CHAPTER 4

# 3-dimensional Topological Quantum Field Theory

In this chapter, following the ideas of [**RT2**] and [**T**], we will show that the algebraic formalism of modular tensor categories is closely related with the topology of 3-manifolds. In particular, we will show that every MTC gives rise to an invariant of compact 3-manifolds. Historically, this was one of the main motivations for the theory of modular tensor categories. We will quote without proofs several important results about 3-manifolds which are used in this construction. The reader is referred to an excellent introductory text [**PS**] and references therein for proofs and detailed discussion.

In this chapter, by "manifold" we mean an oriented compact topological manifold, possibly with boundary; by a "closed manifold" we will mean a manifold without boundary. All maps between manifolds will be continuous and preserving orientation, unless stated otherwise. By $\simeq$ we denote homeomorphism of manifolds, and by $\overline{M}$ we denote the manifold $M$ with reversed orientation. Finally, for an oriented manifold $M$ we endow its boundary $\partial M$ with an orientation in the standard way: $(v_1, \dots, v_k)$ is a positive reper for $\partial M$ if $(v_1, \dots, v_k, n)$ is a positive reper for $M$, where $n$ is the outward normal vector to $\partial M$.

## 4.1. Invariants of 3-manifolds

In this section we construct invariants of closed 3-manifolds. This construction is based on the notion of surgery, which itself is a special case of the operation of gluing, described in the lemma below.

LEMMA 4.1.1. *Let $M_1$ and $M_2$ be two manifolds of the same dimension. Let $N_1$ be a connected component of $\partial M_1$, $N_2$ a connected component of $\partial M_2$ and $f\colon N_1 \xrightarrow{\sim} N_2$ be an orientation reversing homeomorphism.*

(i) *Define $M_1 \cup_f M_2$ by*

$$M_1 \cup_f M_2 := (M_1 \sqcup M_2)/\{(x,y) \mid y = f(x), x \in N_1\}.$$

*Then $M_f \equiv M_1 \cup_f M_2$ is again a manifold. We will say that $M_f$ is obtained by gluing $M_1$ and $M_2$ using the identification $f$ of their boundary components.*

(ii) *If $f' = f \circ \varphi$ for some $\varphi\colon N_1 \xrightarrow{\sim} N_1$ which extends to $M_1 \xrightarrow{\sim} M_1$, then $M_{f'} \simeq M_f$.*

(iii) *$M_f$ depends only on the isotopy class of $f$, i.e., it does not change when we continuously deform $f$.*

PROOF. Only (iii) is not immediately obvious. It follows from the next claim: If $f' \sim f$ then one can write $f' = f \circ \varphi$ for some $\varphi\colon M_1 \xrightarrow{\sim} M_1$ such that $\varphi \neq \mathrm{id}$ only in a neighborhood of $N_1$. Indeed, it suffices to prove this in the case where $M_1$ is the cylinder $[0,1] \times N_1$, in which case it is obvious. $\square$

EXAMPLES 4.1.2. Let $T_1$ and $T_2$ be two solid tori. The boundary of a solid torus is a 2-dimensional torus which can be thought of as a rectangle with identified opposite sides. The sides of the rectangle give two 1-cycles which form a basis in homology (see Figure 4.1).

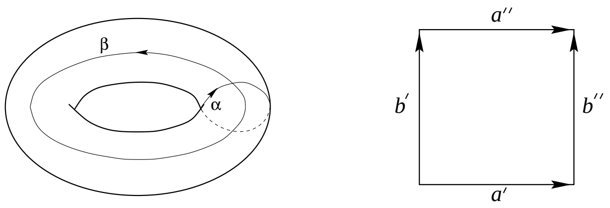
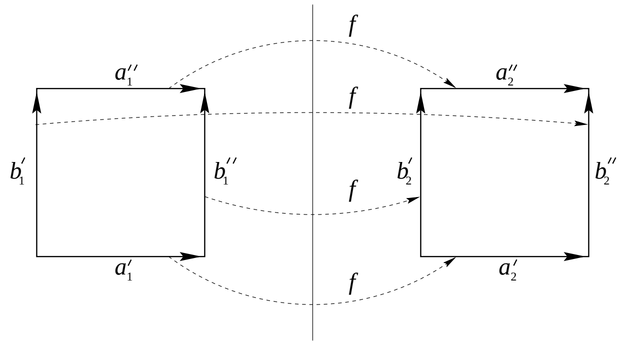



FIGURE 4.1. A torus obtained by gluing opposite sides of a rectangle.

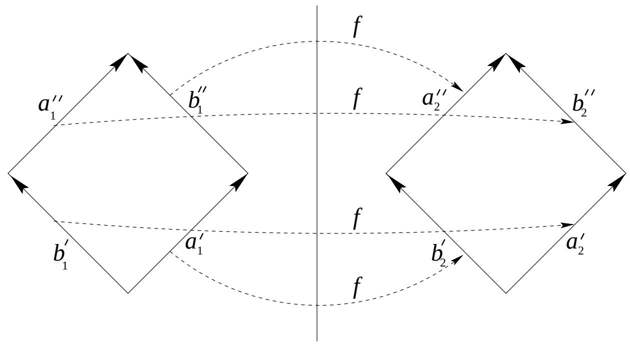
Below we will consider orientation reversing homeomorphisms $f\colon T_1 \xrightarrow{\sim} T_2$ as acting on the corresponding rectangles. We will consider two examples.

(i) $f$ identifies $a_1'$ with $a_2'$, $a_1''$ with $a_2''$, $b_1'$ with $b_2''$ and $b_1''$ with $b_2'$, i.e., it is a reflection in the vertical line:




Then $M_f = T_1 \cup_f T_2 \simeq S^2 \times S^1$. Indeed, $M_f$ has a well-defined projection on the circle $\beta$ which is a common parallel of both tori (on the figure above, $\beta$ is represented by any of the intervals $b_1, \dots, b_2''$). The fiber of this projection is $S^2$, obtained by gluing two copies of a disk along the boundary. Finally, it can be shown that this is indeed a direct product.

(ii) $f$ identifies $a_1'$ with $b_2'$, $a_1''$ with $b_2''$, $b_1'$ with $a_2'$ and $b_1''$ with $a_2''$, i.e., it is a reflection in the diagonal:



Then we claim that $T_1 \cup_f T_2 \simeq S^3$. The easiest way to visualize this is to note that the complement of a solid torus in $S^3 = \mathbb{R}^3 \cup \infty$ is again a solid torus. Hence $S^3$ is a union of two solid tori with boundaries identified by the map $f$.

In general, the spaces $T_1 \cup_f T_2$ obtained by gluing two solid tori are called *lens spaces*, see e.g. [**PS**]. Since such a lens space is defined by the isotopy class of $f$, it is natural to study the group of isotopy classes of homeomorphisms of a 2-dimensional torus.

THEOREM 4.1.3. *Let $T_0$ be a solid torus. Fix a standard basis $\alpha, \beta$ of $\mathrm{H}_1(\partial T_0, \mathbb{Z})$, as shown in Figure* 4.1. (*Given a solid torus $T_0$, $\alpha$ is defined uniquely up to a sign, but $\beta$ is not.*) *Then*:

(i) *For a map $f\colon \partial T_0 \to \partial T_0$, denote by $f_*$ its action in $\mathrm{H}_1(\partial T_0, \mathbb{Z}) \simeq \mathbb{Z}^2$. Then $f \mapsto f_*$ is an isomorphism of the group $\Gamma_{1,0}$ of isotopy classes of homeomorphisms $\partial T_0 \xrightarrow{\sim} \partial T_0$ with $\mathrm{SL}_2(\mathbb{Z})$.* [1]

(ii) *The homeomorphisms of $\partial T_0$ corresponding to the matrices $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ from $\mathrm{SL}_2(\mathbb{Z})$ ($k \in \mathbb{Z}$) can be extended to the whole $T_0$.*

PROOF. (i) For every $f$, the induced automorphism $f_*$ of $H_1(\partial T_0, \mathbb{Z})$ preserves the intersection form. Hence the matrix of $f_*$ in the basis $\alpha, \beta$ belongs to $\mathrm{SL}_2(\mathbb{Z})$. Surjectivity of this map is obvious if we write $\partial T_0 = \mathbb{R}^2/\mathbb{Z}^2$ and note that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{R}^2$; injectivity can be proved by standard topological arguments, see details in [**PS**].

(ii) The homeomorphism of $T_0$ that extends the transformation $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ is the so-called *Dehn twist*: cut the solid torus $T_0$ along the disk with boundary $\alpha$, twist it $k$ times and glue it back (see Figure 4.2 for $k = 1$). □

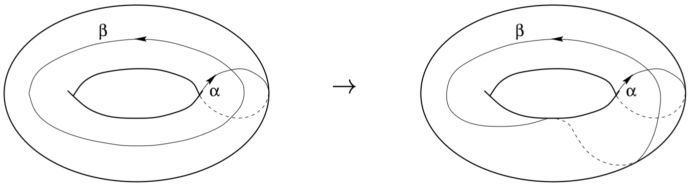



FIGURE 4.2. Dehn twist.

Let us also remind the following standard fact:

THEOREM 4.1.4. *The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the elements*

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

*with the defining relations $s^4 = 1, (st)^3 = s^2$.*

[1] The notation $\Gamma_{1,0}$ is chosen because later we will introduce more general groups $\Gamma_{g,n}$, corresponding to surfaces of genus $g$ with $n$ boundary components.

Let us restrict our attention to 3-manifolds. In this case, there are two classical constructions allowing to get any manifold by gluing of simpler pieces: Heegaard splitting and surgery along links. The approach of this chapter is based on surgery; the Heegaard splitting will be discussed later.

DEFINITION 4.1.5. Let $L$ be a link in $S^3$. Denote by $T_i$ some small tubular neighborhood of the $i$-th component of $L$. Each $T_i$ is a solid torus. Let $T_0$ be a fixed standard solid torus. Suppose we are given (orientation preserving) homeomorphisms $f_i : \partial T_i \xrightarrow{\sim} \partial T_0$. Define a new 3-manifold $M_{L,f}$, called a *surgery* of $S^3$ along the link $L$, by the formula

$$M_{L,f} := \left(S^3 \setminus (\cup_i T_i^{\text{int}})\right) \cup_{f_i} (T_0)^N, \tag{4.1.1}$$

where $N$ is the number of components of $L$ and "int" denotes interior.

In other words, we are cutting from $S^3$ a number of solid tori (possibly linked) and pasting instead new solid tori, but possibly in some twisted way.

THEOREM 4.1.6. *Any connected closed 3-manifold can be obtained as a surgery of $S^3$ along a link.*

Note that the definition above requires that we specify not only the link but also the attachment maps $f_i$. Our next observation is that there is a canonical way to construct $f_i$ if the link is framed (= ribbon). Let $L$ be a framed link in $S^3$ which is directed, i.e., each component of $L$ has an arrow. Let $T_i$ be a tubular neighborhood of the $i$th component, as before. Then the link $L$ determines cycles $\alpha_i$, $\beta_i$ in $\partial T_i$. Instead of giving the formal definition, we draw a picture — see Figure 4.3, where the cycle $\beta_i$ winds the same way as the $i$th component of $L$.

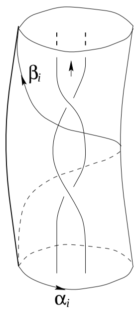



FIGURE 4.3. A ribbon link and the corresponding cycles $\alpha_i$, $\beta_i$ in its tubular neighborhood.

Let $f_i : \partial T_i \xrightarrow{\sim} \partial T_0$ be the homeomorphism which sends the cycle $\alpha_i$ to $-\beta$ and $\beta_i$ to $\alpha$. (By Theorem 4.1.3, such $f_i$ exists and is unique up to isotopy.)

DEFINITION 4.1.7. Let $L$ be a directed framed link in $S^3$. Then we define $M_L = M_{L,f_i}$ to be the surgery of $S^3$ along the link $L$ with the attachment maps $f_i$ described above.

LEMMA 4.1.8. *$M_L$ does not depend on the choice of directions of the components of $L$.*

PROOF. Follows from the fact that $\alpha \mapsto -\alpha$, $\beta \mapsto -\beta$ can be extended to the whole $T_0$ (see Theorem 4.1.3(ii)). □

This construction gives a 3-manifold $M_L$ for any framed link $L$ in $S^3$.

THEOREM 4.1.9. *Any connected compact oriented 3-manifold $M$ without boundary can be obtained as $M_L$ for some framed link $L$ in $S^3$.*

EXAMPLES 4.1.10. (i) When $L = \emptyset$ is the empty link, then $M = S^3$.

(ii) Let $L = \bigcirc$ (no twisting). Then, by Example 4.1.2(ii), $S^3 \setminus T_1 = T_2$ is a solid torus. Then gluing $T_0$ and $T_2$ we obtain $M_L = S^2 \times S^1$ as in Example 4.1.2(i).

(iii) When $L = \bigcirc^1 =$  (framing 1), then $M_L = S^3$. This is by no means obvious; we leave it as an exercise to the reader to deduce it from Example 4.1.2(ii) and Theorem 4.1.3(ii).

The next question is when two different links $L$ and $L'$ in $S^3$ give the same 3-manifold. The answer (highly non-trivial) was found by Kirby. We present it here in a form due to Fenn and Rourke (see [**PS**] and references therein).

THEOREM 4.1.11 (Kirby calculus). *$M_L \simeq M_{L'}$ iff $L'$ can be obtained from $L$ by a sequence of* Kirby–Fenn–Rourke moves *shown in Figure* 4.4 *below and the same with overcrossing and undercrossing interchanged.* (*The number of strands can be arbitrary, including zero.*)

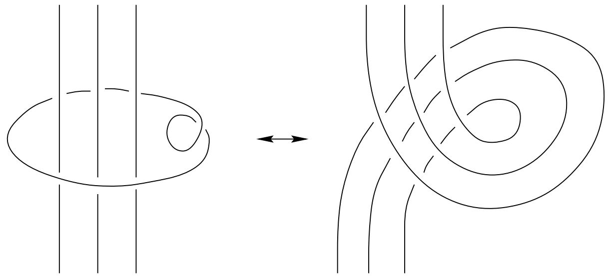


FIGURE 4.4. Kirby–Fenn–Rourke moves.

The proof of this theorem is quite difficult and will not be given here.

THEOREM 4.1.12 (Reshetikhin–Turaev [**RT2**]). *Let $\mathcal{C}$ be a MTC and $L$ be a framed link in $\mathbb{R}^3 \subset S^3$. Define the number $\tau(M_L)$ by the formula*

$$\tau(M_L) := D^{-|L|-1} F^{-1}(L) \left(\frac{p^+}{p^-}\right)^{\sigma(L)/2}, \tag{4.1.2}$$

*where $|L|$ is the number of components of $L$, $D$ is from* (3.1.15), *$p^{\pm}$ from* (3.1.7), *$\sigma(L)$ is the so-called wreath number of $L$* (*see* [**RT2**] *for its definition*), *and $F^{-1}(L)$ is the Reshetikhin–Turaev invariant of the link $L$ from Theorem* 2.3.11 (*we use the convention* (3.1.4): *we take sum over all possible labelings of uncolored strands, with labeling $V_i$ taken with weight $d_i$*).

*Then $\tau(M_L)$ is an invariant of the 3-manifold $M_L$, i.e., it does not depend on the link $L$:*

$$\tau(M_L) = \tau(M_{L'}) \quad \textit{if } M_L \simeq M_{L'}.$$

This invariant can be defined for an arbitrary 3-manifold $M$ and is called *Reshetikhin–Turaev invariant*.

PROOF. If we assume that $p^+/p^- = 1$, then the proof immediately follows from Lemma 3.1.5. The general case is not much more difficult; we refer the reader to the original papers for details. □

EXAMPLES 4.1.13. (i) $\tau(S^3) = D^{-1}$, cf. 4.1.10(i).
(ii) $\tau(S^2 \times S^1) = \tau(M_{\bigcirc}) = F^{-1}(\bigcirc)D^{-2} = 1$, cf. 4.1.10(ii) and (3.1.20).

REMARK 4.1.14. One easily sees that if $L_1, L_2$ are two links in $\mathbb{R}^3$ which are not linked with each other (i.e., they can be separated by a plane), then $M_{L_1 \sqcup L_2} = M_{L_1} \# M_{L_2}$, where we denote by $\#$ the connected sum. Thus, RT invariants satisfy the following multiplicativity property:

$$\tau(M_1 \# M_2) = D\, \tau(M_1)\tau(M_2). \tag{4.1.3}$$

Reshetikhin–Turaev invariants can be generalized to manifolds with ribbon links inside. Let us assume that we have a partially $\mathcal{C}$-colored ribbon link in $\mathbb{R}^3$ which is presented as a union $\Omega \sqcup L$; here $\Omega$ is a $\mathcal{C}$-colored ribbon link (which may contain coupons), and $L$ is an uncolored framed link without coupons. Performing surgery along $L$, we get the manifold $M_L$ with a ribbon link $\Omega_L$ inside. Then Theorem 4.1.11 can be generalized to this situation as follows.

THEOREM 4.1.15 (Reshetikhin–Turaev [**RT2**]). (i) *Any connected closed 3-manifold $M$ with a ribbon link $\Omega$ inside can be obtained as $(M_L, \Omega_L)$.*

(ii) $(M_L, \Omega_L) \simeq (M_{L'}, \Omega'_{L'})$ *iff the link $\Omega' \cup L'$ can be obtained from $\Omega \cup L$ by a sequence of Kirby–Fenn–Rourke moves, where the annulus in Figure* 4.4 *is a part of $L$.*

THEOREM 4.1.16 (Reshetikhin–Turaev [**RT2**]). *Let $\mathcal{C}$ be a MTC and let $\Omega \cup L$ be a partially colored framed link as above. Then*

$$\tau_{\mathcal{C}}(M_L, \Omega_L) := D^{-|L|-1} F^{-1}(L \cup \Omega) \left(\frac{p^+}{p^-}\right)^{\sigma(L)/2} \tag{4.1.4}$$

*is an invariant, i.e., depends only on $(M_L, \Omega_L)$.*

The proof is similar to the proof of the previous theorem. The explicit computation of Reshetikhin–Turaev invariants is not difficult for lens spaces (cf. 4.1.2(iii)) but in general is very complicated.

One may wonder is there a reason why MTCs give invariants of 3-manifolds. The reason is that MTCs give 3-dimensional Topological Quantum Field Theories.

## 4.2. Topological Quantum Field Theory

In this section, we introduce the second main hero of our lectures—topological quantum field theory (TQFT). (The first hero was the modular tensor category.) This notion was introduced in [**W1**], [**At**] and studied extensively in many papers, such as [**Q**]. As before, "manifold" = "compact topological oriented manifold with boundary". We also fix a base field $k$ of characteristic zero; all vector spaces considered in this chapter will be vector spaces over $k$.

DEFINITION 4.2.1. A $(d+1)$-*dimensional Topological Quantum Field Theory* ($d+1$ D TQFT) is the following collection of **data**:

(a) To any $d$-manifold $N$ without boundary assigned a finite dimensional vector space $\tau(N)$.

(b) To any $(d+1)$-manifold $M$ (possibly with boundary) assigned a vector $\tau(M)$ in the vector space $\tau(\partial M)$.

(c) To any homeomorphism of $d$-manifolds $f\colon N \xrightarrow{\sim} N'$ assigned an isomorphism of vector spaces $f_*\colon \tau(N) \xrightarrow{\sim} \tau(N')$.

(d) Functorial isomorphisms

$$\tau(\overline{N}) \xrightarrow{\sim} \tau(N)^*, \tag{4.2.1}$$

$$\tau(\emptyset) \xrightarrow{\sim} k, \tag{4.2.2}$$

$$\tau(N_1 \sqcup N_2) \xrightarrow{\sim} \tau(N_1) \otimes \tau(N_2), \tag{4.2.3}$$

where $\overline{N}$ is the manifold $N$ with the opposite orientation, which are compatible in an obvious sense with each other and with the commutativity, associativity and unit morphisms.

These data are required to satisfy the following **axioms**:

(i) *Functoriality.* If $f\colon M \xrightarrow{\sim} M'$ is a homeomorphism of $(d+1)$-manifolds then $(f|_{\partial M})_*(\tau(M)) = \tau(M')$.

(ii) *Gluing axiom.* Let $M$ be a $(d+1)$-manifold, $\partial M = N_1 \sqcup N_2 \sqcup N_3$, and $f\colon N_1 \xrightarrow{\sim} \overline{N}_2$ be a homeomorphism. Let $M' = M/f$ be the $(d+1)$-manifold obtained from $M$ by identifying $N_1$ with $\overline{N}_2$ using $f$, i.e., by gluing $N_1$ to $N_2$. Then $\tau(M')$ is equal to the image of $\tau(M)$ via the map $\tau(N_1) \otimes \tau(N_2) \otimes \tau(N_3) \to \tau(N_2)^* \otimes \tau(N_2) \otimes \tau(N_3) \to \tau(N_3)$.

(iii) *Normalization axiom.* Let $I$ be an interval and $N$ be a $d$-manifold. Then $\partial(I \times N) = N \sqcup \overline{N}$ and we require that $\tau(I \times N)$ equals the image of $\mathrm{id}_{\tau(N)}$ in $\tau(\overline{N}) \otimes \tau(N) \simeq \tau(N)^* \otimes \tau(N)$.

(iv) *Normalization axiom.* $\tau(S^d) = k$ and $\tau(B^{d+1}) = 1 \in k$, where $B^{d+1}$ is the unit ball in $\mathbb{R}^{d+1}$, and $S^d = \partial B^{d+1}$ is the $d$-sphere.

This completes the definition.

REMARK 4.2.2. For a more pedantic reader, we list here all the compatibility conditions mentioned in part (d) above. Functoriality of the morphisms (4.2.1)–(4.2.3) means that

$$(f \sqcup g)_* = f_* \otimes g_*, \quad \overline{f}_* = (f_*^{-1})^*,$$

where for $f\colon N_1 \xrightarrow{\sim} N_2$ we denote by $\overline{f}$ the same map considered as a homeomorphism $\overline{N_1} \xrightarrow{\sim} \overline{N_2}$, and for a map of vector spaces $\varphi\colon V_1 \to V_2$ we denote by $\varphi^*$ the adjoint map $\varphi^*\colon V_2^* \to V_1^*$.

The compatibility conditions are as follows: to the canonical homeomorphisms

$$N \sqcup \emptyset \xrightarrow{\sim} N,$$

$$N_1 \sqcup N_2 \xrightarrow{\sim} N_2 \sqcup N_1,$$

$$(N_1 \sqcup N_2) \sqcup N_3 \xrightarrow{\sim} N_1 \sqcup (N_2 \sqcup N_3)$$

are assigned the usual isomorphisms of vector spaces

$$\tau(N) \otimes k \xrightarrow{\sim} \tau(N),$$

$$\tau(N_1) \otimes \tau(N_2) \xrightarrow{\sim} \tau(N_2) \otimes \tau(N_1),$$

$$(\tau(N_1) \otimes \tau(N_2)) \otimes \tau(N_3) \xrightarrow{\sim} \tau(N_1) \otimes (\tau(N_2) \otimes \tau(N_3)).$$

Also, to the canonical homeomorphisms

$$\emptyset \xrightarrow{\sim} \overline{\emptyset}, \quad \overline{N_1 \sqcup N_2} \xrightarrow{\sim} \overline{N_1} \sqcup \overline{N_2}$$

are assigned the usual isomorphisms

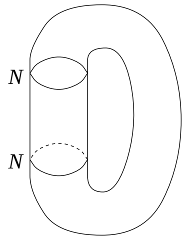
$$k^* \xrightarrow{\sim} k, \quad (\tau(N_1) \otimes \tau(N_2))^* \xrightarrow{\sim} \tau(N_1)^* \otimes \tau(N_2)^*.$$

This completes the list of compatibility conditions.

The axioms of TQFT show in particular that for every $(d+1)$-dimensional manifold $M$ without boundary, the number $\tau(M) \in \tau(\emptyset) = k$ is a topological invariant of $M$. However, not every invariant can be extended to a TQFT (see [**T**], [**Fu**]).

Note that, if $M$ is a $(d+1)$-manifold with $\partial M = \overline{N_1} \sqcup N_2$, then $\tau(M) \in \tau(N_1)^* \otimes \tau(N_2) \simeq \mathrm{Hom}_k(\tau(N_1), \tau(N_2))$. The gluing axiom says that gluing of two such manifolds $M$ gives rise to multiplying the corresponding operators.

Another example: $M = S^1 \times N$ is obtained by gluing the bases of the cylinder $I \times N$ and it is easy to see that $\tau(S^1 \times N) = \mathrm{tr}\, \tau(I \times N) = \dim \tau(N)$. Compare this with (2.3.12):




Next, we have the following important result.

THEOREM 4.2.3. *In any TQFT, we have*:

(i) $(fg)_* = f_* g_*$, $\mathrm{id}_* = \mathrm{id}$.

(ii) *For* $f\colon N_1 \xrightarrow{\sim} N_2$ *the isomorphism* $f_*\colon \tau(N_1) \xrightarrow{\sim} \tau(N_2)$ *depends only on the isotopy class of* $f$.

PROOF. For a homeomorphism $f\colon N_1 \xrightarrow{\sim} N_2$, let $M_f$ be a cylinder $N_1 \times I$ with boundary $\partial M_f$ identified with $\overline{N_1} \sqcup N_2$ using $f$. Then it follows from the normalization axiom that $\tau(M_f) = f_*\colon \tau(N_1) \to \tau(N_2)$. To prove (i), it suffices to notice that the cylinder $M_{fg}$ is homeomorphic to the cylinder obtained by gluing $M_f$ with $M_g$. To prove (ii), note that if $f$ is isotopic to identity then $M_f$ is homeomorphic to the trivial cylinder $M_{\mathrm{id}} = N_1 \times I$; thus, by functoriality, $\tau(M_f) = \mathrm{id}$. We leave the details to the reader. □

COROLLARY 4.2.4. *For every d-manifold* $N$, *a TQFT gives a representation of the group of isotopy classes of homeomorphisms* $f\colon N \xrightarrow{\sim} N$ *in the space* $\tau(N)$.

This group is called the *mapping class group.* We will return to this observation later.

## 4.3. 1+1 dimensional TQFT

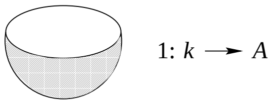
In this section, we consider a toy model of a TQFT: a (1+1) D TQFT. This case is rather trivial; however, the understanding of this example will be crucial for some of our future constructions.

THEOREM 4.3.1. *1+1 D TQFTs are in one-to-one correspondence with finite dimensional* Frobenius algebras*, i.e., commutative associative algebras $A$ with unit and with a linear map* $\mathrm{tr}\colon A \to k$ *such that the bilinear form* $\mathrm{tr}(ab)$ *is non-degenerate.*

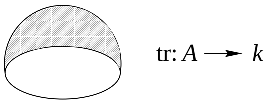
PROOF. 1. *Every 1+1 D TQFT gives a Frobenius algebra.*
There is only one 1-dimensional connected closed manifold: the circle $S^1$ and $\overline{S^1} = S^1$. Let $A$ be the vector space $\tau(S^1)$:

$$A = \tau(\bigcirc).$$

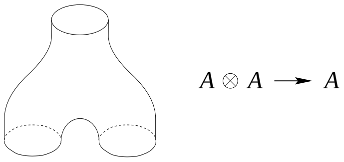
The disk $D^1$ has a boundary $\partial D^1 = S^1$, hence it gives a vector $\tau(D^1) \in A$ which we denote by 1. Since $\tau(\emptyset) := k$, we can consider $\tau(D^1)$ as a map from $k$ to $A$ giving the unit:



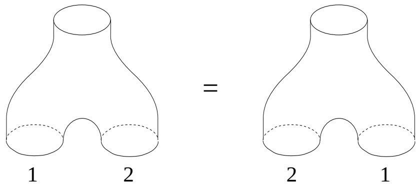

On the other hand, since $\overline{S^1} = S^1$, we can also write $\partial D^1 = \overline{S^1} \sqcup \emptyset$ which gives a map $\mathrm{tr}\colon A \to k$:



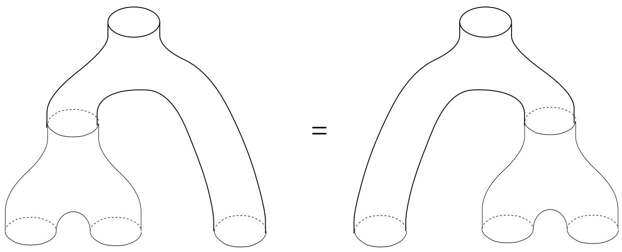

The pair of pants (also called a *trinion*) gives a map $A \otimes A \to A$ which is the multiplication $a \otimes b \mapsto ab$:



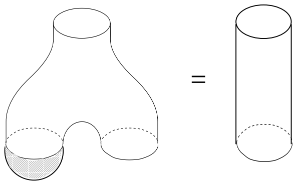

Now the commutativity of multiplication follows from the fact that the flipping of the legs is a homeomorphism:



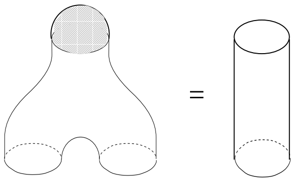

Similarly, associativity follows from the picture



and the gluing axiom. The unit property of 1 is a consequence of the gluing and normalization axioms:



Similarly, the non-degeneracy of the bilinear form $\mathrm{tr}(ab)$ follows from

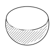


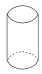
This shows that $A$ is a Frobenius algebra.

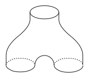
2. *Every Frobenius algebra gives a 1+1 D TQFT.*
Let $A$ be a Frobenius algebra. To the circle $S^1$ we assign $\tau(S^1) := A$ and to a disjoint union of circles a tensor product of $A$ with itself: $\tau(\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{n}) := A^{\otimes n}$.
For $f\colon S^1 \xrightarrow{\sim} S^1$ we let $f_* = \mathrm{id}$ and for $g\colon S^1 \xrightarrow{\sim} \overline{S^1}$ let $g_*$ be the isomorphism $A \xrightarrow{\sim} A^*$ given by the non-degenerate bilinear form $\mathrm{tr}(ab)$.

It is clear how to define $\tau(\text{disk}) \in A$, $\tau(\text{cylinder}) \in A^* \otimes A$ and $\tau(\text{trinion}) \in A^* \otimes A^* \otimes A$. The next lemma allows to extend this to any 2-manifold using the gluing axiom. Let us say that a *cutting* of a 2-dimensional manifold $\Sigma$ is a finite collection of simple non-intersecting curves on $\Sigma$, which are not allowed to intersect with the boundary. Equivalently, we will say that these curves cut the manifold into a union of "pieces", i.e., the connected components of the complement to the curves.

Lemma 4.3.2. *Every 2-manifold with a boundary can be cut into a union of:*

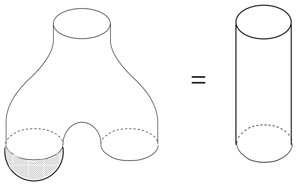
(a) *disks* 

(b) *cylinders* 

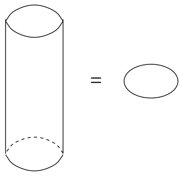
(c) *trinions* 

However, a 2-manifold $M$ can be cut in several different ways. To check that $\tau(M)$ is well-defined, we need the next result.

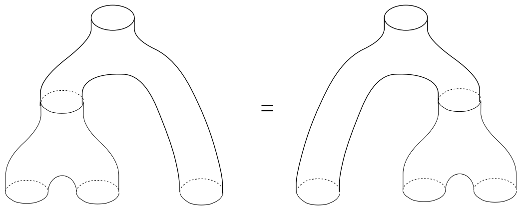
Lemma 4.3.3 ([**HT**]). *Any two ways to cut a 2-manifold $M$ into disks, cylinders and trinions can be related by isotopy of $M$ and a sequence of "simple moves":*

(i) 

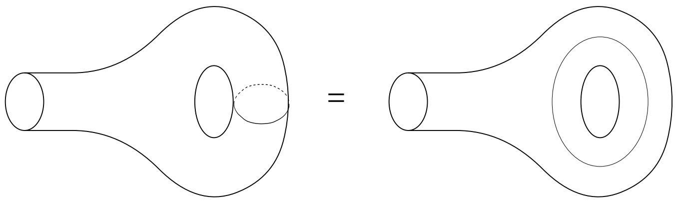
(ii)



(iii)



(iv)



It is easy to check that $\tau$ gives the same result on both sides of (i–iv), therefore it is well defined on any 2-manifold $M$. For example, both sides of (iv) correspond to the vector $\sum_i v_i v^i \in A$ where $\{v_i\}$ and $\{v^i\}$ are dual bases in $A$ with respect to the bilinear form $\mathrm{tr}(ab)$.

This completes the proof of the theorem. □

REMARK 4.3.4. Note that every Frobenius algebra is also a coalgebra: one can define comultiplication $\Delta\colon A \to A \otimes A$ as the adjoint of the multiplication with respect to the bilinear form $\mathrm{tr}(ab)$. However, the relation between comultiplication and multiplication in a Frobenius algebra is different than in a Hopf algebra: instead of $\Delta(ab) = \Delta(a)\Delta(b)$, one has

$$\Delta(ab) = (a \otimes 1)\Delta(b) = (1 \otimes b)\Delta(a).$$

EXAMPLE 4.3.5. Let $X$ be a finite set, $A = \mathcal{F}(X)$ the algebra of $k$-valued functions on $X$ (with respect to multiplication), and $\mathrm{tr}\colon A \to k$ given by $\mathrm{tr}(f) = \sum_{x\in X} f(x)$. This obviously is a Frobenius algebra. Moreover, one easily sees that in this case the comultiplication is given by $\Delta(\delta_x) = \delta_x \otimes \delta_x$ (where $\delta_x$ is the delta function at $x$: $\delta_x(y) = \delta_{x=y}$), and for a connected surface with $n$ boundary components, independent of genus, one has

$$\tau(\Sigma) = \Delta^{n-1}(1) = \sum_{x\in X} \delta_x \otimes \cdots \otimes \delta_x.$$

## 4.4. 3D TQFT from MTC

In this section we will show that every modular tensor category gives rise to a 3D TQFT, which generalizes the invariant of 3-manifolds without boundary constructed in Section 4.1. These ideas were first suggested by Witten [**W2**] (for the modular category arising from representations of $U_q(\mathfrak{sl}_2)$ at roots of unity). Our exposition is based on the construction in [**T**], with some modifications.

Before giving the precise definitions, let us note that we could have replaced in the definition of a TQFT topological manifolds by, say, smooth manifolds, or by manifolds with some additional structure. The only things we need are the notions of boundary, orientation reversing, gluing, and disjoint unions. One can formalize the requirements, but it is hardly necessary. Note that in dimensions 2 and 3 smooth and topological theories are equivalent: every topological manifold can be endowed with a smooth structure, and any two smooth structures are equivalent. Therefore, in this section we will not distinguish between smooth and topological structures.

In this section, we will construct an "extended" TQFT in the sense that we will consider manifolds with some additional structure. Let $\mathcal{C}$ be an abelian category.

DEFINITION 4.4.1. (i) A *$\mathcal{C}$-marked surface* is an oriented compact surface $\Sigma$ with a finite number of points $p_1, \dots, p_m$ on it, and the following data attached to every point $p_i$: a non-zero tangent vector $v_i$, an object $W_i \in \operatorname{Ob}\mathcal{C}$ and a sign $\varepsilon_i = \pm$.

We define a change of orientation of a $\mathcal{C}$-marked surface $\Sigma$ to be the surface $\overline{\Sigma}$ with the same data as $\Sigma$ but with $\varepsilon_i, v_i$ replaced by $-\varepsilon_i, -v_i$.

(ii) A *$\mathcal{C}$-marked 3-manifold* is a pair $(M, T)$, where $M$ is an oriented 3-manifold with a boundary and $T$ is a partially $\mathcal{C}$-colored ribbon tangle in $M$ such that the only uncolored components are annuli, and the colored components may end on the boundary of $M$. Orientation reversal is defined by $\overline{(M,T)} = (\overline{M}, \overline{T})$, where, as usual, $\overline{M}$ is $M$ with reversed orientation, and $\overline{T}$ is $T$ with reversed directions of all strands.

For a $\mathcal{C}$-marked 3-manifold $M$ we will denote by $\partial M$ its topological boundary provided with the following structure of a $\mathcal{C}$-marked surface (see Figure 4.5 below):

— the points $p_i$ are the ends of the ribbon tangle $T$,

— $W_i$ is the color of the corresponding strand of $T$,

— the sign $\varepsilon_i$ is $+$ if the tangle goes outward and $\varepsilon_i = -$ if the tangle goes inward,

— the tangent vector $v_i$ at the point $p_i$ is determined (up to a positive real factor) by the condition that it is tangent to the base of $T$, and the direction is chosen so that the reper $(n_i, v_i)$ has positive orientation (with respect ot the orientation in $\partial M$), where $n_i$ is the unit normal vector to the ribbon (on the "face side").

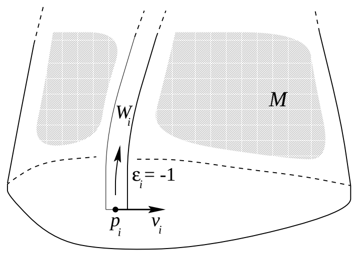



FIGURE 4.5. A $\mathcal{C}$-marked 3-manifold.

It is clear that $\overline{\partial M} = \partial \overline{M}$, and $\overline{\overline{M}} = M$. With this extended notions of 3-manifolds, boundaries, etc., one can also define the notion of 3D TQFT. We will call such TQFT's *"$\mathcal{C}$-extended"*, or for brevity, simply "extended" when there is no ambiguity.

REMARK 4.4.2. Instead of considering manifolds with ribbon tangles inside, we could have considered manifolds with tubular neighborhoods of these tangles removed, and a certain framing and coloring of these tubes. Such a manifold is not even a manifold with a boundary, but a manifold with corners. For this reason, extended TQFT's are sometimes called "TQFT's with corners".

THEOREM 4.4.3 (Turaev [**T**]). *To any MTC $\mathcal{C}$ such that $p^+/p^- = 1$ one can associate a $\mathcal{C}$-extended 3D TQFT which generalizes the Reshetikhin–Turaev invariants $\tau(M, \Omega)$ of closed 3-manifolds defined in Section* 4.1.

REMARK 4.4.4. It was shown in [**Fu**] that every complex-valued invariant of closed 3-manifolds satisfying the multiplicativity $\tau(M_1 \# M_2) = \tau(M_1)\tau(M_2)$ and reality $\tau(\overline{M}) = \overline{\tau(M)}$ properties can be obtained from some (non-extended) TQFT, which is essentially uniquely defined by this invariant. Since Reshetikhin–Turaev invariants, up to a constant, satisfy the multiplicativity property (see (4.1.3)), it is not surprising that they come from some TQFT. The problem is to construct this TQFT explicitly.

The remaining part of this section is devoted to the construction of the TQFT, and thus, the proof of Theorem 4.4.3. The main idea of the proof is to reduce everything to manifolds without boundary (but with some kind of a ribbon link inside) and then use the results of Section 4.1.

**Step 1. Parameterized manifolds.**

Let us start with constructing some supply of "standard" $\mathcal{C}$-marked surfaces. Let us call a *type* $t$ a finite sequence of the form

$$t = ((W_1, \varepsilon_1), (W_2, \varepsilon_2), H, (W_3, \varepsilon_3), H, \dots) \tag{4.4.1}$$

where $W_i$ are objects of $\mathcal{C}$, $\varepsilon_i = \pm$, and $H$ is some formal symbol. We will denote by $g = g(t)$ the number of occurrences of $H$.

For every type $t$ as above, we define some "standard" $\mathcal{C}$-marked surface $\Sigma_t$. First of all, let us define the "standard sphere" to be $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, which we consider as the boundary of the unit ball in $\mathbb{R}^3$. We will also use the "equator" of this sphere, which we define to be the circle given by equation $y = 0$ (see Figure 4.6). The clockwise direction (in the $xz$-plane) of this circle will be referred to as the positive direction. Equivalently, one can view the standard sphere as $S^2 = \mathbb{CP}^1$, with the equator being the completed real axis, and the positive direction given by the positive direction on the real axis. We will always identify these two realizations by stereographic projection, so that the the south pole $(0, 0, -1) \in \mathbb{R}^3$ is identified with $\infty \in \mathbb{CP}^1$.

Now, given a type $t$, we construct $\Sigma_t$ as follows. Let us take the standard sphere. Choose points $p_1 < \cdots < p_n$ on the equator ($n$ is the number of terms in $t$). For every term $(W_i, \varepsilon_i)$ of $t$, assign to the corresponding point the object $W_i$, the sign $\varepsilon_i$, and the vector $v_i$ which goes along the positive direction of the equator. For every occurrence of $H$, glue a handle to the corresponding $p_i$. This gives a $\mathcal{C}$-marked surface $\Sigma_t$ which is defined uniquely up to a unique homeomorphism.

More formally, $\Sigma_t$ can be defined as follows. Define for every $t$ the ribbon tangle $T_t$ which consists of:

(a) One (uncolored) coupon (placed in the bottom of $T_t$).

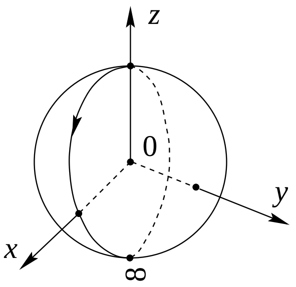



FIGURE 4.6. Standard sphere.

(b) A strand of color $W$ for each occurrence of $(W, \varepsilon)$, which connects the coupon with the top of the tangle. This strand is directed upward if $\varepsilon = +$ and downward if $\varepsilon = -$.

(c) An arc (uncolored and non-directed) for each occurrence of $H$.

Define the handlebody $M_t$ as a neighborhood of the coupon and arcs of $T_t$ in $\mathbb{R}^3$, and let $\Sigma_t = \partial M_t$. One easily sees that this defines $\Sigma_t$ uniquely up to a homeomorphism, and the homeomorphism is unique up to isotopy.

For example, for $t = ((W_1, +), H, (W_2, -))$, the tangle $T_t$ and the surface $\Sigma_t$ are shown in Figure 4.7 below (for technical resons, the tangent vectors $v_i$ are not shown in the figure).

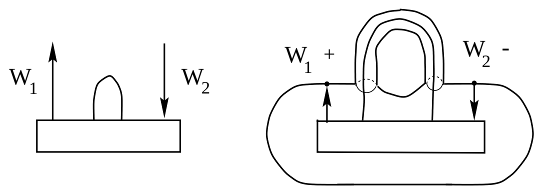



FIGURE 4.7. The tangle $T_t$ and the surface $\Sigma_t$.

Note that some of the surfaces $\Sigma_t$ are homeomorphic. What is more important is that every $\mathcal{C}$-marked surface is (not canonically) isomorphic to at least one of $\Sigma_t$.

For a type $t$, we define $\bar{t}$ to be obtained by reversing the order of terms in the sequence and replacing every $\varepsilon_i$ by $-\varepsilon_i$. For example, for $t = ((W_1, +), H, (W_2, -))$, we have $\bar{t} = ((W_2, +), H, (W_1, -))$. Then we have a canonical homeomorphism $rev_t \colon \overline{\Sigma_t} \xrightarrow{\sim} \Sigma_{\bar{t}}$ given by reflection in the vertical plane $x = 0$, and $rev_t \circ rev_{\bar{t}} = \mathrm{id}$.

In order to construct a TQFT, we will construct first an auxiliary TQFT with "parameterized manifolds". The 2-manifolds in this TQFT will be $\mathcal{C}$-marked surfaces $\Sigma$ together with a parameterization $\varphi$, i.e., an isomorphism

$$\varphi \colon \Sigma \xrightarrow{\sim} \Sigma_{t_1} \sqcup \cdots \sqcup \Sigma_{t_l}. \tag{4.4.2}$$

Homeomorphisms are homeomorphisms preserving these parameterizations (which essentially kills all non-trivial homeomorphisms: the only automorphisms of a parameterized surface are permutations of components of the same type). The parameterized 3-manifolds will be $\mathcal{C}$-marked 3-manifolds equipped with parameterizations of their boundaries.

We claim that if we construct a TQFT for the parameterized manifolds then we automatically get a TQFT for non-parameterized manifolds. Indeed, let us assume that we have constructed a "parameterized" TQFT; in particular, for every pair $(\Sigma, \varphi)$, where $\Sigma$ is a $\mathcal{C}$-marked surface and $\varphi$ is a homeomorphism $\Sigma \xrightarrow{\sim} \Sigma_t$, we have a vector space $\tau(\Sigma, \varphi)$. Let $f$ be a homeomorphism $f\colon \Sigma_t \xrightarrow{\sim} \Sigma_t$. Consider the cylinder $M = \Sigma \times [0,1]$ with the parameterization of its boundary chosen as follows:

$$\partial M = \overline{\Sigma} \sqcup \Sigma \xrightarrow{\varphi \sqcup (f \circ \varphi)} \overline{\Sigma_t} \sqcup \Sigma_t.$$

This gives us an operator

$$f_* = \tau(M)\colon \tau(\Sigma, \varphi) \xrightarrow{\sim} \tau(\Sigma, f \circ \varphi)$$

and it follows from the gluing axiom that $(fg)_* = f_* g_*$, $\mathrm{id}_* = \mathrm{id}$ (compare with the proof of Theorem 4.2.3). Taking tensor product, we can define $f_*$ for a homeomorphism of disjoint union of $\Sigma_t$'s.

This allows us to construct canonical isomorphisms $f_{\varphi,\psi}\colon \tau(\Sigma, \psi) \xrightarrow{\sim} \tau(\Sigma, \varphi)$, which satisfy the compatibility condition $f_{\varphi_1,\varphi_2} f_{\varphi_2,\varphi_3} = f_{\varphi_1,\varphi_3}$. In this case, we can identify all these spaces with each other, thus forming a space $\tau(\Sigma)$ which is canonically isomorphic to each of $\tau(\Sigma, \varphi)$ (compare with the construction in Definition 1.1.11).

Now let $M$ be a $\mathcal{C}$-marked 3-manifold. Choose a parameterization $\varphi$ of its boundary $\Sigma = \partial M$ (see (4.4.2)). Then $\tau(M, \varphi)$ is a vector in the vector space $\tau(\partial M, \varphi)$. Identifying $\tau(\Sigma) = \tau(\Sigma, \varphi)$, we get a vector $\tau(M) \in \tau(\Sigma)$; it is easy to see that this vector does not depend on the choice of $\varphi$.

It is straightforward to check all the axioms of a 3D TQFT. Thus, from every "parameterized" TQFT one can automatically construct a "non-parameterized" TQFT.

**Step 2. Reducing to closed manifolds.**

Now we are going to construct a TQFT based on parameterized manifolds. Let us start by defining the spaces $\tau(\Sigma_t) \equiv \tau(\Sigma_t, \mathrm{id})$. For $t$ given by (4.4.1), let

$$W_t = W_1^{\varepsilon_1} \otimes W_2^{\varepsilon_2} \otimes H \otimes W_3^{\varepsilon_3} \otimes H \otimes \cdots, \tag{4.4.3}$$

where $W^\varepsilon = W$ if $\varepsilon = +$ and $W^*$ if $\varepsilon = -$, and as before, $H = \bigoplus_{i \in I} V_i \otimes V_i^*$—see (2.4.9). Then we define

$$\tau(\Sigma_t) := \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_t) =: \langle W_t \rangle, \tag{4.4.4}$$

For a $\mathcal{C}$-marked surface $\Sigma$ along with a parameterization $\Sigma \xrightarrow{\sim} \Sigma_{t_1} \sqcup \cdots \sqcup \Sigma_{t_l}$, we let $\tau(\Sigma) = \tau(\Sigma_{t_1}) \otimes \cdots \otimes \tau(\Sigma_{t_l})$.

Next, let us construct an isomorphism $\tau(\Sigma_t)^* \xrightarrow{\sim} \tau(\Sigma_{\overline{t}})$. Let $\varphi\colon \mathbf{1} \to W_t$, $\psi\colon \mathbf{1} \to W_{\overline{t}}$. Define $(\varphi, \psi) \in k$ by

$$(\varphi, \psi) = D^{-g}(\mathbf{1} \xrightarrow{\varphi \otimes \psi} W_t \otimes W_{\overline{t}} \xrightarrow{e_t} \mathbf{1}) \tag{4.4.5}$$

where $e_t$ is the tensor product of the evaluation maps $W_i^{\varepsilon_i} \otimes W_i^{-\varepsilon_i} \to \mathbf{1}$ and renormalized evaluation maps $e_H(\eta \otimes \mathrm{id})\colon H \otimes H \to \mathbf{1}$. Here $e_H$ is the evaluation map induced by the identification $H \xrightarrow{\sim} H^*$ (see (2.4.9)), and $\eta|_{V_i \otimes V_i^*} = d_i^{-1}\,\mathrm{id}$. As before, we denote by $d_i$ the (quantum) dimension of $V_i$, and $D$ is given by (3.1.15).

One easily checks that the pairing (4.4.5) is nondegenerate and symmetric. Thus, we have identifications $\tau(\overline{\Sigma_t}) \xrightarrow{\sim} \tau(\Sigma_t)^*$. We extend them to disjoint unions of $\Sigma_t$ in an obvious way.

Now, let us construct for every parameterized $\mathcal{C}$-marked 3-manifold $M$ a vector $\tau(M) \in \tau(\partial M)$. The main idea is, of course, to reduce everything to invariants of closed manifolds. This can be done as follows.

Let us define a *special* link to be a ribbon link $X$ (which may contain coupons), with some of the strands and coupons colored by objects, respectively, morphisms from $\mathcal{C}$ and such that the following condition holds:

— any uncolored strand is either an annulus, or has both ends at the same uncolored coupon, in which case these ends are next to each other,

— all uncolored coupons are of the form $T_t$.

Examples of special links can be found in the next section.

Let $X$ be a special link, with uncolored coupons of types $t_1, \dots, t_k$. Define its Reshetikhin–Turaev invariant $F^{-1}(X) \in (\bigotimes_{i=1}^{k} \langle W_{t_i} \rangle)^*$ as follows. For any collection $\varphi_i \in \langle W_{t_i} \rangle$, let us denote by $T(\varphi_1, \dots, \varphi_k)$ the ribbon link obtained by coloring each uncolored coupon by the corresponding $\varphi$. Then we write

$$F^{-1}(X)(\varphi_1, \dots, \varphi_k) = \sum_c d_c F^{-1}(T(\varphi_1, \dots, \varphi_k)),$$

where the sum is taken over all colorings $c$ of uncolored strands $c\colon U \to I$, and $d_c = \prod_{u \in U} d_{c(u)}$ where $U$ is the set of all uncolored strands. This generalizes the conventions of Chapter 3.

The crucial observation is that every connected $\mathcal{C}$-marked parameterized 3-manifold $M$ can be obtained from $S^3$ by a surgery along some special link $X$. More precisely, let $X$ be a special link in $\mathbb{R}^3 \subset S^3$, with uncolored coupons of types $t_1, \dots, t_k$. Each such coupon determines an embedding of the standard handlebody $M_t$, defined in Step 1, into $\mathbb{R}^3$. Define

$$M_X = M_L \setminus \bigcup M_{t_i},$$

where $L$ is the ribbon link formed by all uncolored annuli of $X$. In other words, we first do surgery along all uncolored annuli and then remove from the obtained closed manifold neighborhoods of uncolored coupons. This gives a $\mathcal{C}$-marked manifold with boundary which is canonically identified with $\sqcup \overline{\Sigma}_{t_i}$.

We let $\tau(M_X) = F^{-1}(X) \in (\bigotimes \langle W_{t_i} \rangle)^* \simeq \bigotimes \langle W_{\overline{t_i}} \rangle$, where we use the pairing (4.4.5) to identify $\langle W_{\overline{t}} \rangle \simeq \langle W_t \rangle^*$. Theorem 4.1.15 implies that this is well-defined. This completes the definition of the TQFT based on parameterized $\mathcal{C}$-marked 3-manifolds. All the compatibility conditions are easy to prove and are left to the reader (or can be found in [**T**]); functoriality is also obvious. Therefore, we have to prove the gluing and the normalization axioms.

**Step 3. Proving the gluing axiom.**

Now, let us prove the gluing axiom, assuming that $p^+/p^- = 1$. Looking at the definitions, we see that it is equivalent to the following statement: if $X$ is a special link which contains two uncolored coupons $T, T'$ of types $t, \overline{t}$ respectively, and $X' = \sqcup_{T,T'}(X)$ is the link obtained from $X$ by "canceling" these two coupons as shown in Figure 4.8 below, then $M_X = M_{X'}$. This statement is of purely topological nature, and we omit its proof.

Finally, the proof of the normalization axiom will be given in the next section, along with other examples. This completes the proof of Theorem 4.4.3.

In the case where $p^+/p^- \neq 1$, the theorem above is incorrect as stated. For example, the gluing axiom holds only up to a multiplicative factor, and instead of the action of the mapping class group in $\tau(\Sigma)$, one would get a projective action.

FIGURE 4.8. "Canceling" of two coupons.

(In the physics language, this is referred to as "anomalies", so for $p^+/p^- \neq 1$, we get a TQFT with anomalies.) As in the case of a projective representation of a group, it is possible to get rid of these factors (anomalies) by a suitable "central extension" of the TQFT. We will discuss this generalization later (see Section 5.7).

## 4.5. Examples

In this section we give several examples of the calculation of the vector spaces and operators for the TQFT defined in the previous section. As before, we fix a MTC $\mathcal{C}$.

First of all, let us get some working experience with special links.

EXAMPLE 4.5.1. Let $T$ be a ribbon tangle, in which all the strands are $\mathcal{C}$-colored except for some annuli (such tangles were discussed in Chapter 3). Such a tangle defines two types $t_{\mathrm{top}}, t_{\mathrm{bot}}$ and a linear map $F^{-1}(T)\colon W_{t_{\mathrm{bot}}} \to W_{t_{\mathrm{top}}}$ (see Theorem 2.3.9).

Let us form a special link $X$ by adding to $T$ two uncolored coupons of type $t_{\mathrm{bot}}, \overline{t_{\mathrm{top}}}$. An example of such ribbon tangle $T$ and the corresponding link $X$ is shown in Figure 4.9.

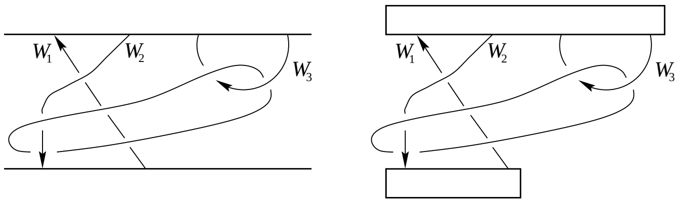



FIGURE 4.9. A ribbon tangle and the corresponding special link.

Then the definition of the previous section gives $F^{-1}(X) \in \langle W_{t_{\mathrm{bot}}}\rangle^* \otimes \langle W_{\overline{t_{\mathrm{top}}}}\rangle^* \simeq \mathrm{Hom}(\langle W_{t_{\mathrm{bot}}}\rangle, \langle W_{t_{\mathrm{top}}}\rangle)$. We claim that this operator is given by $\Phi \mapsto F^{-1}(T)\Phi$. Indeed, it suffices to prove that for $\Phi \in \langle W_{t_{\mathrm{bot}}}\rangle, \Psi \in \langle W_{\overline{t_{\mathrm{top}}}}\rangle$, we have

$$F^{-1}(X)(\Phi, \Psi) = (\mathbf{1} \xrightarrow{\Phi\otimes\Psi} W_{t_{\mathrm{bot}}} \otimes W_{\overline{t_{\mathrm{top}}}} \xrightarrow{F^{-1}(T)\otimes\mathrm{id}} W_{t_{\mathrm{top}}} \otimes W_{\overline{t_{\mathrm{top}}}} \xrightarrow{e_{t_{\mathrm{top}}}} \mathbf{1}).$$

This is immediate from the definition.

The same statement holds if we allow $T$ to be a partially colored ribbon tangle which is allowed to have uncolored strands ending at the top or bottom, as long as the two ends of any such strand are next to each other, and we define

$$F^{-1}(T) = \bigoplus_c d_c F^{-1}(T,c)\colon W_{t_{\mathrm{bot}}} \to W_{t_{\mathrm{top}}},$$

where $c\colon U \to I$ is a coloring (here $U$ is the set of all uncolored strands), and $d_c = \prod d_{c(u)}$ over all annuli and the strands which end at the top (but not the bottom!). In this case, the statement is a little bit less obvious, since one has to check the normalizations.

Thus, we see that as a special case, the operators $F^{-1}(X)$ for special links contain the operators $F^{-1}(T)$ for any partially colored tangle $T$.

Next, let us see how the vectors $\tau(M)$ look in the simplest examples.

EXAMPLE 4.5.2. Let $T$ be a $\mathcal{C}$-colored ribbon tangle such that *bottom*$(T) = \emptyset$, *top*$(T) = t$; thus, $F^{-1}(T)\colon \mathbf{1} \to W_t$ is a vector in $\langle W_t\rangle$. Let $M$ be the unit ball in $\mathbb{R}^3$ with the tangle $T$ placed inside so that the top of $T$ is on the equator of the standard sphere $S^2 = \partial M$. Then it immediately follows from the definition and the previous example that $\tau(M) = F^{-1}(T) \in \langle W_t\rangle$.

More generally, let $T$ be any $\mathcal{C}$-colored ribbon tangle, with *bottom*$(T) = t_1$, *top*$(T) = t_2$, and let $M$ be the domain

$$\{\mathbf{x} \in \mathbb{R}^3 \mid 1 \le \|\mathbf{x}\| \le 2\} \simeq S^2 \times I,$$

with the tangle $T$ placed inside so that the bottom of $T$ is placed on the equator of the inner sphere $\|\mathbf{x}\| = 1$, and the top of $T$ is placed on the equator of the outer sphere $\|\mathbf{x}\| = 2$. Then $F^{-1}(T)\colon \langle W_{t_1}\rangle \to \langle W_{t_2}\rangle$ and

$$\tau(M) \in \langle W_{t_1}\rangle^* \otimes \langle W_{t_2}\rangle \simeq \mathrm{Hom}(\langle W_{t_1}\rangle, \langle W_{t_2}\rangle)$$

is given by $\Phi \mapsto F^{-1}(T)\Phi$ for $\Phi\colon \mathbf{1} \to W_{t_1}$.

The next several examples deal with the case when $g(t) = 1$, so that $\Sigma_t$ is a torus. We will make heavy use of the results of Examples 4.1.2, 4.1.10. We will identify our "standard torus" $\Sigma_t$ with the torus considered in these examples so that the cycles $\alpha, \beta \in \mathrm{H}_1(\Sigma_t)$ shown below correspond to $\alpha, \beta$ shown in Figure 4.1.

EXAMPLE 4.5.3. Let $t = (H)$, so that the handlebody $M_t$ is the solid torus $T$, and $\partial T = \Sigma_t$ is a torus with no marked points. By (4.4.4), $\tau(\Sigma_t) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, H) = \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_i \otimes V_i^*)$. We claim that the vector $\tau(T) \in \tau(\partial T)$ is exactly the image of the identity morphism $\mathrm{id}\colon \mathbf{1} \to \mathbf{1} \otimes \mathbf{1}$ $(\in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_0 \otimes V_0^*))$.

PROOF. The proof is based on the fact that the standard torus can be obtained as a result of surgery of $S^3$ along the special link shown in Figure 4.10.

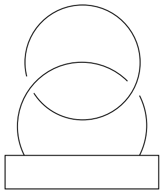
$X =$ 

FIGURE 4.10

Indeed, performing the surgery along the annulus, we get $S^2 \times S^1$ (see Example 4.1.10(ii)); after this, we cut a neighborhood of the coupon, which is isomorphic to a solid torus $T_0$. But by Example 4.1.2(i), the complement of a torus in $S^2 \times S^1$ is again a torus (of course, one also has to check that the attachment maps given by the link $X$ are the same as in Example 4.1.2(i)—we leave it to the reader).

By Example 4.5.1, the corresponding RT invariant $F^{-1}(X) \in \langle H \rangle$ coincides with $F^{-1}(L)$, where $L$ is obtained from the tangle $X$ by removing the coupon. By (3.1.19, 3.1.20), $F^{-1}(L)$ is equal to $\mathrm{id}\colon \mathbf{1} \to V_0 \otimes V_0^* \subset H$. $\square$

REMARK 4.5.4. In the same way one can prove that for the solid handlebody $T_g$ of genus $g$ one has

$$\tau(T_g) = (\mathrm{id}\colon \mathbf{1} \to (\mathbf{1} \otimes \mathbf{1})^{\otimes g}) \in \tau(\partial T_g) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, H^{\otimes g}). \tag{4.5.1}$$

EXAMPLE 4.5.5. Let $t = ((W, -), H)$, so that $\Sigma_t$ is the 2-dimensional torus with one marked point, and $\tau(\Sigma_t) = \langle W^*, H \rangle = \mathrm{Hom}_{\mathcal{C}}(W, H)$. Let $S\colon \Sigma_t \to \Sigma_t$ be the homeomorphism corresponding to the matrix $S \in \mathrm{SL}_2(\mathbb{Z})$ defined in Theorem 4.1.4. Then we claim that $S_*\colon \mathrm{Hom}_{\mathcal{C}}(W, H) \to \mathrm{Hom}_{\mathcal{C}}(W, H)$ coincides with the operator $S_W$ defined in Theorem 3.1.17.

PROOF. Let us consider the manifold $M = \Sigma_t \times I$, so that $\partial M = \Sigma_t \sqcup \overline{\Sigma_t}$, with the parameterization of the boundaries given by $\mathrm{id} \sqcup S$. We claim that $M$ can be obtained from $S^3$ by a surgery along the partially colored link $X$ shown in Figure 4.11.

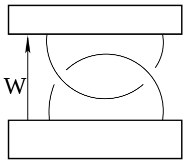
$X =$ 


FIGURE 4.11

Indeed, in this case we do not have to do any surgery at all but just to remove from $S^3$ the two linked solid tori—the neighborhoods of the two coupons. By Example 4.1.2(ii), this gives a cylinder $\Sigma_t \times I$. As before, we leave it to the reader to check that parameterization of the boundaries given by $X$ coincides with the one given in Example 4.1.2(ii). After this, the result follows form Example 4.5.1. $\square$

This explains why we defined the operator $S\colon H \to H$ by this picture—cf. (3.1.32).

EXERCISE 4.5.6. In a similar way, prove that $T_*\colon \mathrm{Hom}_{\mathcal{C}}(W, H) \to \mathrm{Hom}_{\mathcal{C}}(W, H)$ coincides with the operator $T_W$ defined in Theorem 3.1.17.

EXAMPLE 4.5.7. Let us check the normalization axiom. For simplicity, let us consider $t = ((W, -), H)$, so that $\Sigma_t$ is the torus with 1 marked point, $\tau(\Sigma_t) = \mathrm{Hom}_{\mathcal{C}}(W, H)$. Let $M$ be the cylinder $\Sigma_t \times I$, $I = [0, 1]$. Let us check that $\tau(M) = \mathrm{id} \in \mathrm{Hom}(\tau(\Sigma_t), \tau(\Sigma_t))$.

One can represent $M$ as $M_X$ with $X$ from Figure 4.12 below. Indeed, a surgery of $S^3$ along $\bigcirc$ gives $S^2 \times S^1$, as we already discussed in Example 4.1.10(ii). Then we cut two solid tori from $S^2 \times S^1$. But $S^2 \times S^1$ can be also obtained by gluing two solid tori along their boundaries, see Example 4.1.2(ii). Now removing two solid tori from $S^2 \times S^1$, we get $T^2 \times I$.

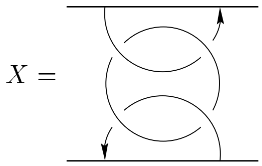



FIGURE 4.12

Note the similarity of the picture for $X$ and the one which defines the matrix $S^2$ in $\mathcal{C}$ (cf. the proof of Theorem 3.1.16).

EXAMPLE 4.5.8. Let

$$t = ((W_1,+),\dots,(W_n,+)),$$
$$t' = ((W_1,+),\dots,(W_{i+1},+),(W_i,+),\dots,(W_n,+)),$$

so that $\Sigma_t, \Sigma_{t'}$ are spheres with $n$ marked points, and

$$\tau(\Sigma_t) = \langle W_t \rangle = \mathrm{Hom}(\mathbf{1}, W_1 \otimes \cdots \otimes W_n),$$
$$\tau(\Sigma_{t'}) = \mathrm{Hom}(\mathbf{1}, W_1 \otimes \cdots \otimes W_{i+1} \otimes W_i \otimes \cdots \otimes W_n).$$

Let $b_i \colon \Sigma_t \xrightarrow{\sim} \Sigma_{t'}$ be the homeomorphism which exchanges $i$-th, $(i+1)$-st marked points as shown in Figure 4.13 below (for convenience, we are not showing the tangent vectors). Then we claim that $(b_i)_* \colon \tau(\Sigma_t) \to \tau(\Sigma_{t'})$ is given by $\Phi \mapsto \sigma_{W_i,W_{i+1}}\Phi$.

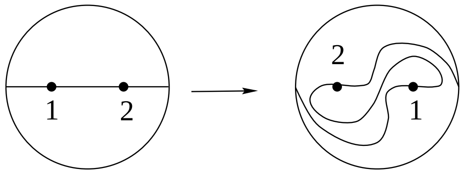



FIGURE 4.13. Braiding homeomorphism.

Indeed, let $M$ be the cylinder $\Sigma_t \times I$, with the parameterization of the boundary given by

$$\varphi \colon \partial M = \overline{\Sigma_t} \sqcup \Sigma_t \xrightarrow{\mathrm{id} \times b_i} \overline{\Sigma_t} \sqcup \Sigma_{t'}.$$

By definition, $(b_i)_*$ coincides with $\tau(M) \in \mathrm{Hom}(\langle W_t \rangle, \langle W_{t'} \rangle)$. To compute $\tau(M)$, note that $M$ is homeomorphic (as a parameterized manifold) to the cylinder $S^2 \times I$ with the trivial parameterization of the boundaries, and with the ribbon tangle shown in Figure 4.14 placed inside (cf. Example 4.5.2). Therefore, by Example 4.5.2, $\tau(M) = \sigma_{W_i,W_{i+1}}$. (This example was used without proof in the Preface.)

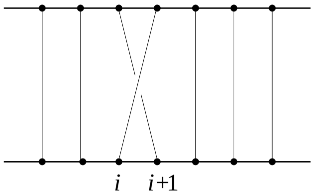



FIGURE 4.14