2.4). Then we impose the following relations:

$$S^2 = Z^{-1} B_{\alpha,c_1}, \tag{5.2.10}$$

$$(ST)^3 = S^2. \tag{5.2.11}$$

The left hand side of relation (5.2.10) is shown in Figure 5.13. An illustration of (5.2.11) can be found in [**BK**, Appendix A].

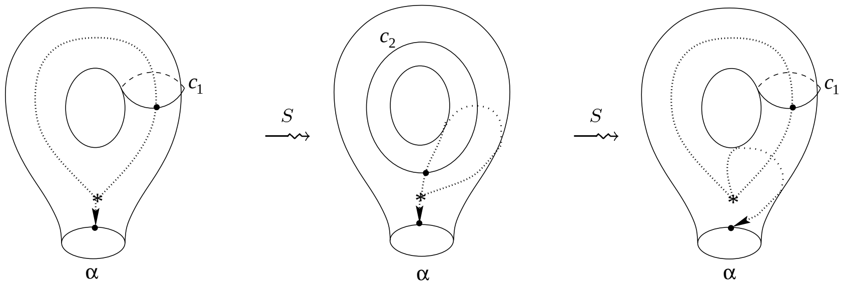


FIGURE 5.13. The relation $S^2 = Z^{-1} B_{\alpha,c_1}$.

MF9: Relation for $g = 1, n = 2$: Let Σ be a marked torus with two holes α, β, isomorphic to the one shown in Figure 5.14. Then we require

$$Z^{-1} B_{\alpha,\beta} F_{c_6}^{-1} F_{c_1} = S^{-1} F_{c_6}^{-1} F_{c_4} T_{c_3} T_{c_4}^{-1} F_{c_4}^{-1} F_{c_5} S F_{c_5}^{-1} F_{c_2} \tag{5.2.12}$$

— see Figure 5.15, where all unmarked arrows are compositions of the form FF^{-1} (see also [**BK**, Appendix B]).

Note that, by their construction, the above relations are invariant under the action of the mapping class group.

REMARK 5.2.7. It is not trivial that relations (5.2.11, 5.2.12) make sense, i.e., that they are indeed closed paths in $\mathcal{M}(\Sigma)$. This is equivalent to checking that the corresponding identities hold in the mapping class group $\Gamma(\Sigma)$. This is indeed so (see, e.g., [**B1, MS2**]). Of course, these relations can also be checked by explicitly drawing the corresponding sequence of cuts and graphs and checking that the final one coincides with the original one, as done in [**BK**].

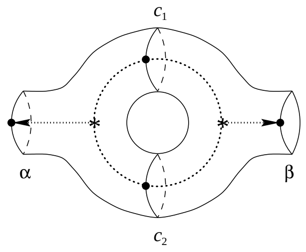


FIGURE 5.14. A marked torus with two holes.

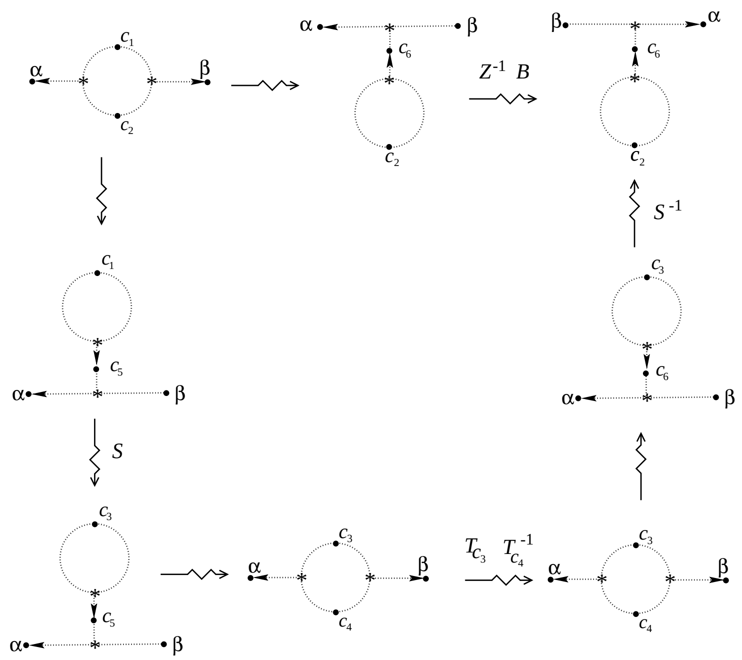


FIGURE 5.15. The relation for $g = 1, n = 2$.

EXAMPLE 5.2.8. Let Σ be a marked torus with one cut c_1 and one hole α (see the left hand side of Figure 5.12). Then we have:

$$(ST)^3 = S^2, \tag{5.2.13}$$

$$S^2T = TS^2, \tag{5.2.14}$$

$$S^4 = T_\alpha^{-1}. \tag{5.2.15}$$

Indeed, (5.2.13) is exactly (5.2.11). Equation (5.2.14) follows from (5.2.10), the Cylinder axiom, and the commutativity of disjoint union, and (5.2.15) easily follows from (5.2.10) and the braiding axiom.

In particular, this implies that the elements $t, s \in \Gamma_{1,1}$ (cf. Example 5.1.11) satisfy relations (5.2.13–5.2.15). In fact, it is known that these are the defining relations of the group $\Gamma_{1,1}$ (see [**B1**]).

Now we can formulate our main result for arbitrary genus.

THEOREM 5.2.9. *Let Σ be an extended surface. Let $\mathcal{M}(\Sigma)$ be the 2-complex with a set of vertices $M(\Sigma)$, edges given by the the Z-, F-, B-, and S-moves, and 2-cells given by relations* MF1–MF9. *Then $\mathcal{M}(\Sigma)$ is connected and simply-connected.*

Again, this theorem was stated (with minor inaccuracies) in [**MS1**], but the proof given there was seriously flawed. An accurate proof was found independently in [**BK**] and, in a different form, [**FG**]. The formulation above is taken from [**BK**].

5.3. Ribbon categories via the Hom spaces

In this section $\mathcal{C}$ will be a semisimple abelian category with representatives of the equivalence classes of simple objects V_i, $i \in I$. We use the notations and conventions of Section 2.4.

In a semisimple abelian category, any object $A \in \mathcal{C}$ is determined by the collection of vector spaces $\mathrm{Hom}(A, \cdot)$. More formally, we have the following well-known lemma.

LEMMA 5.3.1. (i) *Every functor $F\colon \mathcal{C} \to \mathcal{V}ec_f$ is exact* (*recall that we are considering only additive functors*).

(ii) *Let $F\colon \mathcal{C} \to \mathcal{V}ec_f$ be a functor satisfying the following finiteness condition*:

$$F(V_i) = 0 \quad \text{for all but a finite number of } i. \tag{5.3.1}$$

Then F is representable, i.e., there exists an object X_F, unique up to a unique isomorphism, such that $F(A) = \mathrm{Hom}_{\mathcal{C}}(X_F, A)$. Similarly, for a functor $G\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}ec_f$ there exists a unique $Y_G \in \mathcal{C}$ such that $G(A) = \mathrm{Hom}_{\mathcal{C}}(A, Y_G)$.

(iii) *For two functors $F, F'\colon \mathcal{C} \to \mathcal{V}ec_f$ satisfying the finiteness condition above, there is a bijection between the space of functor morphisms $F \to F'$ and $\mathrm{Hom}_{\mathcal{C}}(X_{F'}, X_F)$. A similar statement holds for $G, G'\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}ec_f$.*

Therefore, to construct, say, a functor $F\colon \mathcal{C} \to \mathcal{C}$, it suffices to define a bifunctor $A\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}ec_f$ satisfying suitable finiteness conditions, and then define $F(X)$ by the identity $\mathrm{Hom}(\cdot, F(X)) = A(\cdot, X)$; more formally, one would say "let $F(X)$ be the object representing the functor $A(\cdot, X)$". Similarly, all the functorial isomorphisms can be defined in terms of vector spaces.

Our goal in this section is to rewrite the axioms of a ribbon category in terms of the vector spaces

$$\langle W_1, \dots, W_n \rangle := \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_1 \otimes \cdots \otimes W_n). \tag{5.3.2}$$

This was first done in [**MS1**]. The following definition is essentially taken from [**MS1**]; for this reason, we think it is proper to commemorate their names.

DEFINITION 5.3.2. *Moore–Seiberg data* (MS data for short) for a semisimple abelian category $\mathcal{C}$ is the following collection of data:

Conformal blocks: A collection of functors $\langle\,\rangle\colon \mathcal{C}^{\boxtimes n} \to \mathcal{V}ec_f$ ($n \geq 0$), which are locally finite in the first component: for every $A_1, \dots, A_{n-1} \in \mathcal{C}$, we have $\langle V_i, A_1, \dots, A_{n-1} \rangle = 0$ for all but a finite number of i. (Here $\mathcal{C}^{\boxtimes n}$ denotes the tensor product $\mathcal{C} \boxtimes \cdots \boxtimes \mathcal{C}$ defined in 1.1.15.)

Rotation isomorphisms: Functorial isomorphisms

$$Z\colon \langle A_1, \dots, A_n \rangle \xrightarrow{\sim} \langle A_n, A_1, \dots, A_{n-1} \rangle.$$

R: A symmetric object $R \in \text{ind}-\mathcal{C}^{\boxtimes 2}$ (see Section 2.4).

Gluing isomorphisms: For every $k, l \in \mathbb{Z}_+$ functorial isomorphisms

$$G\colon \langle A_1, \dots, A_k, R^{(1)}\rangle \otimes \langle R^{(2)}, B_1, \dots, B_l\rangle \xrightarrow{\sim} \langle A_1, \dots, A_k, B_1, \dots, B_l\rangle.$$

Commutativity isomorphism: A functorial isomorphism

$$\sigma\colon \langle X, A, B\rangle \xrightarrow{\sim} \langle X, B, A\rangle.$$

These data have to satisfy the axioms MS1–MS7 listed below.

MS1: Non-degeneracy: For every i, there exists an object X such that $\langle X, V_i\rangle \neq 0$.

MS2: Normalization: The functor $\langle\,\rangle\colon \mathcal{C}^0 \equiv \mathcal{V}ec_f \to \mathcal{V}ec_f$ is the identity functor.

MS3: Associativity of G: Let us consider two functorial isomorphisms

$$G'G'', G''G'\colon \langle A_1, \dots, R'^{(1)}\rangle \otimes \langle R'^{(2)}, B_1, \dots, R''^{(1)}\rangle \otimes \langle R''^{(2)}, C_1, \dots, C_n\rangle \xrightarrow{\sim} \langle A_1, \dots, B_1, \dots, C_1, \dots, C_n\rangle,$$

where R', R'' are two copies of R, and G', G'' are the corresponding gluing isomorphisms. Then $G'G'' = G''G'$.

MS4: Rotation axiom: $Z^n = \mathrm{id}\colon \langle A_1, \dots, A_n\rangle \xrightarrow{\sim} \langle A_1, \dots, A_n\rangle$.

MS5: Symmetry of G: For any $m, n \geq 0$ the following diagram is commutative:

$$\begin{tikzcd}
\langle A_1, \dots, A_n, R^{(1)}\rangle \otimes \langle R^{(2)}, B_1, \dots, B_m\rangle \arrow[r, "G"] \arrow[d, "P(Z\otimes Z^{-1})"'] & \langle A_1, \dots, A_n, B_1, \dots, B_m\rangle \arrow[d, "Z^m"'] \\
\langle B_1, \dots, B_m, R^{(2)}\rangle \otimes \langle R^{(1)}, A_1, \dots, A_n\rangle \arrow[r, "G\circ s"] & \langle B_1, \dots, B_m, A_1, \dots, A_n\rangle
\end{tikzcd}.$$

(Here P is the permutation of the two factors in the tensor product and $s\colon R^{\mathrm{op}} \xrightarrow{\sim} R$ is as in Section 2.4.)

MS6: Hexagon axioms: (i) The following diagram is commutative:

$$\begin{tikzcd}
\langle X, A, B, C\rangle \arrow[rr, "\sigma_{A,BC}"] \arrow[dr, "\sigma_{A,B}"'] & & \langle X, B, C, A\rangle \\
& \langle X, B, A, C\rangle \arrow[ur, "\sigma_{A,C}"'] &
\end{tikzcd}$$

where $\sigma_{A,BC}$ is defined as the composition

$$\langle X, A, B, C\rangle \xrightarrow{G^{-1}} \langle X, A, R^{(1)}\rangle \otimes \langle R^{(2)}, B, C\rangle \xrightarrow{\sigma\otimes\mathrm{id}} \langle X, R^{(1)}, A\rangle \otimes \langle R^{(2)}, B, C\rangle \xrightarrow{Z^{-1}G(Z\otimes\mathrm{id})} \langle X, B, C, A\rangle,$$

and $\sigma_{A,B}$ is defined as the composition

$$\langle X, A, B, C\rangle \xrightarrow{G^{-1}Z} \langle C, X, R^{(1)}\rangle \otimes \langle R^{(2)}, A, B\rangle \xrightarrow{\mathrm{id}\otimes\sigma} \langle C, X, R^{(1)}\rangle \otimes \langle R^{(2)}, B, A\rangle \xrightarrow{Z^{-1}G} \langle X, B, A, C\rangle.$$

(ii) The same, but with σ replaced by σ^{-1}.

MS7: Dehn twist axiom: $Z\sigma_{A,B} = \sigma_{B,A}Z\colon \langle A, B\rangle \xrightarrow{\sim} \langle A, B\rangle$, where $\sigma_{A,B} = G(\sigma\otimes\mathrm{id})G^{-1}$ is defined similarly to MS6.

Now we describe how the MS data are related with the tensor structure on the category. Let $\mathcal{C}$ be a semisimple ribbon category. Define:

$$(5.3.3)\qquad \langle A_1, \dots, A_n\rangle = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A_1 \otimes \cdots \otimes A_n),$$

$$(5.3.4)\qquad R = \bigoplus V_i^* \otimes V_i, \qquad \text{cf. (2.4.7)},$$

$$\begin{aligned}(5.3.5)\qquad Z\colon\ \mathrm{Hom}(\mathbf{1}, A_1 \otimes \cdots \otimes A_n) &\xrightarrow{\sim} \mathrm{Hom}({}^*A_n, A_1 \otimes \cdots \otimes A_{n-1})\\ &\xrightarrow{\sim} \mathrm{Hom}(\mathbf{1}, {}^{**}A_n \otimes A_1 \otimes \cdots \otimes A_{n-1})\\ &\xrightarrow{\sim} \mathrm{Hom}(\mathbf{1}, A_n \otimes A_1 \otimes \cdots \otimes A_{n-1}),\end{aligned}$$

$$\begin{aligned}(5.3.6)\qquad G\colon\ \bigoplus \mathrm{Hom}(\mathbf{1}, A_1 \otimes \cdots \otimes A_n \otimes V_i^*) \otimes \mathrm{Hom}(\mathbf{1}, V_i \otimes B_1 \otimes \cdots \otimes B_k)&\\ \xrightarrow{\sim} \mathrm{Hom}(\mathbf{1}, A_1 \otimes \cdots \otimes A_n \otimes V_i^*) \otimes \mathrm{Hom}(V_i^*, B_1 \otimes \cdots \otimes B_k)&\\ \xrightarrow{\sim} \mathrm{Hom}(\mathbf{1}, A_1 \otimes \cdots \otimes A_n \otimes B_1 \otimes \cdots \otimes B_k),&\end{aligned}$$

$$(5.3.7)\qquad \sigma\colon\ \mathrm{Hom}(\mathbf{1}, X \otimes A \otimes B) \xrightarrow{\sim} \mathrm{Hom}(\mathbf{1}, X \otimes B \otimes A).$$

Here we used the rigidity isomorphisms (2.1.13, 2.1.14), the isomorphisms $\delta\colon V \xrightarrow{\sim} V^{**}$, and the fact that in a semisimple category, $\mathrm{Hom}(X, Y) \simeq \bigoplus \mathrm{Hom}(X, V_i) \otimes \mathrm{Hom}(V_i, Y)$.

PROPOSITION 5.3.3. *If $\mathcal{C}$ is a semisimple ribbon category, formulas* (5.3.3)–(5.3.7) *define MS data.*

The proof of this proposition is straightforward: if we use the technique of ribbon graphs developed in Chapter 1, then all the axioms are obvious. □

A natural question is whether this proposition can be reversed, i.e., is it true that every collection of MS data on a semisimple abelian category comes from a structure of a ribbon category. It turns out that it is almost true; to get a precise statement, we must somewhat weaken the rigidity axiom.

Let $\mathcal{C}$ be a monoidal category. We say that an object $V \in \mathrm{Ob}\,\mathcal{C}$ has a weak dual if the functor $\mathrm{Hom}(\mathbf{1}, V \otimes \cdot)$ is representable. In this case, we denote the corresponding representing object by V^*: $\mathrm{Hom}(\mathbf{1}, V \otimes X) = \mathrm{Hom}(V^*, X)$. Obviously, $*$ is functorial: every morphism $f\colon V \to W$ defines a morphism $f^*\colon W^* \to V^*$, provided that V^*, W^* exist.

DEFINITION 5.3.4. A monoidal category $\mathcal{C}$ is called *weakly rigid* if every object has a weak dual and $*\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ is an equivalence of categories.

Of course, every rigid category is weakly rigid; the converse, however, is not true. It is also useful to note that in every weakly rigid category we have a canonical morphism $i_V\colon \mathbf{1} \to V \otimes V^*$, corresponding to $\mathrm{id} \in \mathrm{Hom}(V^*, V^*) = \mathrm{Hom}(\mathbf{1}, V \otimes V^*)$. If the category is rigid, then i_V defined in this way coincides with the one defined by the rigidity axioms.

DEFINITION 5.3.5. A *weakly ribbon* category is a weakly rigid braided tensor category $\mathcal{C}$ endowed with a family of functorial isomorphisms $\theta\colon V \xrightarrow{\sim} V$ satisfying (2.2.8)–(2.2.10).

As discussed in Section 2.2, for a rigid category defining θ satisfying (2.2.8)–(2.2.10) is equivalent to defining $\delta\colon V \xrightarrow{\sim} V^{**}$, so every ribbon category is also weakly ribbon.

EXERCISE 5.3.6. (i) Show that in every semisimple weakly ribbon category, the map $\phi\colon \mathrm{Hom}(V^*, X) \to \mathrm{Hom}(\mathbf{1}, X \otimes V^{**})$ given by $\psi \mapsto (\psi \otimes \mathrm{id}) i_{V^*}$ is an isomorphism.

(ii) Show that in every semisimple weakly ribbon category one can define a family of functorial isomorphisms $\delta\colon V \xrightarrow{\sim} V^{**}$ by the condition that the following diagram be commutative:

$$\begin{array}{ccc} \langle V, X\rangle & \xrightarrow{\simeq} & \mathrm{Hom}(V^*, X) \\ \sigma\Big\downarrow & & \Big\downarrow\phi \\ \langle X, V\rangle & \xrightarrow{\mathrm{id}\otimes\delta} & \langle X, V^{**}\rangle \end{array} .$$

(iii) Show that in every semisimple weakly ribbon category, one has $(\theta_A \otimes \mathrm{id}) f = (\mathrm{id} \otimes \theta_B) f$ for every $f\colon \mathbf{1} \to A \otimes B$. (Hint: use $\theta_V^* = \theta_{V^*}$.)

Note, however, that in general, $(V \otimes W)^* \not\simeq W^* \otimes V^*$, so the axiom $\delta_{V\otimes W} = \delta_V \otimes \delta_W$ does not make sense.

REMARK 5.3.7. The authors do not know any example of a semisimple abelian category which is weakly rigid but not rigid.

Now we can formulate the main theorem of this section.

THEOREM 5.3.8. *Let $\mathcal{C}$ be a semisimple weakly ribbon category with simple objects $V_i, i \in I$. Then formulas* (5.3.3)–(5.3.7), *with δ defined as in Exercise* 5.3.6, *define MS data for $\mathcal{C}$. Conversely, every collection of MS data for a semisimple abelian category $\mathcal{C}$ is obtained in this way.*

PROOF. The first statement of the theorem is parallel to Proposition 5.3.3. The proof is also quite parallel; we just have to check that all the arguments work in a weakly rigid category as well as in a rigid one. This is left to the reader as an exercise; part of it is contained in Exercise 5.3.6. In particular, the identity (2.2.8) $\theta_{V\otimes W} = \sigma_{WV}\sigma_{VW}(\theta_V \otimes \theta_W)$ will give the Rotation axiom, and the identity (2.2.10) $\theta_{V^*} = \theta_V^*$ will give the Dehn twist axiom.

The proof of the converse statement is more complicated. For convenience, we split it into several steps. To simplify the notation, we will write just $\langle \dots, R\rangle \otimes \langle R, \dots\rangle$, omitting the superscripts. Since R is symmetric, this causes no problems. The symmetry of G axiom MS5 implies that the order of the factors is not important for defining G. We will implicitly use this.

Let us start by constructing the duality and tensor product on $\mathcal{C}$ from the MS data.

LEMMA 5.3.9. *Given MS data for $\mathcal{C}$, there exists an involution $*\colon I \to I$ such that $\dim\langle V_i, V_j\rangle = \delta_{i,j^*}$. Also, R is isomorphic (non-canonically) to $\bigoplus V_i \boxtimes V_{i^*}$.*

PROOF. Define $A_{ij} = \dim\langle V_i, V_j\rangle$, and define B_{ij} by $R \simeq \bigoplus B_{ij} V_i \boxtimes V_j$. It follows from the non-degeneracy axiom and the existence of Z that A is a symmetric matrix with no zero rows or columns. From the symmetry of R, we get that B is a symmetric matrix.

Writing the identity $\langle V_i, V_j\rangle = \langle V_i, R^{(1)}\rangle \otimes \langle R^{(2)}, V_j\rangle$ we get the identity $A = ABA$. We leave it to the reader to show that if A, B are symmetric matrices with non-negative integer entries and A has no zero columns, then such an identity is possible only if $A = B$ is a permutation of order 2. (Hint: use $AB = (AB)^2$ to prove that AB either has a zero row or column, or it is the identity matrix.) $\square$

1. Defining the duality functor. Define the functor $*$ by

$$\operatorname{Hom}(V^*, X) = \langle V, X\rangle \tag{5.3.8}$$

(see Lemma 5.3.1). Then the previous lemma immediately implies $V_i^* \simeq V_{i^*}$ (not canonically!). It is easy to see from this that $*$ is an anti-equivalence of categories. In particular, this implies that every object $V \in \mathcal{C}$ is completely determined by the functor $\langle V, \cdot\rangle = \operatorname{Hom}(V^*, \cdot)$.

Note that if the MS data come from the structure of a weakly ribbon category on $\mathcal{C}$ (see Proposition 5.3.3), then the $*$ functor defined above coincides with the one given by the rigidity axioms.

2. $R = \bigoplus V_i^* \boxtimes V_i$**.** To prove this, let us write $R \simeq \sum X_i \boxtimes V_i$ for some $X_i \in \operatorname{ind-}\mathcal{C}$. The isomorphism G gives, in particular, an isomorphism

$$\langle A, V_i^*\rangle \simeq \bigoplus \langle A, X_i\rangle \otimes \langle V_i, V_i^*\rangle.$$

Since $\langle V_i, V_i^*\rangle = \operatorname{Hom}(V_i^*, V_i^*) = k$, we get canonical isomorphisms $\langle A, V_i^*\rangle = \langle A, X_i\rangle$. Thus, we have constructed an isomorphism $R \simeq \bigoplus V_i^* \boxtimes V_i$ such that the isomorphism $G\colon \langle X, Y\rangle \simeq \langle X, R\rangle \otimes \langle R, Y\rangle$ is given by (5.3.6).

3. Tensor product. Define the functor $\otimes\colon \mathcal{C}^{\boxtimes 2} \to \mathcal{C}$ by

$$\langle X, A \otimes B\rangle = \langle X, A, B\rangle, \tag{5.3.9}$$

(it is well defined by the results of Step 1). More generally, define the tensor product of n objects by the following formula:

$$\langle X, A_1 \otimes \cdots \otimes A_n\rangle = \langle X, A_1, \dots, A_n\rangle.$$

Next, define isomorphisms

$$\begin{aligned} A_1 \otimes \cdots \otimes A_i \otimes (B_1 \otimes \cdots \otimes B_k) \otimes A_{i+1} \otimes \cdots \otimes A_n \\ \simeq A_1 \otimes \cdots \otimes A_i \otimes B_1 \otimes \cdots \otimes B_k \otimes A_{i+1} \otimes \cdots \otimes A_n \end{aligned} \tag{5.3.10}$$

as the following composition:

$$\begin{aligned} &\langle X, A_1, \dots, A_i, B_1 \otimes \cdots \otimes B_k, A_{i+1}, \dots, A_n\rangle \\ &\quad \simeq \langle X, A_1, \dots, A_i, R, A_{i+1}, \dots, A_n\rangle \otimes \langle R, B_1 \otimes \cdots \otimes B_k\rangle \\ &\quad \simeq \langle X, A_1, \dots, A_i, R, A_{i+1}, \dots, A_n\rangle \otimes \langle R, B_1, \dots, B_k\rangle \\ &\quad \simeq \langle X, A_1, \dots, A_i, B_1, \dots, B_k, A_{i+1}, \dots, A_n\rangle, \end{aligned}$$

where the isomorphisms are, respectively, G^{-1}, the definition of tensor product, and G.

LEMMA 5.3.10. *Let X be an expression of the form*

$$X = (A_1 \otimes (A_2 \otimes \cdots)) \otimes A_n$$

with any grammatically correct parentheses arrangement (parentheses may enclose any number of factors). Then any two isomorphisms

$$\varphi\colon X \simeq A_1 \otimes \cdots \otimes A_n,$$

obtained as a composition of the morphisms of the form (5.3.10)*, are equal.*

PROOF. Easy induction arguments show that it suffices to prove this statement in the case when we have just two pairs of parentheses. Thus, we need to consider the arrangements of the form $\cdots(\cdots(\cdots)\cdots)\cdots$ and $\cdots(\cdots)\cdots(\cdots)\cdots$. For both of them the statement easily follows from the definitions and the associativity of G. ☐

This shows that $\otimes$ is indeed associative; in particular, we can define associativity constraint $A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ which satisfies the pentagon axiom.

4. Unit. Define the object $\mathbf{1} \in \mathcal{C}$ by

$$\langle \mathbf{1}, X\rangle = \langle X\rangle \tag{5.3.11}$$

(as before, it is well defined due to the results of Step 1).

Define morphisms $\langle A_1, \dots, A_i, \mathbf{1}, A_{i+1}, \dots, A_n\rangle \simeq \langle A_1, \dots, A_i, A_{i+1}, \dots, A_n\rangle$ as the following composition

$$\begin{aligned}\langle A_1, \dots, A_i, \mathbf{1}, A_{i+1}, \dots, A_n\rangle &\simeq \langle A_1, \dots, A_i, R, A_{i+1}, \dots, A_n\rangle \otimes \langle \mathbf{1}, R\rangle \\ &\simeq \langle A_1, \dots, A_i, R, A_{i+1}, \dots, A_n\rangle \otimes \langle R\rangle \simeq \langle A_1, \dots, A_i, A_{i+1}, \dots, A_n\rangle.\end{aligned}$$

Note that this construction remains valid for $n = 0$, in which case, using the normalization axiom, we get

$$\langle \mathbf{1}\rangle = k. \tag{5.3.12}$$

Using the definition of tensor product, we see that the isomorphism

$$\langle X, A_1, \dots, A_i, \mathbf{1}, A_{i+1}, \dots, A_n\rangle \simeq \langle X, A_1, \dots, A_i, A_{i+1}, \dots, A_n\rangle$$

gives rise to an isomorphism

$$A_1 \otimes \cdots \otimes A_i \otimes \mathbf{1} \otimes A_{i+1} \otimes \cdots \otimes A_n \simeq A_1 \otimes \cdots \otimes A_i \otimes A_{i+1} \otimes \cdots \otimes A_n. \tag{5.3.13}$$

LEMMA 5.3.11. *The following diagram, with the horizontal map given by the associativity isomorphism and the two others by the unit isomorphisms* (5.3.13)*, is commutative*:

$$\begin{tikzcd} A \otimes (\mathbf{1} \otimes B) \arrow[rr] \arrow[dr] & & A \otimes \mathbf{1} \otimes B \arrow[dl] \\ & A \otimes B & \end{tikzcd}$$

PROOF. Looking at the definitions, we see that the statement is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} \langle X, A, R'\rangle \otimes \langle R', \mathbf{1} \otimes B\rangle \arrow[r] \arrow[d] & \langle X, A, R'\rangle \otimes \langle R', B\rangle \arrow[r] & \langle X, A, B\rangle \\ \langle X, A, \mathbf{1}, B\rangle \arrow[r] & \langle X, A, R'', B\rangle \otimes \langle \mathbf{1}, R''\rangle \arrow[r] & \langle X, A, R'', B\rangle \otimes \langle R''\rangle \arrow[u] \end{tikzcd}$$

where, as before, R' and R'' are two copies of R. But this easily follows from the associativity of G applied to the space $\langle X, A, R'', R'\rangle \otimes \langle \mathbf{1}, R''\rangle \otimes \langle R', B\rangle$. We leave the details to the reader. ☐

COROLLARY 5.3.12. *The isomorphisms $\mathbf{1} \otimes X \xrightarrow{\sim} X$ and $X \otimes \mathbf{1} \xrightarrow{\sim} X$, given by* (5.3.13)*, satisfy the triangle axiom.*

Combining this fact with the MacLane coherence theorem (Theorem 1.1.9), we see that the MS data indeed defines a structure of a monoidal category on $\mathcal{C}$.

5. Definition of $\langle\,\rangle$. Using the unit isomorphisms (5.3.13), we can identify

$$\langle A_1,\dots,A_n\rangle \xrightarrow{\sim} \langle \mathbf{1},A_1,\dots,A_n\rangle \xrightarrow{\sim} \mathrm{Hom}(\mathbf{1}^*,A_1\otimes\dots\otimes A_n).$$

Next, let us construct an isomorphism $\mathbf{1} \xrightarrow{\sim} \mathbf{1}^*$. Using (5.3.12), we can write $\mathrm{Hom}(\mathbf{1}^*,\mathbf{1}) = \langle\mathbf{1}\rangle = k$. Thus, $1\in k$ gives an isomorphism $\mathbf{1} \xrightarrow{\sim} \mathbf{1}^*$; combining this isomorphism with the previous identity, we can identify

$$\langle A_1,\dots,A_n\rangle \simeq \mathrm{Hom}(\mathbf{1},A_1\otimes\dots\otimes A_n). \tag{5.3.14}$$

6. Commutativity isomorphism. Define the commutativity isomorphism $\sigma\colon A\otimes B\to B\otimes A$ using the following composition:

$$\langle X, A\otimes B\rangle = \langle X,A,B\rangle \xrightarrow{\sigma} \langle X,B,A\rangle = \langle X, B\otimes A\rangle.$$

Then one easily sees that the Hexagon axioms given in Theorem 1.2.5(iii) are immediate corollaries of the Hexagon axioms for MS data. Thus, the MS data defines a structure of a BTC on $\mathcal{C}$.

7. Balancing. Consider the functorial isomorphism

$$\langle V,X\rangle \xrightarrow{\sigma^{-1}} \langle X,V\rangle \xrightarrow{Z} \langle V,X\rangle. \tag{5.3.15}$$

By Lemma 5.3.1, there exists a functorial isomorphism $\theta_V\colon V \xrightarrow{\sim} V$ such that the above composition is given by $\theta_V\otimes \mathrm{id}_X$. One easily checks that $\theta_{\mathbf{1}} = \mathrm{id}$ and that

$$\theta_{W_1}^{-1} = Z\sigma_{W_1,W_2\otimes\dots\otimes W_n} = \sigma_{W_2\otimes\dots\otimes W_n,W_1}Z^{-1}\colon \langle W_1,\dots,W_n\rangle \xrightarrow{\sim} \langle W_1,\dots,W_n\rangle$$

(this is where we need the Dehn twist axiom MS7).

To prove the identity $\theta_{A\otimes B} = \sigma_{B,A}\sigma_{A,B}(\theta_A\otimes\theta_B)$, note that it is equivalent to

$$\sigma_{B,A}\sigma_{A,B}\theta_A\theta_C^{-1}\theta_B = \mathrm{id}\colon \langle A,B,C\rangle \xrightarrow{\sim} \langle A,B,C\rangle, \tag{5.3.16}$$

?! which follows from the identities

$$\begin{aligned}\theta_A^{-1} &= Z\sigma_{A,BC} = Z\sigma_{A,C}\sigma_{A,B},\\ \theta_B^{-1} &= \sigma_{B,A}Z\sigma_{B,C},\\ \theta_C^{-1} &= Z\sigma_{A,C}Z\sigma_{B,C}.\end{aligned}$$

Finally, we leave it to the reader to show that the Dehn twist axiom MF7 is essentially equivalent to the identity $\theta_{V^*} = \theta_V^*$. Thus, the so defined θ satisfies the balancing axioms (2.2.8)–(2.2.10).

This completes the proof of Theorem 5.3.8. $\square$

It would be nice if we had some axiom for MS data which would automatically ensure that the corresponding BTC is rigid. However, the only way of doing it that we know of is explicitly imposing the rigidity condition. (It is claimed in [**MS2**] that rigidity follows from the other axioms; however, at some point, they say "we can check the universality property" without doing it explicitly—we were unable to reconstruct their arguments.)

In the semisimple case the rigidity condition is equivalent to the non-vanishing of certain coefficients, which shows that "almost all" weakly rigid semisimple categories are rigid.

Let $\mathcal{C}$ be a semisimple weakly rigid monoidal category such that $V^{**}\simeq V$ (as discussed above, this holds for any category obtained from MS data). Let $\varphi_i\colon V_i^*\to$

$V_i^* \otimes V_i \otimes V_i^*$ be given by $\varphi_i = \mathrm{id} \otimes i_{V_i}$. Using the associativity isomorphism, we can write

$$\varphi_i = a_i \otimes \mathrm{id} + \sum_{j \neq 0} \psi_j,$$

where a_i are certain morphisms $\mathbf{1} \to V_i^* \otimes V_i$, and ψ_j are some morphisms which are obtained as the composition

$$V_i^* \to V_j \otimes V_i^* \xrightarrow{\psi_j' \otimes \mathrm{id}} (V_i^* \otimes V_i) \otimes V_i^*.$$

Note that since $V_i^* \otimes V_i$ contains $\mathbf{1}$ with multiplicity one, the morphisms a_i lie in a one-dimensional space.

PROPOSITION 5.3.13. *Let $\mathcal{C}$ be a semisimple weakly rigid monoidal category such that $V^{**} \simeq V$, and let $a_i \colon \mathbf{1} \to V_i^* \otimes V_i$ be defined as above. Then $\mathcal{C}$ is rigid iff $a_i \neq 0$ for all $i \in I$.*

PROOF. If $\mathcal{C}$ is rigid, then $e_{V_i} a_i = 1$, which immediately follows from taking composition of φ_i with $e_{V_i} \otimes \mathrm{id}$. Thus, $a_i \neq 0$. Conversely, assume that $a_i \neq 0$. Then define $e_{V_i} \colon V_i^* \otimes V_i \to \mathbf{1}$ by the condition $e_{V_i} a_i = 1$; since $V_i^* \otimes V_i$ contains $\mathbf{1}$ with multiplicity one, this is possible. From this condition, we immediately see that the composition

$$V_i^* \xrightarrow{\mathrm{id} \otimes i_{V_i}} V_i^* \otimes V_i \otimes V_i^* \xrightarrow{e_{V_i} \otimes \mathrm{id}} V_i^*$$

is equal to identity; thus, the second rigidity axiom (2.1.6) is satisfied.

To check the first rigidity axiom, denote the composition

$$V_i \xrightarrow{i_{V_i} \otimes \mathrm{id}} V_i \otimes V_i^* \otimes V_i \xrightarrow{\mathrm{id} \otimes e_{V_i}} V_i$$

by c_i; since $\mathrm{End}(V_i) = k$, c_i is a number. We need to show that $c_i = 1$.

Consider the composition

$$\Phi \colon \mathbf{1} \xrightarrow{i \otimes i} V_i \otimes V_i^* \otimes V_i \otimes V_i^* \xrightarrow{\mathrm{id} \otimes e \otimes \mathrm{id}} V_i \otimes V_i^*.$$

From the second rigidity axiom (already proved), $\Phi = i_{V_i}$. On the other hand, form the definition of c_i, we have $\Phi = c_i i_{V_i}$. This proves $c_i = 1$ and thus, the first rigidity axiom for V_i.

Therefore, if $a_i \neq 0$, then V_i is rigid. But since a direct sum of rigid objects is again rigid, every object in $\mathcal{C}$ is rigid. $\square$

5.4. Modular functor in genus zero and tensor categories

In this section we prove the first main theorem of this chapter, establishing that the axioms of a (weakly) ribbon category are essentially equivalent to the axioms of a modular functor in genus zero.

Let $\mathcal{C}$ be a semisimple abelian category with representatives of the equivalence classes of simple objects V_i, $i \in I$. Let us call a *$\mathcal{C}$-extended modular functor in genus zero* the same data as in Definition 5.1.12 but with the spaces $\tau(\Sigma)$ defined only for Σ of genus zero; therefore, the only gluing allowed is the gluing of two different connected components.

THEOREM 5.4.1 (Moore–Seiberg [**MS1**]). *Let $\mathcal{C}$ be a semisimple weakly ribbon category. Then there is a unique $\mathcal{C}$-extended genus zero modular functor satisfying the properties* (i)–(iii) *below.*

(i) *For the standard sphere* $S_{0,n}$ *(see* (5.2.1)*)*:

$$\tau(S_{0,n}; W_1, \dots, W_n) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_1 \otimes \cdots \otimes W_n) =: \langle W_1, \dots, W_n \rangle. \tag{5.4.1}$$

(ii) $R = \bigoplus V_i^* \otimes V_i$, *and the isomorphism* $s\colon R \xrightarrow{\sim} R^{\mathrm{op}}$ *is given by* (2.4.8).

(iii) *We have*:

$$z_* = Z, \quad b_* = \sigma, \tag{5.4.2}$$

where the homeomorphisms z, b *are defined by* (5.2.2), *and the isomorphisms* Z, σ *are defined by* (5.3.5), (5.3.7). *Also, for every* $k, l \geq 0$, *the composition*

$$\tau(S_{0,k+1}; \dots, R^{(1)}) \otimes \tau(S_{0,l+1}; R^{(2)}, \dots) \to \tau(S_{0,k+1} \sqcup_{k+1,1} S_{0,l+1}) \xrightarrow{(\alpha_{kl})_*} \tau(S_{0,k+l}),$$

where the first arrow is the sewing isomorphism (5.1.1) *and* α_{kl} *is as in* (5.2.3), *coincides with the isomorphism* G *defined by* (5.3.6).

This modular functor is non-degenerate and has the following properties:

(iv) *Let* $t_i\colon S_{0,n} \to S_{0,n}$ *be the Dehn twist around* i^{th} *puncture. Then, under the isomorphism* (5.4.1), $(t_i)_*$ *is given by the twist*

$$\theta_{W_i}\colon \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_1 \otimes \cdots \otimes W_n) \to \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_1 \otimes \cdots \otimes W_n).$$

(v) *If* $\mathcal{C}$ *is rigid, then this modular functor is unitary, with the pairing* (5.1.2)

$$\langle , \rangle_{S_{0,n}}\colon \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_1 \otimes \cdots \otimes W_n) \otimes \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_n^* \otimes \cdots \otimes W_1^*) \to k$$

given by

$$\langle \varphi, \psi \rangle\colon \mathbf{1} \to \mathbf{1} \otimes \mathbf{1} \to W_1 \otimes \cdots \otimes W_n \otimes W_n^* \otimes \cdots \otimes W_1^* \to \mathbf{1}.$$

Here we identify $k = \mathrm{End}(\mathbf{1})$ *and use the fact that for a standard sphere* $S_{0,n}$, *there is a canonical isomorphism* $\overline{S_{0,n}} \xrightarrow{\sim} S_{0,n}$, *which reverses the order of the punctures. This isomorphism is given by the reflection around the imaginary axis.*

Conversely, let τ *be a non-degenerate genus zero* $\mathcal{C}$*-extended MF. Then there is a unique structure of a weakly ribbon category on* $\mathcal{C}$ *such that the above properties* (i)–(iii) *hold.*

PROOF. The proof is based on the comparison of the results of Sections 5.2 and 5.3. Since by Theorem 5.3.8 the structure of a weakly ribbon category on $\mathcal{C}$ is equivalent to what we called MS data, it suffices to show that a non-degenerate genus zero MF defines MS data and vice versa.

Let us assume we are given a collection of MS data. To construct a genus zero MF, let us first consider the pairs (Σ, M), where $M = (C, \{\psi_a\})$ is a parameterization of Σ (see Definition 5.2.1). For each such pair, define the vector space $\tau(\Sigma, M)$ as follows. For every cut c, take a copy R_c of the object R, and define

$$\tau(\Sigma, M) = \bigotimes_a \tau(S_{0,n_a}), \tag{5.4.3}$$

where the index a runs through the set of connected components of $\Sigma \setminus C$, and for each connected component Σ_a, we assign $R_c^{(\varepsilon)}$ to every boundary component of Σ_a which is a cut, where $\varepsilon \in \{1, 2\}$ is chosen so that for one of the occurrences of R_c we take $\varepsilon = 1$ and for the other we take $\varepsilon = 2$ (note that each R_c appears exactly twice in (5.4.3)). Since R is symmetric, it does not matter which occurrence is which.

More explicitly, the same formula can be written as follows. For each cut $c \in C$, choose one of its sides as "positive" and the other as "negative". Then we can define

$$\tau(\Sigma, M) = \bigoplus_{i_c \in I,\, c \in C} \bigotimes_{a} \tau(S_{0,n_a}), \tag{5.4.4}$$

where the sum is taken over all ways to assign an index $i_c \in I$ to every cut $c \in C$, and for each connected component Σ_a of $\Sigma \setminus C$ we assign V_{i_c} to its boundary component if it is the positive side of the cut c, and $V_{i_c}^*$ if it is the negative side of the cut c. This formula depends on the choice of "positive" side for each cut; to identify the formulas corresponding to different choices, one has to use the canonical isomorphism $V_i^* \boxtimes V_i \xrightarrow{\sim} V_{i^*} \boxtimes V_{i^*}^*$ defined in (2.4.8).

For example, if Σ is a sphere with 4 holes which we index by $\alpha, \beta, \gamma, \delta$, and φ is a parameterization with one cut c as in Figure 5.16, then the above formula gives

$$\begin{aligned}\tau(\Sigma, \varphi; W_\alpha, W_\beta, W_\gamma, W_\delta) &= \langle W_\alpha, W_\beta, R^{(1)} \rangle \otimes \langle R^{(2)}, W_\gamma, W_\delta \rangle \\ &= \bigoplus_{i \in I} \langle W_\alpha, W_\beta, V_i \rangle \otimes \langle V_i^*, W_\gamma, W_\delta \rangle.\end{aligned}$$

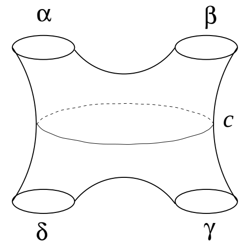


FIGURE 5.16

Of course, every extended surface Σ can be parametrized in many ways. However, if we construct a system of isomorphisms $f_{M,M'} : \tau(\Sigma, M') \xrightarrow{\sim} \tau(\Sigma, M)$, compatible in the following sense: $f_{M,M'} f_{M',M''} = f_{M,M''}$, then we can identify all of these spaces with each other and define the space $\tau(\Sigma)$, which is canonically isomorphic to each of $\tau(\Sigma, M)$ (see a formal definition in the proof of Theorem 4.4.3).

Moreover, such a system of isomorphisms would automatically give a representation of the extended mapping class groupoid $\mathcal{T}eich$, as follows. Let $f \colon \Sigma_1 \xrightarrow{\sim} \Sigma_2$ be a homeomorphism of extended surfaces, and let M_2 be a parameterization of Σ_2. Then f gives rise to a parameterization M_1 of Σ_1 in the obvious way. Moreover, f establishes a one-to-one correspondence between the cuts C_1 on Σ_1 and C_2 on Σ_2, and between the components $(\Sigma_1)_a$ and $(\Sigma_2)_a$. Thus, f gives rise to an identification $\tau(\Sigma_1, M_1) = \bigoplus_{i_c \in I,\, c \in C_1} \bigotimes_a \tau(S_{0,n_a}) = \tau(\Sigma_2, M_2)$. Combining this with the isomorphisms $\tau(\Sigma_1) = \tau(\Sigma_1, M_1)$, $\tau(\Sigma_2) = \tau(\Sigma_2, M_2)$, we get an isomorphism $f_* \colon \tau(\Sigma_1) \xrightarrow{\sim} \tau(\Sigma_2)$. We leave it to the reader to check that this isomorphism does not depend on the choice of M_2 and satisfies $(fg)_* = f_* g_*, \mathrm{id}_* = \mathrm{id}$. Also, it is immediately obvious from (5.4.3) that the so constructed modular functor will satisfy the gluing axiom.

Therefore, our goal is to construct a compatible system of isomorphisms $\tau(\Sigma, M') \xrightarrow{\sim} \tau(\Sigma, M)$. By Theorem 5.2.3, every two parameterizations can be connected by a sequence of simple moves Z, B, F; let us assign to these moves the isomorphisms

Z, σ, G given by the MS data. A comparison of the axioms MF1–MF7 and MS1–MS7 shows that all the relations among the moves Z, B, F also hold for their analogues Z, σ, G; the only relation which is not immediately obvious is the cylinder axiom MF5, but it follows from the functoriality of the morphisms Z, σ, G. Thus, every MS data defines a genus zero MF.

The construction in the opposite direction is quite similar. Assume that we have a genus zero MF. Define the functors $\langle\,\rangle$ and the isomorphisms Z, σ, G as in the statement of the theorem. Again, a comparison of the axioms MF1–MF7 and MS1–MS7 shows that these data satisfy the axioms of MS data. This completes the proof of Theorem 5.4.1. □

?!

EXAMPLE 5.4.2. Consider the surface Σ and the "associativity move" $M \overset{F_c}{\rightsquigarrow} M_0 \overset{F_{c'}^{-1}}{\rightsquigarrow} M'$ shown in Figure 5.17. Assign to the boundary components $\alpha, \dots, \delta$ objects $A, \dots, D$. Then:

$$\begin{aligned} \tau(\Sigma, M) &= \bigoplus_{i \in I} \langle A, B, V_i \rangle \otimes \langle V_i^*, C, D \rangle, \\ \tau(\Sigma, M_0) &= \langle A, B, C, D \rangle, \\ \tau(\Sigma, M') &= \bigoplus_{j \in I} \langle D, A, V_j \rangle \otimes \langle V_j^*, B, C \rangle. \end{aligned}$$

Then the corresponding isomorphisms $\tau(\Sigma, M) \to \tau(\Sigma, M_0) \to \tau(\Sigma, M')$ are given by Figure 5.18 below.

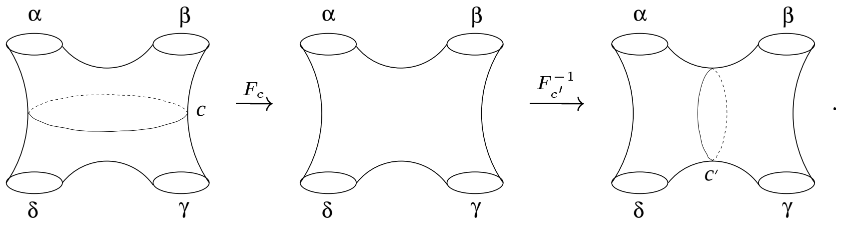


FIGURE 5.17. Associativity move.

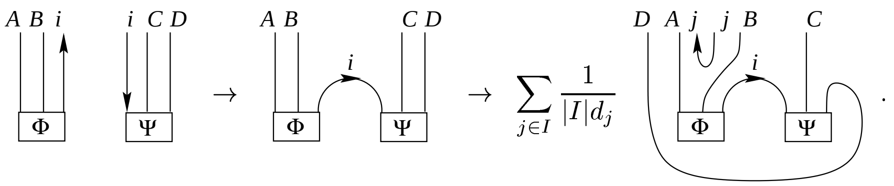


FIGURE 5.18. Associativity isomorphism.

5.5. Modular categories and modular functor for zero central charge

In this section, we will show, developing the ideas of the previous section, that the notion of a modular functor (for arbitrary genus) is equivalent to the notion of a modular tensor category. Recall that for every modular category we have defined the numbers $p^{\pm}$ by (3.1.7). In this section we consider the special case of modular categories with $p^+/p^- = 1$. (For the modular categories coming from conformal field theory this identity holds if the Virasoro central charge of the theory is equal to 0 (cf. Remark 3.1.20), hence the title of this section.)

THEOREM 5.5.1. *Let $\mathcal{C}$ be a modular tensor category with $p^+/p^- = 1$. Then there exists a unique $\mathcal{C}$-extended modular functor τ which satisfies conditions* (i)–(iii) *of Theorem* 5.4.1. *This MF is non-degenerate and satisfies conditions* (iv), (v) *of Theorem* 5.4.1 *and condition* (vi) *below.*

(vi) *Let $S_{1,1}$ be the torus with one hole. Identify*

$$\tau(S_{1,1}; A) = \bigoplus \langle A, V_i, V_i^* \rangle = \bigoplus \mathrm{Hom}(A^*, V_i \otimes V_i^*)$$

using the parameterization of $S_{1,1}$ shown in Figure 5.12. *Let $s\colon S_{1,1} \to S_{1,1}$ be as defined in* (5.1.5). *Then the corresponding*

$$s_* = S\colon \bigoplus \mathrm{Hom}(A^*, V_i \otimes V_i^*) \to \bigoplus \mathrm{Hom}(A^*, V_i \otimes V_i^*) \tag{5.5.1}$$

is given by Theorem 3.1.17.

Conversely, let $\mathcal{C}$ be a semisimple abelian category, and let τ be a non-degenerate $\mathcal{C}$-extended MF. Assume for simplicity that the corresponding structure of a monoidal category on $\mathcal{C}$ (see Theorem 5.4.1) *is rigid. Then $\mathcal{C}$ is a modular tensor category with $p^+ = p^-$; in particular, it has only a finite number of simple objects.*

PROOF. Assume that $\mathcal{C}$ is a modular category. By Theorem 5.4.1, the structure of a modular category on $\mathcal{C}$ defines a genus zero MF. Therefore, we only need to show that this MF can be extended to positive genus. In order to do this, by Theorem 5.2.9, we need to define an isomorphism $S\colon \tau(S_{1,1}, M) \xrightarrow{\sim} \tau(S_{1,1}, M')$, where $S_{1,1}$ is the standard torus and M, M' are the parameterizations shown in Figure 5.12, and then check that relations MF8, MF9 are satisfied.

Note that by definition

$$\tau(S_{1,1}, M; A) = \tau(S_{1,1}, M'; A) = \bigoplus_i \langle A, V_i, V_i^* \rangle = \mathrm{Hom}(A^*, H),$$

where, as before, $H = \bigoplus V_i \otimes V_i^*$. Thus, defining an isomorphism $S\colon \tau(S_{1,1}, M) \xrightarrow{\sim} \tau(S_{1,1}, M')$ is the same as defining a functorial system of isomorphisms $\mathrm{Hom}(A^*, H) \xrightarrow{\sim} \mathrm{Hom}(A^*, H)$ for every object A. By Lemma 5.3.1, this is the same as defining an isomorphism $S\colon H \to H$.

Let us first show that if we define S as in the statement of the theorem, then relations MF8, MF9 are satisfied. Relations MF8 immediately follow from Theorem 3.1.17 and the assumption $p^+ = p^-$.

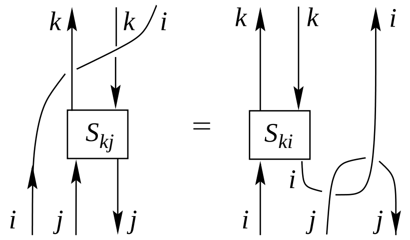
To check relation MF9 for a torus with two holes, let us rewrite it in terms of tensor categories. **?!**

LEMMA 5.5.2. *Let $\mathcal{C}$ be a semisimple ribbon category with finite number of simple objects, and let S be an isomorphism*

$$S = \bigoplus S_{ji}\colon \bigoplus V_i \otimes V_i^* \to \bigoplus V_j \otimes V_j^*. \tag{5.5.2}$$

Then relation MF9 *for S is equivalent to the following condition*:

(5.5.3)



for every $i, j, k \in I$.

The proof of this lemma will be given after the proof of the theorem.

It is easy to check that the operator S defined by (3.1.32) satisfies (5.5.3).

Now, let us prove uniqueness. Assume that we have defined an operator S of the form (5.5.2) such that relations MF8, MF9 are satisfied. Rewrite relation MF9 in the form (5.5.3), put $j = 0$ and note that $S_{k0}\colon \mathbf{1} \to V_k \otimes V_k^*$ is a non-zero multiple of i_{V_k}. This immediately implies that $S_{ki} = a_k S^{\mathrm{st}}_{ki}$ for some non-zero constant a_k, where we temporarily denoted by S^{st} the operator defined by (3.1.32). Equivalently, we can write $S = AS^{\mathrm{st}}$, where the operator $A\colon H \to H$ is "diagonal": $A|_{V_i \otimes V_i^*} = a_i \,\mathrm{id}$. Now, let us use the axiom MF8. In particular, we have $TSTST = S$. Since $S = AS^{\mathrm{st}}$, and A commutes with T, we get $TS^{\mathrm{st}}TAS^{\mathrm{st}}T = S^{\mathrm{st}}$. On the other hand, the operator S^{st} itself satisfies the axiom MF8, and thus, $TS^{\mathrm{st}}TS^{\mathrm{st}}T = S^{\mathrm{st}}$. This implies $A = \mathrm{id}, S = S^{\mathrm{st}}$.

The proof of the converse statement—that a MF defines a structure of a modular category—is trivial. Indeed, the identity $\tau(\Sigma) = \bigoplus \operatorname{End} V_i$ for Σ being a torus without punctures implies that $\mathcal{C}$ has only finitely many simple objects (since $\tau(\Sigma)$ is finite dimensional). Thus, we only have to check that the matrix $\tilde{s}$, defined in (3.1.1), is non-degenerate. But the identity $S = AS^{\mathrm{st}}$ and the invertibility of S imply that S^{st} is invertible. □

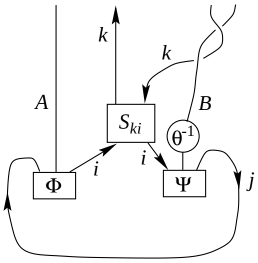
Proof of Lemma 5.5.2. Consider the diagram in Figure 5.15. Let m_1 be the graph in the upper left corner; for convenience, replace the graph m in the lower right corner by $m_2 = F_{c_4}(m)$. Then the vector spaces $\tau(\Sigma, m_1)$ and $\tau(\Sigma, m_2)$ are given by

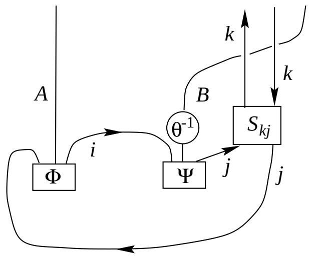
$$\begin{aligned} \tau(\Sigma, m_1) &= \bigoplus_{i,j} \langle V_j^*, A, V_i\rangle \otimes \langle V_i^*, B, V_j\rangle, \\ \tau(\Sigma, m_2) &= \bigoplus_{k} \langle A, V_k, V_k^*, B\rangle, \end{aligned} \tag{5.5.4}$$

where A, B are the objects assigned to the boundary components α, β respectively (see (5.4.4)).

Then relation MF9 can be written as follows: for every $\Phi \otimes \Psi \in \langle V_j^*, A, V_i\rangle \otimes \langle V_i^*, B, V_j\rangle$, we have $f(\Phi \otimes \Psi) = g(\Phi \otimes \Psi)$, where f is the isomorphism given by the composition of moves forming the left side and the bottom of the commutative diagram, and g—by the moves on the top and the right side. We represent this identity pictorially, using Example 5.4.2, Eq. (5.2.9), and the graphical calculus of Section 2.3.

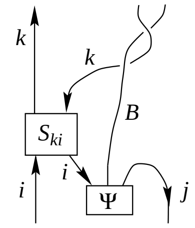
A simple manipulation with figures shows that:

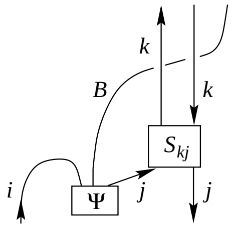
$f(\Phi \otimes \Psi) =$ 

, $g(\Phi \otimes \Psi) =$ 

.

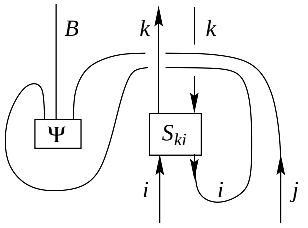
The identity $f(\Phi \otimes \Psi) = g(\Phi \otimes \Psi)\ \forall \Phi$ is equivalent to:

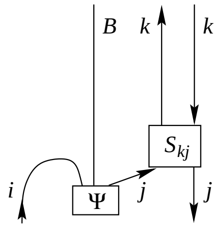


$\doteq$


.

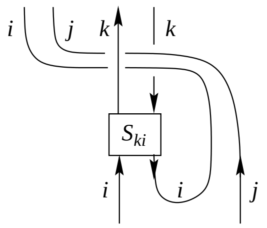
We manipulate this as follows:

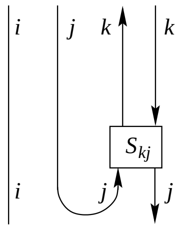


$\doteq$


,

and then cancel Ψ, to get:



$\doteq$


.

From this it is easy to get the statement of the lemma. □

Corollary 5.5.3. *Let $\mathcal{C}$ be an MTC with $p^+ = p^-$. Denote*

$$\tau(g; W_1, \dots, W_n) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, H^{\otimes g} \otimes W_1 \otimes \cdots \otimes W_m)$$

where $H = \bigoplus V_i \otimes V_i^*$. Then we have an action of the pure mapping class group $\Gamma'_{g,n}$ on this space. In particular, for $g = 1, n = 1$ this action coincides with the one defined in Theorem 3.1.17.

Indeed, let $\tau(\Sigma)$ be the modular functor corresponding to $\mathcal{C}$; then it is easy to see, using the gluing axiom, that if Σ is a surface of genus g then $\tau(\Sigma; W_1, \dots, W_n)$ is (not canonically) isomorphic to the space $\tau(g; W_1, \dots, W_n)$ defined above.

REMARK 5.5.4. In fact, Corollary 5.5.3 also holds for modular categories with $p^+/p^- \neq 1$ if we replace the word "action" by "projective action". This will be discussed in Section 5.7.

EXERCISE 5.5.5. Prove the following formula for the dimension of the space $\tau(g)$ for $g \geq 1$ $(n = 0)$:

$$\dim \tau(g) = \sum_{i \in I} \left(\frac{1}{s_{0i}^2} \right)^{g-1}. \tag{5.5.5}$$

Hint: Prove that $\dim \tau(g) = \operatorname{tr}(a^{g-1})$, where $a_{ij} = \dim \tau(g = 1; V_i, V_j^*), i, j \in I$. Then prove that $a = \sum_k N_k N_{k^*}$, where N_k is defined as in Proposition 3.1.12, and use the Verlinde formula to diagonalize a.

5.6. Towers of groupoids

Looking at the previous two sections, one is tempted to say that there is some "universal" set of relations which must hold in any weakly ribbon category, and these relations happen to coincide with the relations for the mapping class group. In this section we sketch the appropriate language in which one can formulate this and other related results. Therefore, we do not really prove any new results here, and we allow ourselves to be somewhat informal.

Let us start by considering our main example: the *Teichmüller tower* $\mathcal{T}eich$. By definition, $\mathcal{T}eich$ is a category with objects all extended surfaces, and morphisms isotopy classes of homeomorphisms of extended surfaces (see Definition 5.1.7(i)). This category is a groupoid, i.e., any morphism in $\mathcal{T}eich$ is invertible. It also has some additional structures which played an important role in the previous sections: the disjoint union and gluing of surfaces. The general definition of a tower of groupoids will be modeled on this example, so let us study it in more detail.

Temporarily, let us denote $\mathcal{T}eich$ by $\mathcal{T}$. Below we list its properties.

(a) $\mathcal{T}$ is a groupoid.

(b) The disjoint union of surfaces $\sqcup\colon \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ and the empty surface $\emptyset \in \operatorname{Ob} \mathcal{T}$ provide $\mathcal{T}$ with the structure of a symmetric tensor category.

(c) There is a functor $A\colon \mathcal{T} \to \mathcal{S}ets$: for a surface Σ, $A(\Sigma) = \pi_0(\partial\Sigma)$ is the set of its boundary components. Here $\mathcal{S}ets$ is the groupoid with objects finite sets, and morphisms bijections. Note that $A(\Sigma_1 \sqcup \Sigma_2) = A(\Sigma_1) \sqcup A(\Sigma_2)$ and $A(\emptyset) = \emptyset$ (canonical isomorphisms). In other words, A is a tensor functor.

(d) There is a gluing operation: for every surface $\Sigma \in \operatorname{Ob} \mathcal{T}$ and an unordered pair $\alpha, \beta \in A(\Sigma)$, we have the surface $G_{\alpha,\beta}(\Sigma) = \sqcup_{\alpha,\beta}(\Sigma)$ obtained by identification of the boundary components α, β (cf. Definition 5.1.12(iv)). The gluing satisfies the following properties:

Compatibility with A: $A(G_{\alpha,\beta}(\Sigma)) = A(\Sigma) \setminus \{\alpha, \beta\}$.

Compatibility with $\sqcup$: if $\alpha, \beta \in A(\Sigma_1)$, there is a canonical functorial isomorphism $G_{\alpha,\beta}(\Sigma_1 \sqcup \Sigma_2) = (G_{\alpha,\beta}\Sigma_1) \sqcup \Sigma_2$.

Associativity: if $\alpha, \beta, \gamma, \delta \in A(\Sigma)$ are distinct, then there exists a canonical functorial isomorphism $G_{\alpha,\beta} G_{\gamma,\delta}(\Sigma) = G_{\gamma,\delta} G_{\alpha,\beta}(\Sigma)$.

Functoriality: for each morphism $f\colon \Sigma \to \Sigma'$, we have an isomorphism $G_f : G_{\alpha,\beta}(\Sigma) \to G_{\alpha',\beta'}(\Sigma')$, where $\alpha' = A(f)(\alpha)$, $\beta' \in A(f)(\beta)$ are the corresponding elements in $A(\Sigma')$. These isomorphisms satisfy $G_{f_1 f_2} = G_{f_1} G_{f_2}$ and $G_{\mathrm{id}} = \mathrm{id}$.

DEFINITION 5.6.1. A *tower of groupoids* (or just a *tower*) is the following collection of data:

(i) A groupoid $\mathcal{T}$;

(ii) A "disjoint union" bifunctor $\sqcup\colon \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ and an object $\emptyset \in \operatorname{Ob}\mathcal{T}$ satisfying the axioms of a symmetric tensor category;

(iii) A "boundary functor": a tensor functor $A\colon \mathcal{T} \to \mathcal{S}ets$;

(iv) A "gluing operation": for every $\Sigma \in \operatorname{Ob}\mathcal{T}$ and an unordered pair $\alpha, \beta \in A(\Sigma)$, we have an object $G_{\alpha,\beta}(\Sigma) \in \mathcal{T}$. The gluing should be associative, functorial and compatible with $\sqcup$ and A (see (d) above).

EXAMPLE 5.6.2. $\mathcal{S}ets$ and $\mathcal{T}eich$ are towers of groupoids.

REMARK 5.6.3. Sometimes it is useful to weaken the above definition by considering towers in which the gluing operation $G_{\alpha,\beta}$ is defined not for all but only for some pairs α, β. In this case, the identities $G_{\alpha,\beta}\sqcup = \sqcup(G_{\alpha,\beta} \times \mathrm{Id})$, $G_{\alpha,\beta}G_{\gamma,\delta} = G_{\gamma,\delta}G_{\alpha,\beta}$ in the definition above should be understood in the following way: if one side is defined, then the other one is also defined and they are equal.

An example of such a "partial" tower is given by the the *Teichmüller tower in genus zero*, $\mathcal{T}eich_0$, in which objects are extended surfaces of genus zero and the functor $G_{\alpha,\beta}$ is defined only if α, β belong to different connected components of Σ.

REMARK 5.6.4. One can give a definition of what it means for a tower of groupoids to be presented by generators and relations (but since this is a little boring, we don't do it here). Then the results of Section 5.2 (and [**BK**]) can be reformulated as giving the generators and relations presentation of the Teichmüller tower $\mathcal{T}eich$. One notes that this presentation is much simpler than the presentations for individual mapping class groups $\Gamma(\Sigma)$. The idea of using the Teichmüller tower with the gluing operation for the study of mapping class groups belongs to Grothendieck [**G**]. More results in this direction can be found in [**HLS**].

Before giving more examples of towers, let us reformulate Definition 5.6.1 in a more functorial way. This will be useful later when we define functors between towers.

Let $\mathcal{T}$ be a tower of groupoids. Then $\mathcal{T}$ is a *fibered category* over $\mathcal{S}ets$. For any finite set S, the fiber $\mathcal{T}_S$ over S is the category with objects all pairs (Σ, φ) where $\Sigma \in \operatorname{Ob}\mathcal{T}$ and $\varphi\colon A(\Sigma) \xrightarrow{\sim} S$ is a bijection. A morphism between two objects $(\Sigma_1, \varphi_1), (\Sigma_2, \varphi_2) \in \operatorname{Ob}\mathcal{T}_S$ is a morphism $f \in \operatorname{Mor}_{\mathcal{T}}(\Sigma_1, \Sigma_2)$ such that $\varphi_1 = \varphi_2 \circ A(f)$. Since both $\mathcal{T}$ and $\mathcal{S}ets$ are groupoids, every fiber $\mathcal{T}_S$ is a groupoid.

A bijection of sets $\psi\colon S \xrightarrow{\sim} S'$ gives rise to a functor $\psi_*\colon \mathcal{T}_S \to \mathcal{T}_{S'}$: on objects $\psi_*(\Sigma, \varphi) = (\Sigma, \psi \circ \varphi)$, and on morphisms $\psi_*(f) = f$. It is obvious that

$$(\phi \circ \psi)_* = \phi_* \circ \psi_*, \quad \mathrm{id}_* = \mathrm{id};$$

in particular, all functors ψ_* are isomorphisms of categories.

Conversely, given a collection of groupoids $\{\mathcal{T}_S\}_{S \in \operatorname{Ob}\mathcal{S}ets}$ together with equivariance functors ψ_* as above, one can reconstruct the groupoid $\mathcal{T}$ and the functor $A\colon \mathcal{T} \to \mathcal{S}ets$.

In terms of these data, $\sqcup$ becomes a collection of functors

$$\sqcup^{S,S'}: \mathcal{T}_S \times \mathcal{T}_{S'} \to \mathcal{T}_{S\sqcup S'},$$

while $\emptyset \in \operatorname{Ob}\mathcal{T}_\emptyset$. They satisfy obvious commutativity, associativity and equivariance conditions.

Similarly, the gluing gives a collection of functors

$$G^S_{\alpha,\beta}: \mathcal{T}_S \to \mathcal{T}_{S\setminus\{\alpha,\beta\}}, \qquad S \in \operatorname{Ob}\mathcal{S}ets,\ \alpha,\beta \in S$$

(the pair α, β is unordered). Indeed, for $(\Sigma, \varphi) \in \operatorname{Ob}\mathcal{T}_S$, we define

$$G^S_{\alpha,\beta}(\Sigma,\varphi) = (\Sigma', \varphi|_{A(\Sigma')}) \quad \text{ where } \Sigma' = G_{\varphi^{-1}\alpha,\varphi^{-1}\beta}(\Sigma)$$

(recall that $A(\Sigma') = A(\Sigma)\setminus\{\varphi^{-1}\alpha, \varphi^{-1}\beta\}$). For a morphism $f: (\Sigma_1, \varphi_1) \to (\Sigma_2, \varphi_2)$ in $\mathcal{T}_S$, we define $G^S_{\alpha,\beta}(f) = G_f$ (recall the functoriality of gluing). Now the properties of gluing can be restated as follows.

Compatibility with A: already incorporated in the definition.

Compatibility with $\sqcup$: for any two sets S, S' and $\alpha, \beta \in S$, there exists a canonical isomorphism of functors $G^{S\sqcup S'}_{\alpha,\beta} \circ \sqcup^{S,S'} = \sqcup^{S\setminus\{\alpha,\beta\},S'} \circ (G^S_{\alpha,\beta} \times \operatorname{Id})$.

Associativity: if $\alpha, \beta, \gamma, \delta \in S$ are distinct then there exists a canonical isomorphism of functors $G^{S\setminus\{\gamma,\delta\}}_{\alpha,\beta} \circ G^S_{\gamma,\delta} = G^{S\setminus\{\alpha,\beta\}}_{\gamma,\delta} \circ G^S_{\alpha,\beta}$.

Functoriality: already incorporated in the requirement that $G^S_{\alpha,\beta}$ are functors.

Finally, there is one more property which follows just from the definition of $G^S_{\alpha,\beta}$.

Equivariance: for any bijection of sets $\psi: S \xrightarrow{\sim} S'$, we have $G^{S'}_{\psi\alpha,\psi\beta} \circ \psi_* = (\psi|_{S\setminus\{\alpha,\beta\}})_* \circ G^S_{\alpha,\beta}$.

DEFINITION 5.6.5. A *tower of groupoids* is a collection of groupoids $\{\mathcal{T}_S\}_{S\in\operatorname{Ob}\mathcal{S}ets}$ equipped with the following structure:

(i) Equivariance functors $\psi_*: \mathcal{T}_S \to \mathcal{T}_{S'}$ for any $\psi \in \operatorname{Mor}_{\mathcal{S}ets}(S, S')$, satisfying $(\phi \circ \psi)_* = \phi_* \circ \psi_*$ and $\mathrm{id}_* = \mathrm{id}$.

(ii) An object $\emptyset \in \operatorname{Ob}\mathcal{T}_\emptyset$ and a collection of functors $\sqcup^{S,S'}: \mathcal{T}_S \times \mathcal{T}_{S'} \to \mathcal{T}_{S\sqcup S'}$, satisfying obvious commutativity, associativity and equivariance conditions.

(iii) A collection of functors $G^S_{\alpha,\beta}: \mathcal{T}_S \to \mathcal{T}_{S\setminus\{\alpha,\beta\}}$, satisfying the above associativity, equivariance and compatibility with $\sqcup$.

PROPOSITION 5.6.6. *Definitions* 5.6.1 *and* 5.6.5 *are equivalent.*

PROOF. It was already sketched above. The details are left to the reader as an exercise. $\square$

DEFINITION 5.6.7. A *tower functor* $\mathcal{F}$ between two towers of groupoids $(\mathcal{T}, \sqcup, A, G)$ and $(\mathcal{T}', \sqcup', A', G')$ is a functor $\mathcal{F}: \mathcal{T} \to \mathcal{T}'$ which preserves all the structure. More precisely:

(i) There is an isomorphism of functors $A \simeq A' \circ \mathcal{F}$. Thus $\mathcal{F}$ gives rise to an equivariant collection of functors $\mathcal{F}^S: \mathcal{T}_S \to \mathcal{T}'_S$, $S \in \operatorname{Ob}\mathcal{S}ets$.

(ii) $\mathcal{F}$ is a tensor functor, i.e., the functors $\mathcal{F} \circ \sqcup$ and $\sqcup' \circ (\mathcal{F} \times \mathcal{F}): \mathcal{T} \times \mathcal{T} \to \mathcal{T}'$ are isomorphic.

(iii) For any finite set S, there is an isomorphism of functors $\mathcal{F}^{S\setminus\{\alpha,\beta\}} \circ G^S_{\alpha,\beta} \simeq {G'}^S_{\alpha,\beta} \circ \mathcal{F}^S: \mathcal{T}_S \to \mathcal{T}'_{S\setminus\{\alpha,\beta\}}$. These isomorphisms are equivariant with respect to bijections of S.

EXERCISE 5.6.8. Spell out property (iii) of Definition 5.6.7 in terms of the gluing operations $G_{\alpha,\beta}$ from Definition 5.6.1.

EXAMPLE 5.6.9. $A\colon \mathcal{T} \to \mathcal{S}ets$ is a tower functor for any tower $\mathcal{T}$.

There is an even more economical way to reformulate the definition of a tower. Looking at the equivariance properties of the collections $\{\mathcal{T}_S\}$ and $\{G^S_{\alpha,\beta}\}$, one can notice that they can be combined if we allow more maps between sets. We introduce a category $\mathcal{S}ets_\sharp$ with the same objects as in $\mathcal{S}ets$ (i.e., finite sets), but with more morphisms: all maps between sets that are composed of bijections and the elementary injections $i^S_{\alpha,\beta}\colon S \setminus \{\alpha,\beta\} \hookrightarrow S$. (This definition was inspired by [**BFM**].) Let $\mathcal{S}ets^\sharp$ be the dual category of $\mathcal{S}ets_\sharp$, i.e., the category with the same objects but with all arrows inverted. All morphisms in $\mathcal{S}ets^\sharp$ are composed of bijections and the elementary morphisms

$$\delta^S_{\alpha,\beta}\colon S \to S \setminus \{\alpha,\beta\}, \qquad S \in \operatorname{Ob}\mathcal{S}ets^\sharp,\ \ \alpha,\beta \in S \text{ (unordered)}.$$

Now if we define

$$(\delta^S_{\alpha,\beta})_* = G^S_{\alpha,\beta}\colon \mathcal{T}_S \to \mathcal{T}_{S\setminus\{\alpha,\beta\}},$$

we will have $(\phi \circ \psi)_* = \phi_* \circ \psi_*$ for $\phi,\psi \in \operatorname{Mor}_{\mathcal{S}ets^\sharp}$. Note that $\mathcal{S}ets^\sharp$ is again a symmetric tensor category with respect to $\sqcup$.

PROPOSITION 5.6.10. *A tower of groupoids is the same as a symmetric tensor category $\mathcal{T}$ fibered over $\mathcal{S}ets^\sharp$ such that all fibers $\mathcal{T}_S$ ($S \in \operatorname{Ob}\mathcal{S}ets^\sharp$) are groupoids. In other words, we have parts* (i) *and* (ii) *of Definition* 5.6.5 *with $\mathcal{S}ets$ replaced with $\mathcal{S}ets^\sharp$.*

In this language a *tower functor* $\mathcal{F}$ between two towers is just a collection of functors $\mathcal{F}^S\colon \mathcal{T}_S \to \mathcal{T}'_S$, equivariant with respect to $\operatorname{Mor}_{\mathcal{S}ets^\sharp}$, and such that the corresponding functor $\mathcal{F}\colon \mathcal{T} \to \mathcal{T}'$ is a tensor functor. A *natural transformation* Φ between two tower functors $\mathcal{F},\mathcal{G}\colon \mathcal{T} \to \mathcal{T}'$ is a $\operatorname{Mor}_{\mathcal{S}ets^\sharp}$-equivariant collection of natural transformations Φ^S between the functors $\mathcal{F}^S,\mathcal{G}^S$. Then, as usual, $\mathcal{F}\colon \mathcal{T} \to \mathcal{T}'$ is called an *equivalence of towers* if there exists a tower functor $\mathcal{F}'\colon \mathcal{T}' \to \mathcal{T}$ such that the tower functors $\mathcal{F}\mathcal{F}'$ and $\mathcal{F}'\mathcal{F}$ are isomorphic to Id.

After introducing all this abstract nonsense let us now give some examples and applications.

EXAMPLE 5.6.11. Let $\mathcal{C}$ be an abelian category and $R \in \text{ind}-\mathcal{C}^{\boxtimes 2}$ be a symmetric object.[3] We define the tower of groupoids $\mathcal{F}un(\mathcal{C})$ as follows.

Objects: all pairs (S,F) where S is a finite set and F is a functor $\mathcal{C}^{\boxtimes S} \to \mathcal{V}ec_f$.

Morphisms: $\operatorname{Mor}((S_1,F_1),(S_2,F_2))$ consists of all pairs (f,φ) where $\varphi\colon S_1 \xrightarrow{\sim} S_2$ is a bijection, $f\colon F_1 \xrightarrow{\sim} \varphi_*F_2$ is an isomorphism of functors, and φ_*F_2 is the composition $\mathcal{C}^{\boxtimes S_1} \xrightarrow{\varphi_*} \mathcal{C}^{\boxtimes S_2} \xrightarrow{F_2} \mathcal{V}ec_f$.

Boundary functor: $A(S,F) = S$.

Disjoint union: $(S_1 \sqcup S_2, F_1 \otimes F_2\colon \mathcal{C}^{\boxtimes(S_1\sqcup S_2)} \to \mathcal{V}ec_f)$, and similarly for morphisms. The object $\emptyset$ is the obvious one.

Gluing: given by $G_{\alpha,\beta}(S) = S\setminus\{\alpha,\beta\}$ and $G_{\alpha,\beta}(F) = F(\dots,R^{(1)},\dots,R^{(2)},\dots)$, where $R^{(1)},R^{(2)}$ are put in the places corresponding to the indices α,β.

[3] Here and below we use the same notation as in Section 2.4.

DEFINITION 5.6.12. Let $\mathcal{C}$ be an abelian category and $R \in \text{ind}-\mathcal{C}^{\boxtimes 2}$ be a symmetric object. A *representation* of a tower $\mathcal{T}$ in $\mathcal{C}$ is a tower functor $\rho\colon \mathcal{T} \to \mathcal{F}un(\mathcal{C})$.

The following theorem, which follows immediately from the definitions, elucidates the notion of a modular functor.

THEOREM 5.6.13. *A $\mathcal{C}$-extended modular functor is the same as a representation τ of the Teichmüller tower $\mathcal{T}eich$ in $\mathcal{C}$ with the additional normalization condition $\tau(S^2) = \text{id}\colon \mathcal{C}^0 = \mathcal{V}ec_f \to \mathcal{V}ec_f$.*

In a similar way one can rewrite the notion of MS data (see Section 5.3). In order to introduce the corresponding tower of groupoids $\mathcal{MS}$, we will first need the following definition.

DEFINITION 5.6.14. A *marking graph* is a graph m without cycles (a "forest") with the following additional data:

(i) The vertices of m are split into two subsets, "internal" and "external"

$$\text{Vertices}(m) = \text{Int}(m) \sqcup \text{Ext}(m),$$

so that every external vertex is 1-valent, and there are no edges connecting two external vertices.

(ii) For every internal vertex $v \in \text{Int}(m)$, an order on the set of all edges ending at v is given.

REMARK 5.6.15. The marking graphs with 3-valent internal vertices are essentially the same as "Bratelli diagrams" used in physics literature.

Graphs of this type appeared in our discussion of parameterizations of extended surfaces (see Section 5.2). In the figures, we use $*$ for internal vertices and $\bullet$ for external vertices. To show the order, we draw the edges in a clockwise order and mark the first edge by an arrow.

We define a CW complex $\mathcal{M}_0$ in a way parallel to the definition of $\mathcal{M}(\Sigma)$ for genus 0 (see Section 5.2). The vertices of $\mathcal{M}_0$ are all marking graphs. We define the simple moves Z, B, F by Figures 5.5, 5.6 and 5.7, respectively. The relations in $\mathcal{M}_0$ are obtained from MF1–MF7 by forgetting the surfaces.

EXAMPLE 5.6.16. The *Moore–Seiberg tower* $\mathcal{MS}$ is the tower of groupoids defined as follows.

Objects: all marking graphs.

Morphisms: $\text{Mor}(m_1, m_2)$ consists of all paths in the CW complex $\mathcal{M}_0$ connecting m_1 with m_2, modulo homotopy. (In other words, as a groupoid $\mathcal{MS}$ is the fundamental groupoid of $\mathcal{M}_0$.)

Boundary functor: $A(m) = \text{Ext}(m)$.

Disjoint union and $\emptyset$: obvious.

Gluing: if $\alpha, \beta \in \text{Ext}(m)$ are in different connected components, then we define $G_{\alpha,\beta}(m)$ to be the graph obtained by identifying the vertices α and β. The order at the new internal vertex $\alpha = \beta$ is given by $e_\alpha < e_\beta$ where e_α is the edge of m ending at α.

Note that $\mathcal{MS}$ is a "partial" tower in the sense of Remark 5.6.3.

THEOREM 5.6.17. *Let $\mathcal{C}$ be a semisimple abelian category. Then MS data for $\mathcal{C}$ is the same as a non-degenerate representation ρ of the Moore–Seiberg tower $\mathcal{MS}$ in $\mathcal{C}$ with the additional normalization condition $\rho(*) = \text{id}\colon \mathcal{V}ec_f \to \mathcal{V}ec_f$, where $*$ is the marking graph with one vertex and no edges.*

PROOF. Given a collection of MS data, let us construct a representation ρ of the tower $\mathcal{MS}$. For a marking graph m, define the functor $\rho(m)\colon \mathcal{C}^{\boxtimes \operatorname{Ext}(m)} \to \mathcal{V}ec_f$ similarly to (5.4.3). In other words, if W_v are the objects assigned to the external vertices $v \in \operatorname{Ext}(m)$, then we let

$$\rho(m)(\{W_v\}) = \bigotimes_{u \in \operatorname{Int}(m)} \langle X_{e_u^1}, \dots, X_{e_u^{k_u}} \rangle,$$

where $e_u^1, \dots, e_u^{k_u}$ are the edges adjacent to u, in the order defined by u, and $X_e = W_v$ if e connects u with an external vertex v, or $X_e = R$ if e connects two internal vertices.

The definition of the functorial isomorphisms which we assign to the morphisms of graphs is obvious. We also have obvious isomorphisms $\rho(m_1 \sqcup m_2) \simeq \rho(m_1) \otimes \rho(m_2)$ and $\rho(G_{\alpha,\beta}(m)) \simeq G_{\alpha,\beta}(\rho(m))$; in the latter isomorphism both sides coincide with $\rho(m)(\dots, R^{(1)}, \dots, R^{(2)}, \dots)$.

Now, a comparison of the relations MS1–MS7 and the relations MF1–MF7, used in the definition of $\mathcal{M}_0$, shows that the so defined ρ is indeed a representation of $\mathcal{MS}$.

Conversely, given a representation ρ of the tower $\mathcal{MS}$, define the MS data as follows:

$$\langle W_1, \dots, W_n \rangle = \rho(m_n)(W_1, \dots, W_n)$$

where m_n is the "standard" marking graph, with one internal vertex and n external vertices. Again, it is clear how to define the isomorphisms Z, σ, G and check that all the relations are satisfied. □

It is clear by its definition that the tower $\mathcal{MS}$ is just the projection on the level of marking graphs of another tower $\mathcal{PT}eich_0$: the *parametrized Teichmüller tower in genus zero.* On its hand, $\mathcal{PT}eich_0$ is the genus zero part of a tower $\mathcal{PT}eich$ which appeared implicitly in Section 5.2 and which we now proceed to define.

EXAMPLE 5.6.18. The *parameterized Teichmüller tower* $\mathcal{PT}eich$ is the tower of groupoids defined as follows.

Objects: all pairs (Σ, M), where Σ is an extended surface and $M = (C, \{\psi_a\})$ is a parameterization of Σ (see Definition 5.2.1).

Morphisms: $\operatorname{Mor}((\Sigma_1, M_1), (\Sigma_2, M_2))$ consists of all pairs (f, φ) where $f\colon \Sigma_1 \xrightarrow{\sim} \Sigma_2$ is a homeomorphism of extended surfaces and φ is a path in $\mathcal{M}(\Sigma_2)$ connecting $f(M_1)$ with M_2. The composition of morphisms is given by $(f, \varphi) \circ (g, \psi) = (f \circ g, \varphi \circ f(\psi))$.

Boundary functor: $A(\Sigma, M) = A(\Sigma) = \pi_0(\partial\Sigma)$ — the set of boundary components of Σ.

Disjoint union and $\emptyset$: the usual ones.

Gluing: $G_{\alpha,\beta}(\Sigma, M) = (\sqcup_{\alpha,\beta}(\Sigma), \sqcup_{\alpha,\beta} M)$, where $\sqcup_{\alpha,\beta}(\Sigma)$ is obtained from Σ by gluing the boundary components α, β, and the parameterization $\sqcup_{\alpha,\beta} M$ is obtained from M by adding $\alpha = \beta$ as a new cut and keeping the homeomorphisms ψ_a unchanged.

Note that by Theorem 5.2.9 the path φ is uniquely defined by f, so we could as well omit φ from the above definition of morphisms. However, it will be useful for us to have the definition in this form.

Now we can reformulate the main results of the previous sections in a much more transparent way.

THEOREM 5.6.19. (i) *The towers of groupoids $\mathcal{T}eich$ and $\mathcal{PT}eich$ are equivalent. Similarly, their genus zero parts $\mathcal{T}eich_0$ and $\mathcal{PT}eich_0$ are equivalent.*
(ii) *The towers $\mathcal{PT}eich_0$ and $\mathcal{MS}$ are equivalent.*

PROOF. (i) To prove the first statement, consider the obvious forgetting functor $\mathcal{PT}eich \to \mathcal{T}eich$. It suffices to check that this functor is bijective on morphisms. By Theorem 5.2.9, for every two parameterizations M, M' of an extended surface Σ there exists a unique path in $\mathcal{M}(\Sigma)$ connecting them. Thus, in a pair $(f, \varphi) \in \mathrm{Mor}_{\mathcal{PT}eich}$, the path φ is uniquely determined by f, which is equivalent to saying that the forgetting functor gives a bijection $\mathrm{Mor}_{\mathcal{PT}eich} \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{T}eich}$. The proof for genus zero is completely parallel.

(ii) To prove the second statement, consider the functor $\mathcal{PT}eich_0 \to \mathcal{MS}$ which assigns to the pair (Σ, M) the marking graph of M. Obviously, every marking graph can be obtained in this way. Thus, it suffices to prove that this functor gives a bijection of the spaces of morphisms. This is immediate from comparing the moves and relations and the following rigidity lemma.

LEMMA 5.6.20. *Let Σ be an extended surface, $M \in M(\Sigma)$ be a parameterization, and m the corresponding marking graph. Let $f : \Sigma \xrightarrow{\sim} \Sigma$ be a homeomorphism which preserves the graph m pointwise.*[4] *Then f is homotopic to identity.*

This completes the proof of Theorem 5.6.19. □

A comparison of the theorems above makes the relation between genus zero modular functors and weakly ribbon structures on a semisimple category obvious.

5.7. Central extension of modular functor

In Section 5.5 we have constructed a $\mathcal{C}$-extended modular functor (MF) starting from any modular tensor category $\mathcal{C}$ satisfying $p^+/p^- = 1$. As with TQFT constructed from $\mathcal{C}$, the gluing axiom fails when $p^+/p^- \neq 1$. There are two approaches to deal with the general case.

First, we can content ourselves with a modification of the gluing axiom, which says that it holds only up to a multiplicative factor. This is similar to the notion of a projective representation of a group.

The second approach is to try to construct a kind of a "central extension" of the modular functor. This was done independently by several authors; our exposition follows an unpublished manuscript [**BFM**] by Beilinson, Feigin, and Masur.

We begin with some preliminaries. Let V be a symplectic real vector space of dimension $2g$, $g \in \mathbb{N}$. Let Λ_V be the set of all Lagrangian subspaces of V, i.e., maximal isotropic subspaces of V. This is a compact manifold. Let T_V be the *Poincaré groupoid* of Λ_V; by definition, objects of this groupoid are points of Λ_V and morphisms are homotopy classes of paths connecting two points. It is convenient to define T_V for $V = 0$ as the category with only one object 0 and $\mathrm{Hom}_{T_0}(0, 0) = \mathbb{Z}$.

The proof of the following lemma is straightforward and will be omitted.

[4] It is not sufficient to require that $f(m) = m$, as f could interchange components of m.

LEMMA 5.7.1. (i) *For any two symplectic vector spaces V_1, V_2, there exists a canonical map $\Lambda_{V_1} \times \Lambda_{V_2} \to \Lambda_{V_1 \oplus V_2}$.*

(ii) *Let $N \subset V$ be an isotropic subspace, i.e., such that the restriction of the symplectic form on N is 0. Then the space $N^\perp/N$ is symplectic, and there exists a canonical map $\Lambda_{N^\perp/N} \to \Lambda_V$ which assigns to a Lagrangian subspace $L \subset N^\perp/N$ the subspace $\pi^{-1}(L) \subset N^\perp \subset V$, where $\pi\colon N^\perp \to N^\perp/N$ is the natural projection. The induced map of fundamental groupoids $T_{N^\perp/N} \to T_V$ is an equivalence.*

COROLLARY 5.7.2. *For any point $a \in \Lambda_V$, the fundamental group $\pi_1(\Lambda_V, a)$ is isomorphic to $\mathbb{Z}$.*

Corollary 5.7.2 implies that the group $\mathbb{Z}$ acts freely on $\mathrm{Mor}_{T_V}(L_1, L_2)$ for any $L_1, L_2 \in \Lambda_V$. (In other words, $\mathrm{Mor}_{T_V}(L_1, L_2)$ is a $\mathbb{Z}$-*torsor*.) Hence we have a non-canonical identification

$$\mathrm{Mor}_{T_V}(L_1, L_2) \xrightarrow{\sim} \mathbb{Z}. \tag{5.7.1}$$

Let us choose such identifications for all $L_1, L_2 \in \Lambda_V$. If $\varphi\colon L_1 \to L_2$ and $\psi\colon L_2 \to L_3$ are two morphisms in T_V, corresponding to numbers $m, n \in \mathbb{Z}$, then in general $\psi\varphi\colon L_1 \to L_3$ corresponds to some $p \neq m + n$. The difference

$$\mu(L_1, L_2, L_3) := p - m - n \tag{5.7.2}$$

is called the *Maslov index* of the subspaces L_1, L_2, L_3.

Let Σ be an extended surface, as in Section 5.1. We denote by $cl(\Sigma)$ the surface without boundary obtained from Σ by gluing disks to all boundary circles, and let

$$H(\Sigma) := \mathrm{H}_1(cl(\Sigma), \mathbb{R}). \tag{5.7.3}$$

The intersection form makes $H(\Sigma)$ a symplectic space of dimension $2g$ where g is the genus of Σ (i.e., of $cl(\Sigma)$). Introduce the notations

$$\Lambda_\Sigma := \Lambda_{H(\Sigma)}, \quad T_\Sigma := T_{\Lambda_\Sigma}. \tag{5.7.4}$$

When Σ is of genus zero, we have $H(\Sigma) = 0$ and Λ_Σ is a point. In this case, it is convenient to define T_Σ as the category with only one object 0 and $\mathrm{Hom}_{T_\Sigma}(0,0) = \mathbb{Z}$.

The next lemma is left as an exercise.

LEMMA 5.7.3. (i) *There exists a canonical map $a\colon \Lambda_{\Sigma_1} \times \Lambda_{\Sigma_2} \to \Lambda_{\Sigma_1 \sqcup \Sigma_2}$. (However, it is not a homeomorphism.)*

(ii) *Let the surface Σ be obtained by sewing two surfaces along one boundary component: $\Sigma = \Sigma_1 \sqcup_{\alpha,\beta} \Sigma_2$. Then $H(\Sigma_1 \sqcup \Sigma_2) \simeq H(\Sigma)$. Therefore, there exists a canonical homeomorphism $g_{\alpha,\beta}\colon \Lambda_{\Sigma_1 \sqcup \Sigma_2} \xrightarrow{\sim} \Lambda_\Sigma$.*

(iii) *Let Σ be obtained from Σ' by gluing two boundary circles α_1, α_2 in the same connected component: $\Sigma = \sqcup_{\alpha_1,\alpha_2} \Sigma'$. These two circles give a cycle $\alpha \in H(\Sigma)$. Then we claim that $H(\Sigma') \simeq \alpha^\perp/\mathbb{R}\alpha$. Therefore, we have a canonical map $g_{\alpha_1,\alpha_2}\colon \Lambda_{\Sigma'} \to \Lambda_\Sigma$ which induces an equivalence $T_{\Sigma'} \xrightarrow{\sim} T_\Sigma$.*

EXERCISE 5.7.4. Let Σ be an extended surface, and let C be a cut system on Σ, i.e., a finite set of closed simple non-intersecting curves on Σ such that the connected components Σ_a of $\Sigma \setminus C$ have genus zero (cf. Definition 5.2.1). By Lemma 5.7.3, this defines a map $\prod \Lambda_{\Sigma_a} \to \Lambda_\Sigma$. Since, by definition, each Λ_{Σ_a} is a point, this map gives an element $y_C \in \Lambda_\Sigma$. Show that y_C is the subspace in $\mathrm{H}_1(cl(\Sigma), \mathbb{R})$ spanned by the classes $[c], c \in C$.

Now we can define the "central extension" of the Teichmüller tower which was defined in Section 5.6.

DEFINITION 5.7.5. The central extension $\widetilde{\mathcal{T}eich}$ of the Teichmüller tower $\mathcal{T}eich$ is the tower of groupoids defined as follows.

Objects: all pairs (Σ, y), where Σ is an extended surface and and $y \in \Lambda_\Sigma$.

Morphisms: $\mathrm{Mor}((\Sigma_1, y_1), (\Sigma_2, y_2))$ consists of all pairs (f, ϕ), where $f\colon \Sigma_1 \xrightarrow{\sim} \Sigma_2$ is an orientation preserving homeomorphism and $\phi \in \mathrm{Mor}_{T_{\Sigma_2}}(f_* y_1, y_2)$. Here $f_*\colon \Lambda_{\Sigma_1} \to \Lambda_{\Sigma_2}$ is the map induced from f.

Boundary functor: $A(\Sigma, y) = \pi_0(\partial\Sigma)$ is the set of boundary components of Σ.

Disjoint union: $(\Sigma_1, y_1) \sqcup (\Sigma_2, y_2) = (\Sigma_1 \sqcup \Sigma_2, a(y_1 \oplus y_2))$, where $a\colon \Lambda_{\Sigma_1} \times \Lambda_{\Sigma_2} \to \Lambda_{\Sigma_1 \sqcup \Sigma_2}$ is as in Lemma 5.7.3(i). The object $\emptyset$ is the obvious one.

Gluing: $G_{\alpha,\beta}(\Sigma, y) = (\sqcup_{\alpha,\beta}(\Sigma), g_{\alpha,\beta}(y))$, where $g_{\alpha,\beta}\colon \Lambda_\Sigma \to \Lambda_{\sqcup_{\alpha,\beta}(\Sigma)}$ is as in Lemma 5.7.3(ii), (iii).

This groupoid is a central extension of the usual Teichmüller groupoid in the following sense: we have an obvious functor $\widetilde{\mathcal{T}eich} \to \mathcal{T}eich$ compatible with all the operations, and for each $(\Sigma, y) \in \mathrm{Ob}\,\widetilde{\mathcal{T}eich}$, the kernel of the map $\mathrm{Aut}_{\widetilde{\mathcal{T}eich}}(\Sigma, y) \to \mathrm{Aut}_{\mathcal{T}eich}(\Sigma)$ is $\mathrm{Aut}_{T_\Sigma}(y) = \mathbb{Z}$ (see (5.7.1)). In other words, denoting for an extended surface Σ and $y \in \Lambda_\Sigma$ the extended mapping class group by

$$\hat{\Gamma}(\Sigma, y) := \mathrm{Aut}_{\widetilde{\mathcal{T}eich}}(\Sigma, y), \tag{5.7.5}$$

(up to an isomorphism, this does not depend on the choice of y), we can write the following exact sequence:

$$0 \to \mathbb{Z} \to \hat{\Gamma}(\Sigma, y) \to \Gamma(\Sigma) \to 0. \tag{5.7.6}$$

Note that for Σ of genus zero, Λ_Σ is a point, and we have a canonical isomorphism $\hat{\Gamma}(\Sigma) = \mathbb{Z} \times \Gamma(\Sigma)$, i.e., the above exact sequence splits. For positive genus, this is not so.

EXAMPLE 5.7.6. Let $\Sigma = S_{1,1}$ be the torus with one puncture, and let α, β be the meridian and the parallel of the torus, so that $H(\Sigma) = \mathbb{R}[\alpha] \oplus \mathbb{R}[\beta]$ (see Figure 5.19). Then $\Lambda_\Sigma = \mathbb{RP}^1 = S^1$. Let $s, t \in \Gamma_{1,1}$ be the elements of the mapping class group defined in Example 5.1.11.

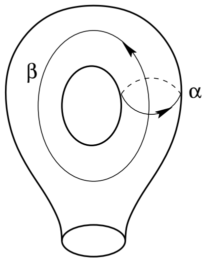


FIGURE 5.19

For $y = [\alpha]$ we will describe the central extension $\hat{\Gamma}(\Sigma, y)$. Note that $t_*([\alpha]) = [\alpha]$, $s_*[\alpha] = [\beta]$. Let us choose a path ϕ in Λ_Σ connecting the points $[\beta]$ and $[\alpha]$.

Now, define elements $\hat{t}, \hat{s}, \hat{c} \in \hat{\Gamma}(\Sigma, y)$ by $\hat{t} = (t, \mathrm{id})$, $\hat{s} = (s, \phi)$, $\hat{c} = (c, \mathrm{id})$, where $c = s^2$ acts on $H(\Sigma)$ by $v \mapsto -v$, and thus, acts trivially on Λ_Σ. Then we claim that the group $\hat{\Gamma}(\Sigma, y)$ is generated by the elements $\hat{t}, \hat{s}, \hat{c}, \gamma$ with the relations

$$\hat{s}^2 = \gamma\hat{c}, \quad (\hat{s}\hat{t})^3 = \hat{s}^2, \quad \gamma, \hat{c} \text{ are central,} \tag{5.7.7}$$

where $\gamma = (\mathrm{id}, \gamma)$ is the generator of the fundamental group $\pi_1(\Lambda_\Sigma, y) = \mathbb{Z}$.

Similarly, if we consider a torus without punctures, then the mapping class group $\Gamma(S_{1,0}, y)$ is generated by the same elements with the additional relation $\hat{c}^2 = 1$. The proof of both of these statements is left to the reader as an exercise.

REMARK 5.7.7. One sees that for $\Sigma = S_{1,1}$, the exact sequence (5.7.6) trivially splits. For $\Sigma = S_{1,0}$, we have $\Gamma(\Sigma) = \mathrm{SL}_2(\mathbb{Z})$, and one can check that the above exact sequence does not split, but it "splits over $\mathbb{Q}$": if we denote by $\hat{\Gamma}(\Sigma, y)_{\mathbb{Q}} = \hat{\Gamma}(\Sigma, y) \times_{\mathbb{Z}} \mathbb{Q}$ the group obtained by adding to $\hat{\Gamma}(\Sigma)$ fractional powers of γ, then the exact sequence

$$0 \to \mathbb{Q} \to \hat{\Gamma}(\Sigma, y)_{\mathbb{Q}} \to \Gamma(\Sigma) \to 0$$

does split. However, it can be shown that for $g > 1$ the exact sequence (5.7.6) for $\Gamma_{g,0}$ does not split even over $\mathbb{Q}$.

Now we can formulate the notion of a modular functor with a central charge. Recall that we have defined the notion of a representation of a tower of groupoids in an abelian category $\mathcal{C}$ (see Definition 5.6.12), and the modular functor can be defined as a representation of the Teichmüller tower (see Theorem 5.6.13).

DEFINITION 5.7.8. Let $\mathcal{C}$ be an abelian category. A *$\mathcal{C}$-extended modular functor with (multiplicative) central charge $K \in k^\times$* is a representation of the tower $\widetilde{\mathcal{T}eich}$, with the additional normalization condition $\tau(S^2) = k$, and such that for every extended surface Σ and $y \in \Lambda_\Sigma$ the generator γ of $\mathrm{Aut}_{\mathcal{T}_\Sigma}(y) = \mathbb{Z} \subset \mathrm{Aut}_{\widetilde{\mathcal{T}eich}}(\Sigma, y)$ acts as multiplication by K.

For those readers who do not like the language of towers of groupoids, this definition can be spelled out explicitly as follows.

DEFINITION 5.7.9. A *modular functor with (multiplicative) central charge $K \in k^\times$* is the following collection of data:

(i) Let Σ be a compact oriented surface with boundary, with a point and an object of $\mathcal{C}$ attached to any boundary circle, and let $y \in \Lambda_\Sigma$. To any such (Σ, y) the modular functor assigns a finite dimensional vector space $\tau(\Sigma, y)$.

(ii) To any morphism $\tilde{f} \colon (\Sigma, y) \to (\Sigma', y')$ the modular functor assigns an isomorphism of the corresponding vector spaces $\tilde{f}_* \colon \tau(\Sigma, y) \xrightarrow{\sim} \tau(\Sigma', y')$.

(iii) Functorial isomorphisms $\tau(\emptyset) \xrightarrow{\sim} k$, $\tau(\Sigma_1 \sqcup \Sigma_2, y_1 \oplus y_2) \xrightarrow{\sim} \tau(\Sigma_1, y_1) \otimes \tau(\Sigma_2, y_2)$.

(iv) A symmetric object $R \in \mathrm{ind}{-}\mathcal{C}^{\boxtimes 2}$ (see Section 2.4).

(v) **Gluing isomorphism:** Let Σ' be the surface obtained from Σ by cutting Σ along a circle. Then we require that there is an isomorphism

$$\tau(\Sigma', y; R^{(1)}, R^{(2)}) \to \tau(\Sigma, g(y)) \tag{5.7.8}$$

where g is as in Lemma 5.7.3(ii), (iii).

These data have to satisfy the same axioms as in Definition 5.1.12 and the following additional relation. Note that for every (Σ, y) the group $\pi_1(\Lambda_\Sigma, y)$ is canonically isomorphic to $\mathbb{Z}$. (The orientation of Σ gives a choice for the sign of the generator γ.) Then we require that $\gamma_*: \tau(\Sigma, y) \to \tau(\Sigma, y)$ be a multiplication by K.

THEOREM 5.7.10. *Any modular tensor category gives rise to a modular functor with central charge $K = p^+/p^-$. Conversely, if τ is a $\mathcal{C}$-extended modular functor with central charge K, then it defines on $\mathcal{C}$ a structure of a weakly ribbon category. If this category is rigid, then $\mathcal{C}$ is a modular category with $p^+/p^- = K$.*

PROOF. The proof is similar to the proof in the case of zero central charge ($p^+ = p^-$). It is based on an analogue of Theorem 5.2.9, giving the set of moves and relations among the parameterizations. However, now we have to extend the notion of parameterization as follows.

Let Σ be an extended surface and $y \in \Lambda_\Sigma$. An *extended parameterization* $\hat{M}$ is a pair (M, φ), where M is a parameterization of Σ (see Definition 5.2.1), and $\varphi \in \mathrm{Mor}_{T_\Sigma}(y, y_M)$, where $y_M \in \Lambda_\Sigma$ is the Lagrangian subspace defined by the cut system C of M (see Example 5.7.4).

Since the moves B, F, Z do not change y_M, we can lift each of them to a move between extended parameterizations by letting $\hat{B} = (B, \mathrm{id})$, etc. We also have a new move $\gamma: (M, \varphi) \rightsquigarrow (M, \gamma \circ \varphi)$, where γ is the generator of $\mathrm{Aut}_{T_\Sigma}(y_M, y_M) = \mathbb{Z}$. Finally, the move S can be lifted to a move $\hat{S}$ as in Example 5.7.6. Then each of relations MF1–MF7 makes sense as a relation among the moves $\hat{Z}, \dots, \hat{F}$. As for relations MF8, MF9, they can be uniquely lifted to relations among the moves between the extended parameterizations by replacing $Z, \dots S$ by $\hat{Z}, \dots, \hat{S}$ and inserting an appropriate power of γ to make it into a closed loop in $\hat{M}(\Sigma)$. We will denote the corresponding axioms by MF$\hat{8}$, MF$\hat{9}$. Let us also add an axiom MF$\hat{10}$ requiring that γ be central. Then it is easy to deduce from Theorem 5.2.9 that the corresponding 2-complex $\hat{\mathcal{M}}(\Sigma)$ is connected and simply-connected.

Now to show that every MTC defines a modular functor, we can follow the same approach as before, i.e., first define $\tau(\Sigma, y, \hat{M})$, and then assign to every move $\hat{E}: \hat{M} \rightsquigarrow \hat{M}'$ an isomorphism $\tau(\Sigma, y, \hat{M}) \to \tau(\Sigma, y, \hat{M}')$ so that all the relations MF1–MF$\hat{10}$ are satisfied.

Let us define $\tau(\Sigma, y, \hat{M}) = \tau(\Sigma, M)$ (thus, it does not depend on the choice of y and φ) and assign to the moves $\hat{Z}, \hat{B}, \hat{F}$ the same isomorphisms as before (i.e., Z, σ, G). Assign to γ the isomorphism given by multiplication by p^+/p^-. Finally, assign to $\hat{S}$ the operator $S/\sqrt{p^+/p^-}$, where S is defined in Theorem 3.1.17. Explicit calculation shows that for so defined $\hat{S}$, relations MF$\hat{8}$, MF$\hat{9}$ are satisfied. For MF$\hat{8}$, this calculation essentially coincides with the one done in Example 5.7.6.

The proof in the opposite direction is absolutely parallel to the one for the genus zero case; thus, we omit it. □

5.8. From 2D MF to 3D TQFT

Starting from a modular tensor category $\mathcal{C}$ with $p^+/p^- = 1$, we have constructed a $\mathcal{C}$-extended 3-dimensional Topological Quantum Field Theory (Section 4.4) and a $\mathcal{C}$-extended 2-dimensional modular functor (Section 5.1). We have also showed that conversely, if $\mathcal{C}$ is a semisimple abelian category then any $\mathcal{C}$-extended 2-dimensional modular functor gives rise to a structure of a modular category on $\mathcal{C}$ (provided that the rigidity condition is satisfied).

Schematically, we have:

$$
\begin{tikzcd}
 & \mathcal{C}\text{-extended 3D TQFT} \\
\text{MTC } \mathcal{C} \arrow[ur] \arrow[dr, Leftrightarrow] & \\
 & \mathcal{C}\text{-extended 2D MF}
\end{tikzcd}
.
$$

This indicates that there must be also a direct construction relating ($\mathcal{C}$-extended) 3D TQFT with ($\mathcal{C}$-extended) 2D MF.

3D TQFT $\to$ 2D MF. This implication has already been discussed before: in fact, the axioms of 2D MF (except the gluing axiom) are part of the axioms of 3D TQFT, cf. Remark 5.1.2. To prove that the gluing axiom also follows from the axioms of 3D TQFT, we again use the version of extended surface from Definition 5.1.10.

Let Σ'_V be the surface obtained from a surface Σ by cutting a circle from it and labeling the two new boundary components with objects V and V^*, as in Definition 5.1.12 (see Figure 5.20).

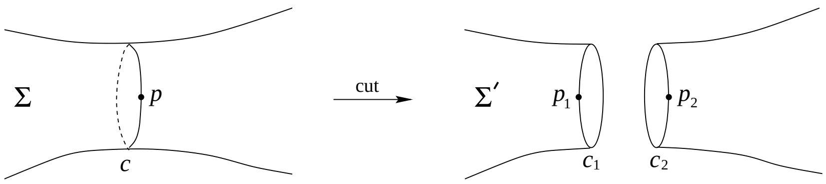


FIGURE 5.20

In accordance with the proof of Proposition 5.1.8, instead of Σ'_V we consider the surface $\Sigma'' = \Sigma''_V$ obtained from Σ'_V by replacing the boundary circles with marked points with tangent vectors at them. We can shrink Σ'', so that it is "inside" Σ, as in Figure 5.21 below.

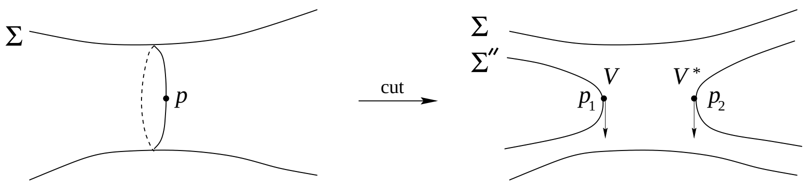


FIGURE 5.21

Then we "fill in the space between Σ and Σ''", i.e., we consider a 3-manifold M with boundary $\partial M = \Sigma \sqcup \overline{\Sigma''}$ (see Figure 5.22). This M is a $\mathcal{C}$-marked 3-manifold, hence it gives a vector

$$\tau(M) \in \tau(\partial M) \simeq \mathrm{Hom}_k(\tau(\Sigma''), \tau(\Sigma)).$$

Considered as a map $\tau(\Sigma'_V) \to \tau(\Sigma)$, this gives the required gluing map (5.1.1). One can easily check that this definition is correct and satisfies all the properties of Definition 5.1.12.

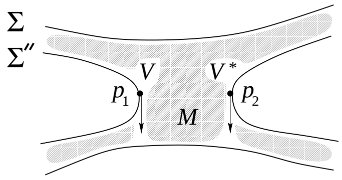


FIGURE 5.22

2D MF → 3D TQFT. This implication is much more difficult and, to the best of our knowledge, no complete construction of it is known. There are two approaches: the first one, due to L. Crane [**C**] (see also [**Ko**]), is based on the Heegaard splitting; the second one, due to M. Kontsevich and to I. Frenkel (unpublished), is based on Morse theory.

Following Crane [**C**], we will construct (non-extended) 3D TQFT starting from a $\mathcal{C}$-extended 2D MF. We do not know how to extend this construction to a $\mathcal{C}$-extended 3D TQFT.

We will use the following well-known theorem in topology (for references, see [**Cr**]).

THEOREM 5.8.1 (Reidemeister–Singer). *Let M be a connected closed oriented 3-manifold. Then*:

(i) *M can be presented as a result of gluing of two solid handlebodies*:

$$M = M_\varphi = H_1 \sqcup_\varphi H_2,$$

where $\varphi\colon \partial H_1 \xrightarrow{\sim} \overline{\partial H_2}$. Such a presentation is called a Heegaard splitting.

(ii) *Two such M_φ and $M_{\varphi'}$ are homeomorphic iff $\varphi\colon \partial H_1 \xrightarrow{\sim} \overline{\partial H_2}$ can be obtained from $\varphi'\colon \partial H_1' \xrightarrow{\sim} \overline{\partial H_2'}$ by a sequence of the following moves*:

(a) *$H_1 = H_1'$, $H_2 = H_2'$, φ' is isotopic to φ.*

(b) *$H_1 = H_1'$, $H_2 = H_2'$, $\varphi' = y \circ \varphi \circ x$, where $x \in N_{H_1}$, $y \in N_{H_2}$ and*

$$N_H := \{\textit{homeomorphisms of } \partial H \textit{ which extend to } H\}.$$

(c) Stabilization. *Let $H_1' = H_1 \# T$, $H_2' = H_2 \# T$, where T is a solid torus and $\#$ denotes a connected sum of topological spaces (see Figure 5.23 below). Let $\varphi' = \varphi \# s$, where $s\colon \partial T \xrightarrow{\sim} \partial T$ is the homeomorphism of the 2-torus which has a matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the standard basis $\{\alpha, \beta\}$ of $\mathrm{H}_1(\partial T, \mathbb{R})$. Then $M_{\varphi'} \simeq M_\varphi \# S^3 \simeq M_\varphi$.*

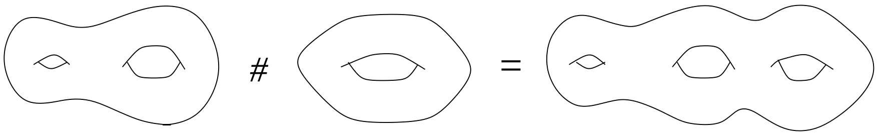


FIGURE 5.23. Connected sum of 3-manifolds.

Now suppose that we have a $\mathcal{C}$-extended modular functor. Let H be a solid handlebody whose boundary ∂H is a surface of genus g. We will construct a vector $v_0(H) \in \tau(\partial H)$ as follows.

Choose some non-intersecting "cuts", i.e., disks embedded in H, which cut H into contractible pieces. This also gives a system of cuts on ∂H and thus, a decomposition of ∂H into spheres with holes: $\partial H = \bigcup \Sigma_a$. Consider all possible labelings $i : \{\text{cuts}\} \to I$ of the cutting circles by simple objects of $\mathcal{C}$ (see Figure 5.24).

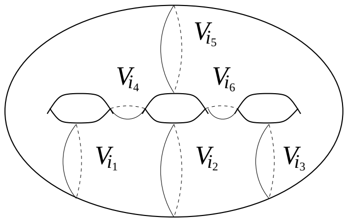


FIGURE 5.24

Then, by the gluing axiom,

$$\tau(\partial H) \simeq \bigoplus_i \bigotimes_a \tau(\Sigma_a; \{V_{i_c}^{\varepsilon}\}_{c \subset \partial \Sigma_a}).$$

Here Σ_a are the components of ∂H, the notation $c \subset \partial \Sigma_a$ means that the cut c is one of the boundary components of Σ_a, and V^ε is either V or V^* chosen so that every V_{i_c} appears in the tensor product once as V_{i_c} and once as $V_{i_c}^*$.

Let us choose all $i_c = 0$, i.e., all $V_{i_c} = \mathbf{1}$. Then $\tau(\Sigma_a; \mathbf{1}, \dots, \mathbf{1}) = k$. Therefore, this gives a vector

$$v_0(H) = \bigotimes_a (1 \in \tau(\Sigma_a; \mathbf{1}, \dots, \mathbf{1})) \in \tau(\partial H).$$

(compare with Remark 4.5.4).

THEOREM 5.8.2 (Crane [**C**]). *The vector $v_0(H)$ does not depend on the choice of the cuts. Moreover, $v_0(H)$ is N_H-invariant.*

PROOF. Obviously, any two systems of cuts of H into a union of solid balls can be related to one another by a sequence of the following moves:

(a) the action of N_H, and (b) the F-move.

It is easy to see that $v_0(H)$ does not change under the move (b). As for (a), one needs a description of the generators of N_H. Such a description is known [**Su**]. Then one checks that $v_0(H)$ is invariant under these generators—this is not difficult—we refer to [**C**], [**Ko**] for the details.

The fact that $v_0(H)$ is N_H-invariant follows from (a). □

Now we will use Theorems 5.8.1 and 5.8.2 to construct invariants of closed 3-manifolds.

Let $M = M_\varphi = H_1 \sqcup_\varphi H_2$ be as in 5.8.1. The map $\varphi \colon \partial H_1 \xrightarrow{\sim} \overline{\partial H_2}$ gives an isomorphism of vector spaces $\varphi_* \colon \tau(\partial H_1) \xrightarrow{\sim} \tau(\overline{\partial H_2}) = \tau(\partial H_2)^*$. We define

$$\tau(M) := D^{g-1} \, (\varphi_*(v_0(H_1)), v_0(H_2)), \tag{5.8.1}$$

where $D = s_{00}^{-1}$ is defined by (3.1.15).

The prefactor D^{g-1} is chosen in order that $\tau(M)$ be invariant under the stabilization move 5.8.1(c). Indeed, let $H' = H \# T$. Then $\partial H' = \partial H \# \partial T$, where ∂T is the 2-torus. By the construction of $v_0(H')$ it is clear that

$$v_0(H') = v_0(H) \otimes v_0(T).$$

Then

$$\begin{aligned}\tau(M') &= D^g\left((\varphi\#s)_*(v_0(H_1)\otimes v_0(T)), v_0(H_2')\right)\\ &= \tau(M)D\left(s_* v_0(T), v_0(T)\right) = \tau(M)Ds_{00} = \tau(M).\end{aligned}$$

Therefore, we have constructed an invariant τ of closed 3-manifolds. To construct 3D TQFT, we have to define $\tau(M)$ for any 3-manifold M with boundary. To do so, we need a variant of Heegaard splitting for 3-manifolds with boundary. There is such a theorem, due to Motto [**Mo**]. His result is similar to what we had before, only one has to consider not only handlebodies but also "hollow handlebodies". A hollow handlebody is a handlebody with some parts of its interior cut out. Hence, it has both "inner" and "outer" boundary. We glue two such hollow handlebodies by identifying their outer boundaries, the remaining inner boundaries give the boundary of the resulting 3-manifold.

Then we can repeat the above construction of $\tau(M)$ for manifolds M with boundary. This gives the implication

$$\mathcal{C}\text{-extended 2D MF} \to \text{(non-extended) 3D TQFT}.$$

In order to go one step further, i.e., to construct a $\mathcal{C}$-extended 3D TQFT, one needs an analog of Heegaard splitting and Reidemeister–Singer theorem for manifolds with boundary and marked points. To the best of our knowledge, such a result is not available at the moment. Hopefully, this is only a temporary difficulty. Finally, let us note that if we start with a non-extended 2D MF, without gluing axiom, the construction of 3D TQFT would fail.