CHAPTER 3

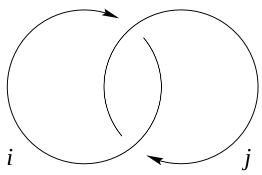
Modular Tensor Categories

In this chapter, we introduce one more refinement of the notion of a tensor category — that of a modular tensor category. By definition, this is a semisimple ribbon category with a finite number of simple objects satisfying a certain non-degeneracy condition. It turns out that these categories have a number of remarkable properties; in particular, we prove that in such a category one can define a projective action of the group $\mathrm{SL}_2(\mathbb{Z})$ on an appropriate object, and that one can express the tensor product multiplicities (fusion coefficients) via the entries of the S-matrix (this is known as Verlinde formula).

We also give two examples of modular tensor categories. The first one, the category $\mathcal{C}(\mathfrak{g}, \varkappa), \varkappa \in \mathbb{Z}_+$, is a suitable semisimple subquotient of the category of representation of the quantum group $U_q(\mathfrak{g})$ for q being root of unity: $q = e^{\pi \mathrm{i}/m\varkappa}$. The second one is the category of representations of a quantum double of a finite group G, or equivalently, the category of G-equivariant vector bundles on G. (We do not explain here what is the proper definition of Drinfeld's category $\mathcal{D}(\mathfrak{g}, \varkappa)$ for $\varkappa \in \mathbb{Z}_+$, which would be a modular category — this will be done in Chapter 7.)

3.1. Modular tensor categories

In this section we will study ribbon categories with some additional properties. Let $\mathcal{C}$ be a semisimple ribbon category. We will use the same notation as in Section 2.4. Define the numbers $\tilde{s}_{ij} \in k = \operatorname{End} \mathbf{1}$ $(i, j \in I)$ by the following picture:

(3.1.1) $\tilde{s}_{ij} =$ 

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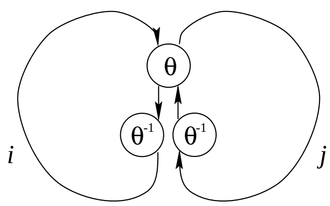
Here and below, we will often label strands of tangles by the indices $i \in I$ meaning by this V_i. Note that (2.3.17) implies

(3.1.2) $\tilde{s}_{ij} =$ 

$$= \theta_i^{-1}\theta_j^{-1} \operatorname{tr} \theta_{V_i^* \otimes V_j} = \theta_i^{-1}\theta_j^{-1} \sum_{k \in I} N_{i^* j}^k \theta_k d_k.$$

Also, it is easy to see that

$$\tilde{s}_{ij} = \tilde{s}_{ji} = \tilde{s}_{i^*j^*} = \tilde{s}_{j^*i^*}, \quad \tilde{s}_{i0} = d_i = \dim V_i. \tag{3.1.3}$$

DEFINITION 3.1.1. A *modular* (*tensor*) *category* (MTC for short) is a semisimple ribbon category $\mathcal{C}$ satisfying the following properties:

(i) $\mathcal{C}$ has only a finite number of isomorphism classes of simple objects: $|I| < \infty$.

(ii) The matrix $\tilde{s} = (\tilde{s}_{ij})_{i,j\in I}$, where $\tilde{s}_{ij}$ is defined by (3.1.1), is invertible.

REMARK 3.1.2. If $\mathcal{C}$ is symmetric, one can change overcrossing and undercrossing, hence $\tilde{s}_{ij} = d_i d_j$. Unless $|I| = 1$, this matrix $\tilde{s}$ is singular, therefore $\mathcal{C}$ is not modular.

REMARKS 3.1.3. (i) Many authors (for example, Turaev [**T**]) impose weaker conditions, not necessarily requiring semisimplicity in our sense. We are only interested in the simplest case; thus the above definition is absolutely sufficient for our purposes. We refer the reader to [**Ke**], [**Lyu2**] for a discussion of the non-semisimple case.

(ii) The name "modular" is justified by the fact that in this case we can define a projective action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on certain objects in our category, as we will show below. To the best of our knowledge, this construction first appeared (in rather vague terms) in a paper of Moore and Seiberg [**MS2**]; later it was formalized by Lyubashenko [**Lyu1**] and others. Our exposition follows the book of Turaev [**T**].

(iii) The appearance of the modular group in tensor categories may seem mysterious; however, there is a simple geometrical explanation, based on the fact that to each modular tensor category one can associate a 2+1-dimensional Topological Quantum Field Theory. This also shows that in fact we have an action of the mapping class group of any closed oriented 2-dimensional surface on the appropriate objects in MTC. This is the key idea of the book [**T**], and will be discussed in detail in Chapter 4.

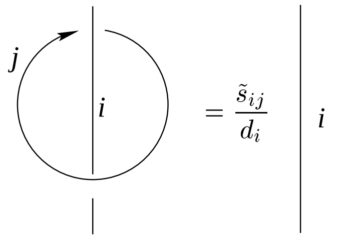
From now on, let us adopt the following convention:

(3.1.4) If some (closed) strand in a picture is left unlabeled then we assume summation over all labels $i \in I$ each taken with the weight $d_i = \dim V_i$.

Since $d_{i^*} = d_i$, we can drop the arrow of such a strand. Recall also that we omit the upward arrow when there is no ambiguity. Then we have the following propositions. (Their statements and proofs can be written explicitly in terms of σ, i, e, δ, etc., but we will prefer to use the pictorial presentation.)

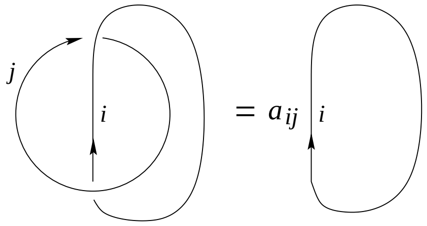
LEMMA 3.1.4. *In any semisimple ribbon category we have*

(3.1.5)



Recall that by Lemma 2.4.1, $d_i \neq 0$.

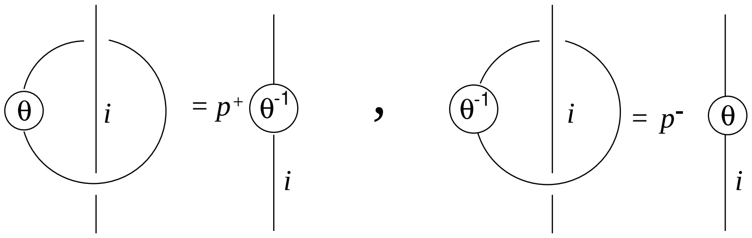
PROOF. The left hand side is an element of $\mathrm{End}(V_i) = k$, i.e., it is equal to $a_{ij}\,\mathrm{id}_{V_i}$ for some $a_{ij} \in k$. Taking a trace (i.e., closing the diagram), we obtain



The left hand side is equal to $\tilde{s}_{ij}$, while the right hand side to $a_{ij}d_i$. □

LEMMA 3.1.5. *We have the following identities*:

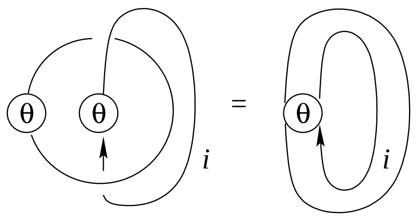
(3.1.6)



where

$$p^{\pm} := \sum_{i \in I} \theta_i^{\pm 1} d_i^2. \tag{3.1.7}$$

PROOF. We will consider only the case of plus sign, the case of minus sign is similar. Again the left hand side is an element of $\mathrm{End}(V_i) = k$, we take the trace of this element and multiply it with θ_i. Then, using (2.3.17), we get



Now decompose the tensor product $V_j \otimes V_i$ as in (2.4.1) to get

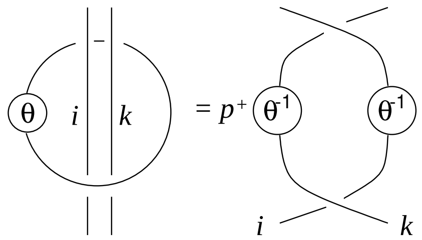
$$\theta_i \,\mathrm{tr}(\mathrm{lhs}) = \sum_j d_j \,\mathrm{tr}_{V_j \otimes V_i}\, \theta = \sum_{j,k} N_{ji}^k d_j d_k \theta_k.$$

Using (2.4.3) and (2.4.6), we obtain

$$\theta_i \,\mathrm{tr}(\mathrm{lhs}) = \sum_k \Big(\sum_j N_{ik^*}^{j^*} d_j\Big) d_k \theta_k = \sum_k d_i d_{k^*} d_k \theta_k = \Big(\sum_k \theta_k d_k^2\Big) d_i = p^+ d_i,$$

as desired. □

COROLLARY 3.1.6.



PROOF. Since any object is a direct sum of simple ones, (3.1.6) holds if we replace V_i by any object V. Apply this identity for $V = V_i \otimes V_k$ and use (2.3.17). □

THEOREM 3.1.7. *Define the matrices* $\tilde{s} = (\tilde{s}_{ij})$, $t = (t_{ij})$ *and* $c = (c_{ij})$ *("charge conjugation matrix") by* (3.1.1) *and*

$$t_{ij} = \delta_{ij}\theta_i, \tag{3.1.8}$$

$$c_{ij} = \delta_{ij^*}. \tag{3.1.9}$$

Then we have:

$$(\tilde{s}t)^3 = p^+\tilde{s}^2, \tag{3.1.10}$$

$$(\tilde{s}t^{-1})^3 = p^-\tilde{s}^2 c, \tag{3.1.11}$$

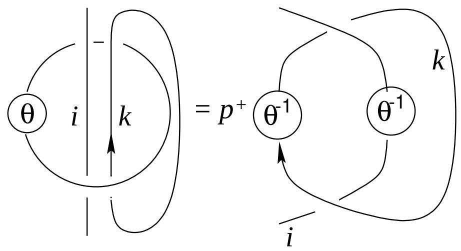
$$ct = tc, \quad c\tilde{s} = \tilde{s}c, \quad c^2 = 1, \tag{3.1.12}$$

where $p^{\pm}$ *are defined by* (3.1.7). *Moreover, when* $\tilde{s}$ *is invertible, we have*

$$\tilde{s}^2 = p^+p^-c. \tag{3.1.13}$$

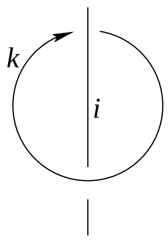
PROOF. The fact that c commutes with $\tilde{s}$ and t follows from (3.1.3) and (2.4.5); and $c^2 = 1$ because $i^{**} = i$. To prove the non-trivial relations (3.1.10, 3.1.11), consider first the identity

(3.1.14)

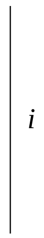


obtained from Corollary 3.1.6. The right hand side is equal to

$p^+\theta_i^{-1}\theta_k^{-1}$

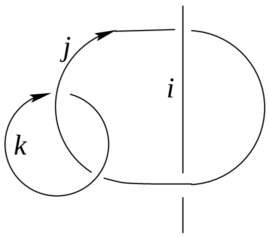


$= p^+\theta_i^{-1}\theta_k^{-1}\dfrac{\tilde{s}_{ik}}{d_i}$



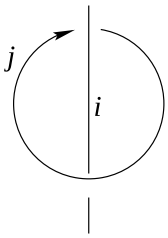
where we used Lemma 3.1.4. We can rewrite the left hand side of (3.1.14) as

$$\sum_j d_j\theta_j$$

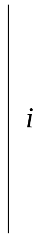


Applying Lemma 3.1.4 twice we obtain

$$\sum_j d_j\theta_j\frac{\tilde{s}_{jk}}{d_j}$$



$$= \sum_j \theta_j\tilde{s}_{jk}\frac{\tilde{s}_{ij}}{d_i}$$



This gives the identity

$$\sum_j \tilde{s}_{ij}\theta_j\tilde{s}_{jk} = p^+\theta_i^{-1}\tilde{s}_{ik}\theta_k^{-1}$$

which is equivalent to

$$\tilde{s}t\tilde{s} = p^+t^{-1}\tilde{s}t^{-1},$$

proving (3.1.10). Similarly, using the analogue of Corollary 3.1.6 with minus sign, one can prove

$$\tilde{s}t^{-1}\tilde{s} = p^-t\tilde{s}tc,$$

which implies (3.1.11).

When the matrix $\tilde{s}$ is non-singular, it is a matter of pure algebra to deduce Eq. (3.1.13) from (3.1.10)–(3.1.12). ☐

COROLLARY 3.1.8. *In an MTC, p^+ and p^- are non-zero.*

Now assume that the category $\mathcal{C}$ is modular, and introduce the notation

$$D := \sqrt{p^+p^-}, \quad \zeta := (p^+/p^-)^{1/6} \tag{3.1.15}$$

(assuming that they exist in k, otherwise we can always pass to a certain algebraic extension). Define the renormalized matrix

$$s := \tilde{s}/D. \tag{3.1.16}$$

Then we can rewrite the relations from Theorem 3.1.7 as follows:

$$(st)^3 = \sqrt{\frac{p^+}{p^-}}s^2 = \zeta^3s^2, \quad s^2 = c, \quad ct = tc, \quad c^2 = 1. \tag{3.1.17}$$

Recalling the well-known description of $\mathrm{SL}_2(\mathbb{Z})$ as the group generated by the elements

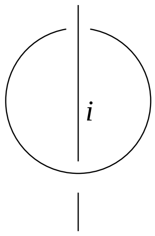
$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{3.1.18}$$

with relations $(st)^3 = s^2, s^4 = 1$, we see that the matrices s, t give a projective representation of $\mathrm{SL}_2(\mathbb{Z})$. (The fact that $s^2t = ts^2$ follows from $(st)^3 = s^2$.)

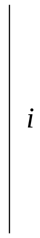
REMARK 3.1.9. Of course, one easily sees that we can replace the matrix t by t/ζ and get a true representation of $\mathrm{SL}_2(\mathbb{Z})$ rather than a projective one. In fact, since $\mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Q}) = 0$, every projective representation of $\mathrm{SL}_2(\mathbb{Z})$ over a field k of characteristic 0 can be trivialized in some algebraic extension of k. However, we prefer not to do it: later we will show that any MTC gives rise to projective representations of more general groups (mapping class groups), of which $\mathrm{SL}_2(\mathbb{Z})$ is the simplest example, and these representations can not be trivialized. Moreover, if we renormalize t now, it will make things only worse later.

COROLLARY 3.1.10. *In an MTC, we have*:

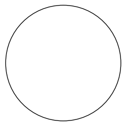
(3.1.19)



$= p^+p^-\delta_{i,0}$



(3.1.20) $p^+p^- = \sum d_i^2 =$



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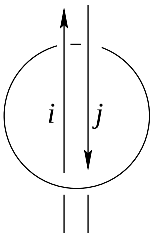
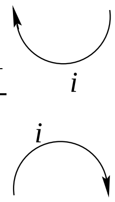
(3.1.21)



$= \delta_{ij}\dfrac{p^+p^-}{d_i}$



PROOF. Let us prove the first identity. As before, it suffices to prove that the traces of both sides are equal. By Lemma 3.1.4 the left hand side of (3.1.19) is equal to $\sum_j d_j\tilde{s}_{ij}/d_i \,\mathrm{id}_{V_i}$. Taking a trace, we obtain

$$\sum_j d_j\tilde{s}_{ij} = \sum_j \tilde{s}_{0j}\tilde{s}_{ij} = (\tilde{s})^2_{0i} = p^+p^-c_{0i} = p^+p^-\delta_{i,0}.$$

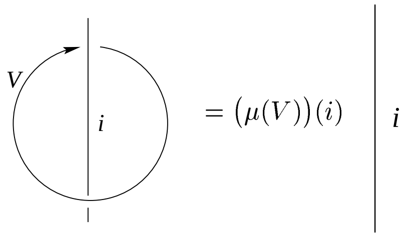
The second identity (3.1.20) easily follows from (3.1.19). The proof of (3.1.21) is similar to the above, using twice Lemma 3.1.4. □

We note that equation (3.1.20), along with the definition of s, give the following formulas for the number $D = \sqrt{p^+p^-}$:

$$D = \sqrt{\sum \dim^2 V_i} = s_{00}^{-1}. \tag{3.1.22}$$

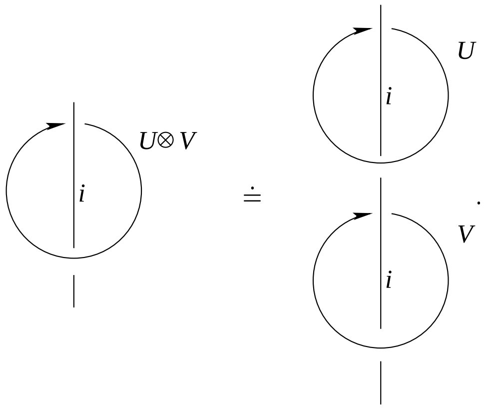
We can easily describe the Grothendieck ring of a modular tensor category. As before, let $\mathcal{C}$ be an MTC and let $K(\mathcal{C})$ be the Grothendieck ring of $\mathcal{C}$ (see Definition 2.1.9). Then the algebra $K = K(\mathcal{C}) \otimes_{\mathbb{Z}} k$ is a finite dimensional commutative associative algebra with a basis $x_i = \langle V_i \rangle$, $i \in I$, and a unit $1 = x_0$. This algebra is frequently called the *fusion algebra*, or *Verlinde algebra*.

THEOREM 3.1.11. *Let $\mathcal{C}$ be an MTC, $K = K(\mathcal{C}) \otimes_{\mathbb{Z}} k$, and let $F(I)$ be the algebra of k-valued functions on the set I. Define a map $\mu\colon K \to F(I)$ by the picture*:



Then μ is an algebra isomorphism.

PROOF. It is immediate from the results of Section 2.3 that μ is an algebra homomorphism. Indeed,



Choose a basis in $F(I)$ consisting of renormalized delta-functions: $\epsilon_i(j) = \delta_{ij}/s_{0i}$. Then it follows from Lemma 3.1.4 and the obvious identity $\tilde{s}_{ij}/d_i = s_{ij}/s_{0i}$ that the map μ is given by

$$\mu(x_j) = \sum_i s_{ij}\epsilon_i. \tag{3.1.23}$$

Since the matrix s_{ij} is invertible, this completes the proof. □

The importance of this result is that it gives a new basis $\mu^{-1}(\epsilon_i)$ in K in which the multiplication becomes diagonal. For brevity, let us write $\epsilon_i \in K$ instead of $\mu^{-1}(\epsilon_i)$. Then (3.1.23) and $\epsilon_i\epsilon_j = \delta_{ij}\epsilon_i/s_{0i}$ imply that

$$x_i\epsilon_j = \epsilon_j\, s_{ij}/s_{0j}. \tag{3.1.24}$$

Comparing this with the usual formula for the multiplication in the basis x_i:

$$x_i x_j = \sum_k N_{ij}^k x_k, \tag{3.1.25}$$

we get the following proposition.

PROPOSITION 3.1.12. *For a fixed i let N_i be the matrix of multiplication by x_i in the basis $\{x_j\}$, i.e., $(N_i)_{ab} = N_{ib}^a$, and let D_i be the following diagonal matrix: $(D_i)_{ab} := \delta_{ab} s_{ia}/s_{0a}$. Then*

$$sN_i s^{-1} = D_i. \tag{3.1.26}$$

This proposition is usually formulated by saying that "the s-matrix diagonalizes the fusion rules". Another reformulation is the following. Define in K another operation, $*$ (convolution), by the formula

$$x_i * x_j = \delta_{ij} x_i / s_{0i}. \tag{3.1.27}$$

Then:

$$s(xy) = s(x) * s(y), \tag{3.1.28}$$

$$s(x * y) = s(x)s(y). \tag{3.1.29}$$

Therefore, the matrix s can be considered as some kind of a Fourier transform.

Finally, Proposition 3.1.12 immediately implies the following famous formula for the coefficients N_{ij}^k, which was conjectured in [**Ve**] and proved in [**MS1**].

THEOREM 3.1.13 (Verlinde formula).

$$N_{ij}^k = \sum_r \frac{s_{ir} s_{jr} s_{k^* r}}{s_{0r}}. \tag{3.1.30}$$

Before giving the proof, let us note that as a consequence the right hand side of (3.1.30) is a non-negative integer, which is a non-trivial and unexpected fact.

PROOF. Rewrite formula (3.1.26) as $sN_i = D_i s$, or

$$\sum_a N_{ij}^a s_{ar} = \frac{s_{ir} s_{jr}}{s_{0r}}. \tag{3.1.31}$$

Multiplying this identity by s_{rk^*} and summing over r, we get (3.1.30). □

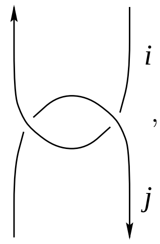
REMARK 3.1.14. If the base field $k = \mathbb{C}$, and the category $\mathcal{C}$ is Hermitian, that is, if it can be endowed with a complex conjugation functor $\bar{\ }$ satisfying certain compatibility conditions [**T**, Sect. II.5], then it can be shown that the matrices s, t are unitary (see [**Ki**]).

Let $\mathcal{C}$ be a modular tensor category. Recall the object $H = \bigoplus V_i \otimes V_i^* \in \mathcal{C}$ defined in (2.4.9). As was mentioned in Section 2.4, we have canonical isomorphisms $H \simeq H^*$ and $H \simeq \bigoplus V_i^* \otimes V_i$. It also follows from the definition that $\dim H = D^2 = \sum (\dim V_i)^2$.

DEFINITION 3.1.15. Define elements $S, T, C \in \operatorname{End} H$ as follows. Write

$$S = \bigoplus_{i,j \in I} S_{ij}, \quad S_{ij} \colon V_j \otimes V_j^* \to V_i \otimes V_i^*$$

and similarly $T = \bigoplus T_{ij}$, $C = \bigoplus C_{ij}$. Then:

(3.1.32) $$S_{ij} := \frac{d_i}{D}$$ 

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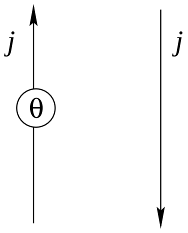
(3.1.33) $$T_{ij} := \delta_{ij}$$ 

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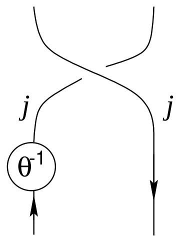
(3.1.34) $$C_{ij} := \delta_{ij^*}$$ 

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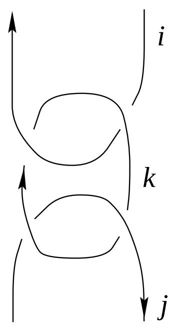
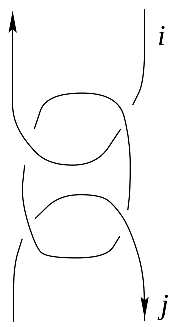
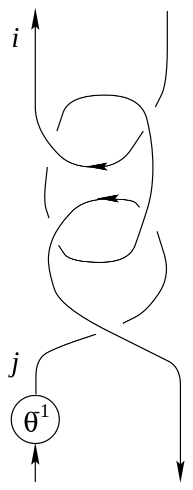
We have the following generalization of Theorem 3.1.7.

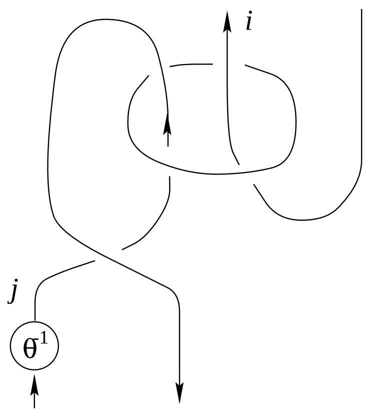
THEOREM 3.1.16. *$S^2 = C$, $C^2 = S^4 = \theta_H^{-1}$, $(ST)^3 = \sqrt{p^+/p^-}S^2$ and the element C is central in* $\operatorname{End} H$.

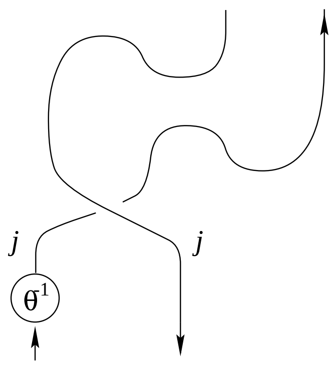
PROOF. Let us first check the identity $S^2 = C$. We have:

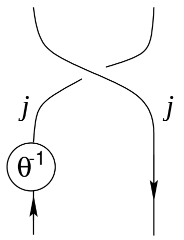
$$(S^2)_{ij} = \sum_k S_{ik}S_{kj} = \sum_k \frac{d_i}{D}\frac{d_k}{D}$$ 

$$= \frac{d_i}{D^2}$$ 

$= \frac{d_i}{D^2}$ 

$= \frac{d_i}{D^2}$ 

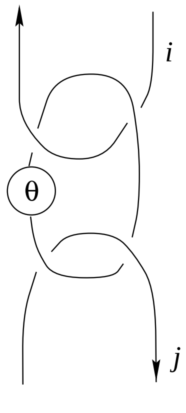
$= \delta_{ij^*}$ 

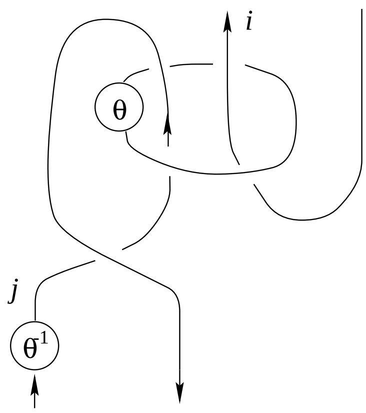
$= \delta_{ij^*}$ 

$= C_{ij}$

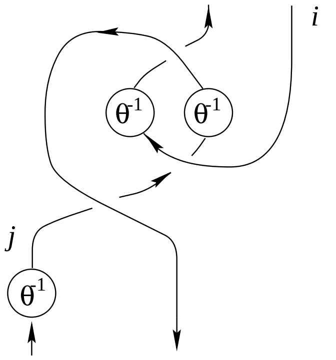
using (3.1.21) and $p^+p^- = D^2$, $d_i = d_{i^*}$.

Similarly, $(STS)_{ij} = \sum_{k,l} S_{ik}T_{kl}S_{lj} = \sum_k S_{ik}(\theta_k \otimes \mathrm{id})S_{kj}$ is equal to

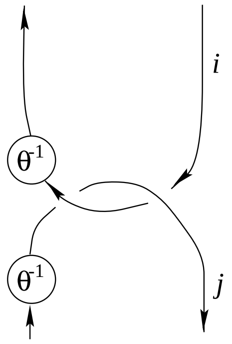
$\frac{d_i}{D^2}$ 

$= \frac{d_i}{D^2}$ 

$$= \frac{d_i}{D^2}p^+$$



$$= \sqrt{\frac{p^+}{p^-}}\frac{d_i}{D}$$



which equals $\sqrt{p^+/p^-}(T^{-1}ST^{-1})_{ij}$; now using Corollary 3.1.6 instead of Corollary 3.1.10. This proves that $(ST)^3 = \sqrt{p^+/p^-}S^2$.

Finally, using (2.3.17), it is easy to see that $(C^2)_{ij} = \delta_{ij}\theta^{-1}_{V_i\otimes V_i^*} = (\theta_H^{-1})_{ij}$. □

We cannot say that S, T give a projective representation of the modular group in H, since θ_H is not a constant. However, θ_H becomes a constant after restriction to an isotypic component of H. Equivalently, let us fix a simple object U in our category and consider the space

$$\mathrm{Hom}(U, H) = \bigoplus_{i\in I} \mathrm{Hom}(U, V_i \otimes V_i^*).$$

This is a vector space over k, and $\theta_H|_{\mathrm{Hom}(U,H)} = \theta_U\,\mathrm{id}_{\mathrm{Hom}(U,H)}$, $\theta_U \in k$.

THEOREM 3.1.17. *Define the maps* $S_U, T_U\colon \mathrm{Hom}(U,H) \to \mathrm{Hom}(U,H)$ *by*

$$\begin{aligned} S_U &\colon \Phi \mapsto S\Phi, \\ T_U &\colon \Phi \mapsto T\Phi. \end{aligned}$$

Then S_U, T_U *satisfy the following relations*

$$\begin{aligned} S_U^4 &= \theta_U^{-1}, \\ T_U S_U^2 &= S_U^2 T_U, \\ (S_U T_U)^3 &= \sqrt{\frac{p^+}{p^-}} S_U^2, \end{aligned}$$

and thus give a projective representation of the group $\mathrm{SL}_2(\mathbb{Z})$ *in* $\mathrm{Hom}(U, H)$.

EXAMPLE 3.1.18. Let $U = \mathbf{1}$ be the unit object in $\mathcal{C}$. Then we have a canonical identification $\mathrm{Hom}(\mathbf{1}, V_i \otimes V_i^*) \simeq k$, and thus we have a canonical basis $\{\chi_i\}$ of $\mathrm{Hom}(\mathbf{1}, H)$. In this case, the action of the modular group defined in Theorem 3.1.17 in the basis $\{\chi_i\}$ is given by s, t defined by (3.1.16) and (3.1.8).

The next theorem was proved by Vafa in the context of Conformal Field Theory.

THEOREM 3.1.19 (Vafa [**V2**]). *In any modular tensor category the numbers* θ_i *and* $\zeta = (p^+/p^-)^{1/6}$ *are roots of unity (regardless of the base field* k*).*

PROOF. We will use the following observation: if

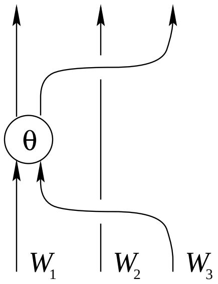
$$\prod_{j\in I} \theta_j^{M_{ij}} = 1, \qquad i \in I,$$

with a non-singular integer matrix M_{ij}, then all θ_j are roots of unity. Indeed, we can diagonalize the matrix M_{ij} by rows and columns operations.

For fixed objects W_1, W_2, W_3 in $\mathcal{C}$, define the following endomorphisms of $W_1 \otimes W_2 \otimes W_3$:

$$\theta_1 := \theta_{W_1} \otimes \mathrm{id} \otimes \mathrm{id}, \quad \theta_2 := \mathrm{id} \otimes \theta_{W_2} \otimes \mathrm{id}, \quad \theta_3 := \mathrm{id} \otimes \mathrm{id} \otimes \theta_{W_3},$$

$$\theta_{12} := \theta_{W_1 \otimes W_2} \otimes \mathrm{id}, \quad \theta_{23} := \mathrm{id} \otimes \theta_{W_2 \otimes W_3}, \quad \theta_{13} :=$$



$$\theta_{123} := \theta_{W_1 \otimes W_2 \otimes W_3}.$$

Then it is easy to check that

$$\theta_{12}\theta_{13}\theta_{23} = \theta_{123}\theta_1\theta_2\theta_3 \tag{3.1.35}$$

(this identity is sometimes called the *lantern identity*). Consider this identity for $W_1 = V_i$, $W_2 = V_i^*$, $W_3 = V_i$. It gives rise to an identity of operators in the vector space

$$U_i = \mathrm{Hom}(V_i, V_i \otimes V_i^* \otimes V_i)$$

which is non-zero since it contains $i_{V_i} \otimes \mathrm{id}_{V_i}$. We take determinant of both sides of this identity.

To compute $\det \theta_{12}|_{U_i}$, we use the decompositions of $V_i \otimes V_i^*$ and $V_j \otimes V_i$ as direct sums of simple objects:

$$V_i \otimes V_i^* = \sum_j N_{ii^*}^j V_j, \quad V_j \otimes V_i = \sum_k N_{ji}^k V_k,$$

and (2.4.4, 1.1.2). We obtain

$$\det \theta_{12}|_{U_i} = \prod_j \theta_j^{N_{ii^*}^j N_{ji}^i}.$$

Similarly, we compute the determinants of other θ's and get the identity

$$\prod_j \theta_j^{A_{ij}} = \theta_i^{4 \dim U_i},$$

where $A_{ij} = 2N_{ii^*}^j N_{ij}^i + N_{ii}^j N_{ji^*}^i$. Using that $\dim U_i = (1/3)\sum_j A_{ij} > 0$, it is easy to see that the matrix $A_{ij} - 4\delta_{ij} \dim U_i$ is nonsingular. It follows that all θ_i are roots of unity.

Since $\det t = \prod_i \theta_i$, $\det t$ is a root of unity. On the other hand, $s^4 = 1$ implies that $\det s$ is a 4th root of unity. Therefore, it follows from $(st)^3 = \zeta^3 s^2$ that ζ is a root of unity. □

REMARK 3.1.20. In MTCs coming from Conformal Field Theory (CFT), when the base field is $\mathbb{C}$, one usually writes

$$\theta_i = e^{2\pi \mathrm{i} \Delta_i}, \quad \zeta = e^{2\pi \mathrm{i} c/24}. \tag{3.1.36}$$

The numbers Δ_i are called the *conformal dimensions* and c is called the (*Virasoro*) *central charge* of the theory. In this language Vafa's theorem asserts that the conformal dimensions and the central charge of the theory are rational numbers; this is one of the reasons why such CFTs are called *rational*.

One can also easily prove the following result.

THEOREM 3.1.21. *All the numbers $s_{ij}/s_{0j} = \tilde{s}_{ij}/d_j$ are algebraic integers.*

PROOF. By Verlinde formula (3.1.26), these numbers are the eigenvalues of the matrix N_i with integer entries. □

3.2. Example: Quantum double of a finite group

We will give the simplest example of a modular tensor category—the category of finite dimensional representations of the Hopf algebra $D(G)$, which is the quantum double of the group algebra $k[G]$ of a finite group G. It is interesting that this example appeared in two seemingly unrelated areas—the theory of characters of reductive groups over finite fields [**L5, L6**] and the orbifold constructions in Conformal Field Theory [**DVVV, KT**].

Let us first fix the notation. Let G be a finite group. Recall that its *group algebra* $k[G]$ over a field k is a Hopf algebra with a k-basis $\{x\}_{x\in G}$ and

multiplication	$x \otimes y \mapsto xy, \qquad x, y \in G,$
unit	$e \qquad$ (the unit element of G),
comultiplication	$\Delta(x) = x \otimes x, \qquad x \in G,$
counit	$\varepsilon(x) = 1,$
antipode	$\gamma(x) = x^{-1}.$

This Hopf algebra is cocommutative. A representation of $k[G]$ is the same as a representation of G. By Maschke's theorem, the category $\mathcal{R}ep_f\, k[G]$ of finite dimensional representations is semisimple.

The Hopf algebra dual to $k[G]$ is isomorphic to the function algebra $F(G)$ of the group G. It has a k-basis $\{\delta_g\}_{g\in G}$ consisting of delta functions:

$$\delta_g(x) = \delta_{g,x} = \begin{cases} 1 & \text{for } g = x, \\ 0 & \text{for } g \neq x. \end{cases}$$

It has

multiplication	$\delta_g \delta_h = \delta_{g,h}\delta_g, \qquad g, h \in G,$
unit	$1 = \sum_{g\in G} \delta_g,$
comultiplication	$\Delta(\delta_g) = \sum_{g_1 g_2 = g} \delta_{g_1} \otimes \delta_{g_2}, \qquad g \in G,$
counit	$\varepsilon(\delta_g) = \delta_{g,e},$
antipode	$\gamma(\delta_g) = \delta_{g^{-1}}.$

A representation of $F(G)$ is the same as a G-graded vector space (since $\{\delta_g\}_{g\in G}$ are projectors).

Applying Drinfeld's quantum double construction [**Dr3**] it is easy to describe explicitly the quantum double $D(G)$ of $k[G]$. As a vector space, $D(G) = F(G) \otimes_k$

$k[G]$. It is a Hopf algebra with

multiplication	$(\delta_g \otimes x)(\delta_h \otimes y) = \delta_{gx,xh}(\delta_g \otimes xy),$	$x, y, g, h \in G,$
unit	$1 = \sum_{g\in G} \delta_g \otimes e,$	
comultiplication	$\Delta(\delta_g \otimes x) = \sum_{g_1 g_2 = g} (\delta_{g_1} \otimes x) \otimes (\delta_{g_2} \otimes x),$	$g, x \in G,$
counit	$\varepsilon(\delta_g \otimes x) = \delta_{g,e},$	
antipode	$\gamma(\delta_g \otimes x) = \delta_{x^{-1}g^{-1}x} \otimes x^{-1}.$	

The Hopf algebra $D(G)$ is quasitriangular with

R-matrix $\quad R = \sum_{g\in G} (\delta_g \otimes e) \otimes (1 \otimes g).$

(Of course, once we know the above formulas, they can be easily checked directly.)

Note that $F(G)$ and $k[G]$ embed in $D(G)$ as k-algebras and $D(G)$ is their semidirect product:

$$D(G) = F(G) \rtimes k[G], \tag{3.2.1}$$

with

$$x\delta_g x^{-1} = \delta_{xgx^{-1}} \quad \text{for} \;\; g, x \in G. \tag{3.2.2}$$

Let $\mathcal{R}ep_f D(G)$ be the category of finite dimensional representations of $D(G)$ as a k-algebra. By the above remarks, a representation V of $D(G)$ is the same as a G-module with a G-grading $V = \bigoplus_{g\in G} V_g$ satisfying $xV_g \subset V_{xgx^{-1}}$, $x, g \in G$. In other words, objects of $\mathcal{R}ep_f D(G)$ are finite dimensional G-equivariant vector bundles over G. We will show that the category $\mathcal{R}ep_f D(G)$ is semisimple and will describe its simple objects.

For $V \in \operatorname{Ob} \mathcal{R}ep_f D(G)$ and $v \in V$ the submodule generated by v is

$$D(G)v = \sum_{g\in G} k[G]\delta_g v = \sum_{g\in G} \bigoplus_{xgx^{-1}\in\overline{g}} xZ(g)\delta_g v,$$

where $\overline{g}$ denotes the conjugasy class and $Z(g)$ the centralizer of g in G. Note that $k[Z(g)]\delta_g v$ is an irreducible representation π of $Z(g)$. Hence

$$V_{\overline{g},\pi} := k[G]\delta_g v = \bigoplus_{xgx^{-1}\in\overline{g}} x\pi, \tag{3.2.3}$$

is an irreducible $D(G)$-module which depends only on the conjugacy class $\overline{g}$ and the isomorphism class of the irreducible representation π of $Z(g)$. The action of $D(G)$ on $V_{\overline{g},\pi}$ is given explicitly by:

$$(\delta_f \otimes h)(xv) = \delta_{f,hxgh^{-1}x^{-1}}\, hxv \quad \text{for} \;\; f, h, x \in G, \; v \in \pi. \tag{3.2.4}$$

This shows that the category $\mathcal{R}ep_f D(G)$ is semisimple with simple objects $V_{\overline{g},\pi}$ labeled by pairs $(\overline{g}, \pi)$, where $\overline{g} \in \overline{G}$ is a conjugacy class in G and $\pi \in \widehat{Z(g)}$ is an isomorphism class of irreducible representation of the centralizer $Z(g)$ of some element $g \in \overline{g}$ (π is independent of the choice of g).

In what follows we will use the orthogonality relations of irreducible characters of a finite group G:

$$\frac{1}{|G|}\sum_{h\in G}\operatorname{tr}_{\pi^*}(h)\operatorname{tr}_{\pi'}(hg)=\frac{\operatorname{tr}_\pi(g)}{\operatorname{tr}_\pi(e)}\delta_{\pi,\pi'},\qquad \pi,\pi'\in\widehat{G},\ g\in G, \tag{3.2.5}$$

$$\frac{1}{|Z(g)|}\sum_{\pi\in\widehat{G}}\operatorname{tr}_{\pi^*}(g)\operatorname{tr}_\pi(h)=\delta_{\overline{g},\overline{h}},\qquad h,g\in G. \tag{3.2.6}$$

Also recall that $|\overline{g}||Z(g)|=|G|$.

THEOREM 3.2.1. *$\mathcal{R}ep_f D(G)$ is a modular tensor category with simple objects $V_{\overline{g},\pi}$ labeled by $(\overline{g},\pi)$, $\overline{g}\in\overline{G}$, $\pi\in\widehat{Z(g)}$ $(g\in\overline{g})$. We have:*

$$V^*_{\overline{g},\pi}\simeq V_{\overline{g^{-1}},\pi^*}, \tag{3.2.7}$$

$$t_{(\overline{g},\pi),(\overline{g'},\pi')}=\delta_{(\overline{g},\pi),(\overline{g'},\pi')}\frac{\operatorname{tr}_\pi(g)}{\operatorname{tr}_\pi(e)}, \tag{3.2.8}$$

$$s_{(\overline{g},\pi),(\overline{g'},\pi')}=\frac{1}{|Z(g)||Z(g')|}\sum_{\substack{h\in G\\ hg'h^{-1}\in Z(g)}}\operatorname{tr}_\pi(hg'^{-1}h^{-1})\operatorname{tr}_{\pi'}(h^{-1}g^{-1}h). \tag{3.2.9}$$

The numbers $p^{\pm}$ from (3.1.7) *are equal to the order of G.*

The s-matrix (3.2.9) was first introduced by Lusztig [**L5**] (see also [**L6, L7**]) under the names "non-abelian Fourier transform" and "exotic Fourier transform". Then it appeared in [**DVVV**] and [**KT**] in connection with "orbifolds". Dijkgraaf, Pasquier and Roche [**DPR**] considered a generalization of the above construction which is also related to orbifolds. They introduced a quasi-Hopf algebra $D^c(G)$, depending on a cohomology class $c\in \mathrm{H}^3(G,\mathrm{U}(1))$, which reduces to $D(G)$ when $c=1$.

PROOF OF THEOREM 3.2.1. Eq. (3.2.7) follows easily from the definitions (note that $Z(g^{-1})=Z(g)$ and $\operatorname{tr}_{\pi^*}(h)=\operatorname{tr}_\pi(h^{-1})$).

To prove (3.2.8), we compute the twists θ using the results of Proposition 2.2.4 and Lemma 2.2.5. Since $\gamma^2=\mathrm{id}$, it follows that $\delta_V=\mathrm{id}$, cf. (2.2.11). Hence,

$$\theta=u^{-1}=\sum_{h\in G}\delta_h\otimes h. \tag{3.2.10}$$

As g is central in $Z(g)$, it acts as a constant $=\operatorname{tr}_\pi(g)/\operatorname{tr}_\pi(e)$ on the representation π; hence by (3.2.4), $\theta_{\overline{g},\pi}=\operatorname{tr}_\pi(g)/\operatorname{tr}_\pi(e)$.

To prove (3.2.9), we will use (3.1.2). We compute for $x,x'\in G$, $v\in\pi^*$, $v'\in\pi'$:

$$\begin{aligned}\theta_{V^*_{\overline{g},\pi}\otimes V_{\overline{g'},\pi'}}(xv\otimes x'v')&=\Delta(u^{-1})(xv\otimes x'v')\\ &=\sum_{\substack{h\in G\\ h_1h_2=h}}(\delta_{h_1}\otimes h)(xv)\otimes(\delta_{h_2}\otimes h)(x'v')\\ &=\sum_{\substack{h\in G\\ h_1h_2=h}}\delta_{h_1,hxg^{-1}x^{-1}h^{-1}}hxv\otimes\delta_{h_2,hx'g'x'^{-1}h^{-1}}hx'v'\\ &=(fxv\otimes fx'v'),\qquad \text{where } f=xg^{-1}x^{-1}x'g'x'^{-1}.\end{aligned}$$

Hence,

$$\operatorname{tr}\theta_{V^*_{\overline{g},\pi}\otimes V_{\overline{g'},\pi'}} = \sum_{\substack{xg^{-1}x^{-1}\in\overline{g^{-1}}\\ x'g'x'^{-1}\in\overline{g'}\\ x^{-1}x'g'x'^{-1}x\in Z(g^{-1})}} \operatorname{tr}_{\pi^*}(g^{-1}x^{-1}x'g'x'^{-1}x)\operatorname{tr}_{\pi'}(x'^{-1}xg^{-1}x^{-1}x'g')$$

$$= \frac{\operatorname{tr}_{\pi^*}(g^{-1})}{\operatorname{tr}_{\pi^*}(e)}\frac{\operatorname{tr}_{\pi'}(g')}{\operatorname{tr}_{\pi'}(e)}\frac{1}{|Z(g)||Z(g')|}\sum_{\substack{h\in G\\ hg'h^{-1}\in Z(g)}} \operatorname{tr}_{\pi^*}(hg'h^{-1})\operatorname{tr}_{\pi'}(h^{-1}g^{-1}h),$$

which proves (3.2.9).

The computation of $p^{\pm}$ is straightforward (using (3.2.5, 3.2.6)), and is left to the reader. □

3.3. Quantum groups at roots of unity

We will show that the category of representations of a quantum group at root of unity is a modular tensor category.

We will use the notation and definitions from Section 1.3. Recall that the quantum group $U_q(\mathfrak{g})$ was defined over the field $\mathbb{C}_q$ where q is a formal variable (Definition 1.3.1). We also defined a version of the quantum group ("the quantum group with divided powers") which makes sense for $q \in \mathbb{C}$ (see (1.3.18)).

In this section we will consider the case $q = e^{\pi i/m\varkappa}$ ($\varkappa \in \mathbb{Z}_+$ and m is from (1.3.17)), and we will abbreviate $U_q(\mathfrak{g})|_{q=e^{\pi i/m\varkappa}}$ to $U_q(\mathfrak{g})$. As usual, we let $q^a = e^{a\pi i/m\varkappa}$ for any $a \in \mathbb{Q}$. Let $\mathcal{C}(\mathfrak{g},\varkappa)$ be the category of finite dimensional representations of $U_q(\mathfrak{g})$ over $\mathbb{C}$ with weight decomposition:

$$V = \bigoplus_{\lambda\in P} V^\lambda, \qquad q^h|_{V^\lambda} = q^{(h,\lambda)}\operatorname{id}_{V^\lambda},$$
$$e_i^{(n)}(V^\lambda) \subset V^{\lambda+n\alpha_i}, \quad f_i^{(n)}(V^\lambda) \subset V^{\lambda-n\alpha_i}.$$

Note that our definition of weight decomposition is stronger than just requiring that all q^h be diagonalizable: the action of q^h does not allow one to distinguish between V^λ and $V^{\lambda+2m\varkappa\mu}$, $\mu \in P$.

THEOREM 3.3.1. *$\mathcal{C}(\mathfrak{g},\varkappa)$ is a ribbon category over $\mathbb{C}$.*

PROOF. The associatity, unit, etc., follow from the fact that $U_q(\mathfrak{g})$ is a Hopf algebra (cf. Examples 1.2.8(iii), 2.1.4). For the commutativity we need that the R-matrix can be defined over $U_q(\mathfrak{g})_{\mathbb{Z}}$, which was proved by Lusztig, see [**L2**]. □

DEFINITION 3.3.2. Let $\lambda \in P_+$ be a dominant integer weight of $\mathfrak{g}$. The *Weyl module* V_λ of $U_q(\mathfrak{g})$ is defined by

$$V_\lambda = (V_\lambda)_{\mathbb{Z}} \otimes_{\mathcal{A}} \mathbb{C},$$

where $\mathcal{A} = \mathbb{Z}[q^{\pm 1/|P/Q|}]$ and $(V_\lambda)_{\mathbb{Z}} = U_q(\mathfrak{g})_{\mathbb{Z}} v_\lambda \subset (V_\lambda)_{\mathbb{C}_q}$ is the $U_q(\mathfrak{g})_{\mathbb{Z}}$-submodule of $(V_\lambda)_{\mathbb{C}_q}$ generated by the highest weight vector.

This means that we choose a basis of $(V_\lambda)_{\mathbb{C}_q}$ such that the action of $U_q(\mathfrak{g})_{\mathbb{Z}}$ has coefficients from $\mathbb{Z}[q^{\pm 1/|P/Q|}]$ and then we can put q a complex number. This description shows that the weight subspaces of V_λ are the same as those of $(V_\lambda)_{\mathbb{C}_q}$.

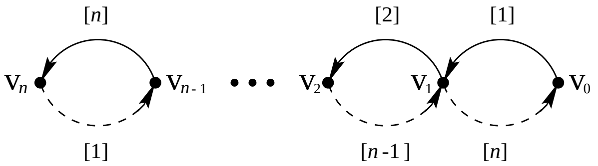
For example, let us consider first the case when $\mathfrak{g} = \mathfrak{sl}_2$. The weight lattice of $\mathfrak{sl}_2$ can be identified with $\mathbb{Z}$, so the Weyl modules are

$$V_n = \sum_{i=0}^{n} \mathbb{C}v_i\,, \qquad n \in \mathbb{Z}_+.$$

Here v_0 is the highest weight vector and $v_i = f^{(i)}v_0$. The action of $U_q(\mathfrak{sl}_2)$ is given by (recall that $[k] := (q^k - q^{-k})/(q - q^{-1})$):

$$q^h v_i = q^{n-2i}v_i, \quad ev_i = [n-i+1]v_{i-1}, \quad fv_i = [i+1]v_{i+1},$$

see the figure (f is represented by solid lines and e by dashed ones).



The coefficients of the above action are in $\mathbb{Z}[q^{\pm 1}]$, so it makes sense for $q \in \mathbb{C}^\times$. We will assume that $q \neq \pm 1$.

EXERCISE 3.3.3. Write the action of $e^{(k)}$ and $f^{(k)}$ in this basis.

Let $q = e^{\pi \mathrm{i}/\varkappa}$, $\varkappa \in \mathbb{Z}_+$. Then the module V_n may be reducible since $[k] = 0$ when $\varkappa$ divides k. For example, for $n = 3$, $\varkappa = 3$, the basis elements v_1 and v_2 span a submodule V_3'. This claim does not follow simply from the fact that V_3' is invariant under the operators e and f, because for example $e^{(3)}$ is a new operator different from $e^3/[3]!$ (since $[3] = 0$). We leave the proof as an exercise (not too difficult). The submodule V_3' is not a direct summand, hence V_3 is not semisimple.

THEOREM 3.3.4. (i) *The module V_n is irreducible for $n < \varkappa$.*
(ii) $\dim_q V_n = [n+1] = 0$ *if and only if $\varkappa$ divides $n+1$.*

The proof of this theorem is straightforward. In particular, this theorem implies that

(3.3.1) $\qquad$ For $0 \le n \le \varkappa - 2$, V_n is irreducible and $\dim_q V_n \neq 0$,

which is obvious because in this case all q-factorials are non-zero. (In fact, one has a stronger statement: V_n is irreducible iff $n < \varkappa$ or $n = l\varkappa - 1$, $l \in \mathbb{Z}_+$, see [**AP**].)

We will need a similar result for an arbitrary semisimple finite dimensional Lie algebra $\mathfrak{g}$. Recall the number m from (1.3.17). We let $q = e^{\pi \mathrm{i}/m\varkappa}$, $\varkappa \in \mathbb{Z}$, and assume that $\varkappa \ge h^\vee$, where $h^\vee = \langle \rho, \theta \rangle + 1$ is the dual Coxeter number, ρ is the half sum of positive roots, and θ is the highest root of $\mathfrak{g}$.

THEOREM 3.3.5. $\dim_q V_\lambda = 0$ *if and only if $\lambda + \rho \in H_{\alpha,l}$ for some $\alpha \in \Delta_+$, $l \in \mathbb{Z}$, where $H_{\alpha,l}$ is the hyperplane*

$$H_{\alpha,l} := \{x \in \mathfrak{h}^* \mid \langle x, \alpha \rangle = l\varkappa\}.$$

PROOF. By (2.3.13) we have an explicit formula for $\dim_q$:

$$\dim_q V_\lambda = \operatorname{tr}_{V_\lambda} q^{2\rho} = \chi_\lambda(q^{2\rho}), \tag{3.3.2}$$

where χ_λ is the character of the representation V_λ. Here and below we use the notation $e^\lambda(q^\mu) = q^{\langle\langle \lambda, \mu \rangle\rangle}$ and extend it to $f(q^\mu)$ for $f \in \mathbb{C}[P]$, where P is the weight lattice of $\mathfrak{g}$.

We have the *Weyl formula* for χ_λ:

$$\chi_\lambda(q^{2\rho}) = \frac{1}{\delta(q^{2\rho})} \sum_{w\in W} (-1)^{l(w)} q^{\langle\langle w(\lambda+\rho),2\rho\rangle\rangle}, \tag{3.3.3}$$

where $l(w)$ is the length of w, and δ is the *Weyl denominator*

$$\delta = \prod_{\alpha\in\Delta_+} (e^{\alpha/2} - e^{-\alpha/2}) = \sum_{w\in W} (-1)^{l(w)} e^{w(\rho)}. \tag{3.3.4}$$

(This equality is the Weyl denominator formula.)

We can rewrite (3.3.3) as

(3.3.5)

$$\chi_\lambda(q^{2\rho}) = \frac{1}{\delta(q^{2\rho})} \sum_{w\in W} (-1)^{l(w)} q^{2\langle\langle \lambda+\rho,w(\rho)\rangle\rangle} = \frac{\delta(q^{2(\lambda+\rho)})}{\delta(q^{2\rho})} = \prod_{\alpha\in\Delta_+} \frac{[\langle\langle\alpha,\lambda+\rho\rangle\rangle]}{[\langle\langle\alpha,\rho\rangle\rangle]},$$

where, as usual, $[n]$ denotes the q-number.

Note that $\langle\langle\alpha,\rho\rangle\rangle \le \langle\langle\theta,\rho\rangle\rangle = m(h^\vee - 1) < m\varkappa$, thus the denominator is non-zero. The numerator is 0 exactly when $\lambda+\rho$ belongs to some $H_{\alpha,l}$. □

Let us define the *affine Weyl group* W^a to be the group generated by reflections with respect to the hyperplanes $H_{\alpha,l}$. It contains the Weyl group W of $\mathfrak{g}$ which is generated by reflections with respect to the hyperplanes $H_{\alpha,0}$. Recall the following standard facts (see e.g. [**K1**]).

THEOREM 3.3.6. (i) *W^a is a Coxeter group generated by the simple reflections s_i $(i = 1,\dots,\operatorname{rank}\mathfrak{g})$ and the reflection s_0 with respect to the hyperplane $H_{\theta,1}$.*

(ii) *$W^a = W \ltimes \varkappa Q^\vee$ where $Q^\vee$ is the coroot lattice embedded in $\mathfrak{h}^*$ using the form $\langle,\rangle$; $\varkappa Q^\vee$ acts on $\mathfrak{h}^*$ by translations.*

(iii) *A fundamental domain for the shifted action $w.\lambda := w(\lambda+\rho)-\rho$ of W on $\mathfrak{h}^*$ is the Weyl chamber*

$$\overline{C} = \{\lambda\in\mathfrak{h}^* \mid (\lambda+\rho,\alpha_i^\vee)\ge 0,\ (\lambda+\rho,\theta^\vee)\le\varkappa\}. \tag{3.3.6}$$

For example, for $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{h}^*$ is a line and $\overline{C}$ is the closed interval $[-1,\varkappa-1]$.

We will need a simple technical lemma.

LEMMA 3.3.7. (i) *Let $f\in\mathbb{C}[P]^{\pm W}$ be W invariant (respectively anti-invariant). Then $f(q^{2\mu})$ is (anti)symmetric with respect to the action of W^a on μ.*

(ii) *Conversely, if $f(q^{2\mu}) = f(q^{2\mu'})$ for all $f\in\mathbb{C}[P]^W$ then $\mu' = w(\mu)$ for some $w\in W^a$.*

PROOF. (i) The (anti)symmetry with respect to W is obvious. It suffices to check that $f(q^{2\mu})$ is symmetric with respect to translations from $\varkappa Q^\vee$, i.e.,

$$f(q^{2(\mu+\varkappa\alpha^\vee)}) = f(q^{2\mu}), \qquad \alpha^\vee\in Q^\vee.$$

This follows from the equation

$$e^\lambda(q^{2(\mu+\varkappa\alpha^\vee)}) = q^{2\langle\langle\lambda,\mu\rangle\rangle} q^{2\varkappa\langle\langle\mu,\alpha^\vee\rangle\rangle}$$

and the fact that $2\varkappa\langle\langle\mu,\alpha^\vee\rangle\rangle = 2\varkappa m\langle\mu,\alpha^\vee\rangle \in 2\varkappa m\mathbb{Z}$.

(ii) The proof of the converse statement is left to the reader as an exercise; the crucial step is proving that certain matrices are non-singular. We will give an example of a calculation of this sort later (see the proof of Theorem 3.3.20). □

COROLLARY 3.3.8. *If we define "*$\dim_q V_\lambda$*" for all* $\lambda \in P$ *as* $\delta(q^{2(\lambda+\rho)})/\delta(q^{2\rho})$, *then it is* W^a*-antisymmetric with respect to the shifted action on* λ.

PROOF. Follows from Lemma 3.3.7 and the fact that δ is a W-antisymmetric element in $\mathbb{C}[P]$ (see (3.3.4)). □

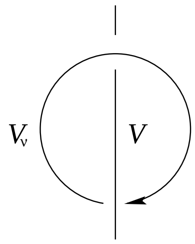
THEOREM 3.3.9. *Let* $C = \{\lambda \in P_+ \mid (\lambda+\rho, \theta^\vee) < \varkappa\}$. *Then for* $\lambda \in C$ *we have* $\dim_q V_\lambda > 0$ *and* V_λ *is irreducible.*

(In fact, one can describe exactly when V_λ is irreducible (see [**APW**]) but we will not need it.)

PROOF. The fact that $\dim_q V_\lambda > 0$ follows from Eq. (3.3.5). The irreducibility of V_λ follows from the so-called "linkage principle" (in a weak form):

> V_λ can have a subquotient with highest weight λ' only if $\lambda' = w(\lambda)$ for some $w \in W^a$.

To prove it, introduce operators $K_\nu \colon V \to V$ (where $\nu \in P_+$, V is any module) by the picture



Since K_ν is a morphism in the category $\mathcal{C}(\mathfrak{g}, \varkappa)$, it commutes with the action of $U_q(\mathfrak{g})$ on V. If v_λ is a highest weight vector in V, it is easy to see that $K_\nu(v_\lambda) = \chi_\nu(q^{2(\lambda+\rho)})v_\lambda$. Indeed, let $\{v_i\}$ and $\{v^i\}$ be dual bases in V_ν and V_ν^*. Using 1.2.8(iii), 2.3.4 and 2.2.4, we compute:

$$\begin{aligned}
K_\nu \colon v_\lambda &\overset{i}{\mapsto} \sum_i v_\lambda \otimes v_i \otimes v^i \\
&\overset{\sigma}{\mapsto} \sum_i q^{\langle\langle \lambda, \mathrm{wt}\, v_i \rangle\rangle}(v_i + \cdots) \otimes v_\lambda \otimes v^i \\
&\overset{\sigma}{\mapsto} \sum_i q^{2\langle\langle \lambda, \mathrm{wt}\, v_i \rangle\rangle} v_\lambda \otimes (v_i + \cdots) \otimes v^i \\
&\overset{\delta}{\mapsto} \sum_i q^{2\langle\langle \lambda+\rho, \mathrm{wt}\, v_i \rangle\rangle} v_\lambda \otimes (v_i + \cdots) \otimes v^i \\
&\overset{e}{\mapsto} \Big(\sum_i q^{2\langle\langle \lambda+\rho, \mathrm{wt}\, v_i \rangle\rangle}\Big) v_\lambda = \chi_\nu(q^{2(\lambda+\rho)})v_\lambda,
\end{aligned}$$

where "$+\cdots$" denotes terms with lower weight than v_i.

The operators K_ν are central and act by constant on v_λ, therefore for subquotients we have

$$\chi_\nu(q^{2(\lambda+\rho)}) = \chi_\nu(q^{2(\lambda'+\rho)}).$$

Because all χ_ν, $\nu \in P_+$, span $\mathbb{C}[P]^W$, it follows from Lemma 3.3.7(ii) that $\lambda' = w(\lambda)$ for some $w \in W^a$.

This completes the proof of the theorem. □

Note that $\mathcal{C}(\mathfrak{g},\varkappa)$ is a very complicated category; in particular, it is not semisimple. We want to extract a semisimple part with simple objects V_λ, $\lambda \in C$. As an indication that this is possible, we give without proof the following fact (see [**AP**] and references therein).

PROPOSITION 3.3.10. *For $\lambda, \mu \in C$ we have*

$$V_\lambda \otimes V_\mu \simeq \Big(\bigoplus_{\nu\in C} N^\nu_{\lambda\mu} V_\nu\Big) \oplus Z$$

for some module Z with $\dim_q Z = 0$.

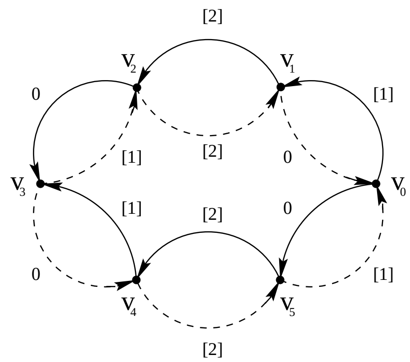
However, it is not possible to declare all modules of $\dim_q = 0$ to be 0. For example, for $\mathfrak{g} = \mathfrak{sl}_2$ we have $\dim_q(V_{\varkappa-2} \oplus V_\varkappa) = 0$, while both $V_{\varkappa-2}$ and $V_\varkappa$ are modules with non-zero q-dimension and $V_{\varkappa-2}$ is simple.

The correct construction was found by Andersen and Paradowski [**AP**] and is based on the use of an auxiliary category of tilting modules, which is interesting in its own right.

DEFINITION 3.3.11. A module T over $U_q(\mathfrak{g})$ is called *tilting* if both T and T^* have composition series with factors V_λ, $\lambda \in P_+$. Let $\mathcal{T}$ be the full subcategory of $\mathcal{C}(\mathfrak{g},\varkappa)$ consisting of all tilting modules.

EXAMPLE 3.3.12. (i) If $\lambda \in C$ then $V_\lambda \simeq V_{\lambda^*}$ for $\lambda^* = -w_0(\lambda)$, where w_0 is the longest element in W. Therefore the module V_λ is tilting. However, for a general $\lambda \in P_+$, V_λ may not be tilting.

(ii) Let $\mathfrak{g} = \mathfrak{sl}_2$, $q = e^{\pi i/3}$, so $[3] = 0$. Consider the Weyl module V_3 over $U_q\mathfrak{sl}_2$. We add two more vectors to it and extend the action of $\mathfrak{sl}_2$ as shown in the figure for the elements e and f (f is represented by solid lines and e by dashed ones).



(The reader can define as an exercise the action of $e^{(k)}, f^{(k)}$ for $k > 0$.) We obtain a module $T = \sum_{i=0}^{5} \mathbb{C}v_i$. It is easy to see that the vectors v_0, v_1, v_2, v_3 generate a submodule isomorphic to V_3 and the factor by it is isomorphic to V_1. It can be easily shown that $T^* \simeq T$, hence the module T is tilting. Note that T is not a direct sum of V_3 and V_1.

The following important theorem was proved by Andersen and Paradowski (see [**AP**] and references therein).

THEOREM 3.3.13 ([**AP**]). (i) *The category of tilting modules $\mathcal{T}$ is closed under $*$, $\oplus$, $\otimes$ and direct summands.*

(ii) *For every $\lambda \in P_+$ there exists a unique indecomposable tilting module T_λ such that its weight subspace $(T_\lambda)^\mu$ is 0 unless $\mu \leq \lambda$ and $(T_\lambda)^\lambda = \mathbb{C}$.*

(iii) *For* $\lambda \in C$ *we have* $T_\lambda = V_\lambda$, *while for* $\lambda \notin C$ *we have* $\dim_q T_\lambda = 0$. *Hence* $\dim_q T \geq 0$ *for all* $T \in \operatorname{Ob} \mathcal{T}$.

We will not give a proof of the theorem. We only note that, for example, it is rather difficult to show that $\mathcal{T}$ is closed under $\otimes$.

COROLLARY 3.3.14. $\mathcal{T}$ *is a ribbon category.*

Note that $\mathcal{T}$ is not an abelian category since it is not closed under quotients.

DEFINITION 3.3.15. A tilting module T is called *negligible* if $\operatorname{tr}_q f = 0$ for any $f \in \operatorname{End} T$. (In particular, $\dim_q T = 0$.)

LEMMA 3.3.16. T *is negligible iff* $T = \bigoplus_{\lambda \notin C} n_\lambda T_\lambda$ *for some* $n_\lambda \in \mathbb{Z}_+$.

PROOF. Follows easily from Theorem 3.3.13. Indeed, it is enough to show that T_λ is negligible iff $\lambda \notin C$. Since T_λ is indecomposable and $\dim_{\mathbb{C}} T_\lambda < \infty$, every endomorphism f of T_λ in some homogeneous basis has the form $f = c\,\mathrm{id} + $upper triangular. Then $\operatorname{tr}_q f = c \dim_q T_\lambda$. □

DEFINITION 3.3.17. A morphism $f\colon T_1 \to T_2$ is called *negligible* if $\operatorname{tr}_q(fg) = 0$ for all $g\colon T_2 \to T_1$.

Note that if T_1 or T_2 is negligible then any morphism $f\colon T_1 \to T_2$ is negligible.

LEMMA 3.3.18. (i) *If* T *is negligible, then so are* T^*, $T \otimes T'$ *for any* T', *and direct summands of* T.

(ii) *If* f *is negligible, then so are* f^*, $f \otimes g$, fg *and* gf *for any* g.

The proof being obvious is omitted.

DEFINITION 3.3.19. Let $\mathcal{C}^{\mathrm{int}} \equiv \mathcal{C}^{\mathrm{int}}(\mathfrak{g}, \varkappa)$ ($\varkappa \in \mathbb{Z}, \varkappa \geq h^\vee$) be the category with objects tilting modules and morphisms

$$\operatorname{Hom}_{\mathcal{C}^{\mathrm{int}}}(V, W) = \operatorname{Hom}_{\mathcal{T}}(V, W)/\text{negligible morphisms}.$$

We list some properties of the category $\mathcal{C}^{\mathrm{int}} \equiv \mathcal{C}^{\mathrm{int}}(\mathfrak{g}, \varkappa)$:

1. $T \in \operatorname{Ob} \mathcal{T}$ is negligible iff it is isomorphic to 0 in $\mathcal{C}^{\mathrm{int}}$.
2. $\mathcal{C}^{\mathrm{int}}$ is a ribbon category.
3. Any object V in $\mathcal{C}^{\mathrm{int}}$ is isomorphic to $\bigoplus_{\lambda \in C} n_\lambda V_\lambda$.
4. $\mathcal{C}^{\mathrm{int}}$ is a semisimple abelian category and $\dim_{\mathcal{C}^{\mathrm{int}}} V > 0$ if $V \not\simeq 0$.

These properties show that $\mathcal{C}^{\mathrm{int}}$ is the category we wanted. It is a semisimple ribbon category with a finite number of simple objects. A natural question is whether this category is modular. We will show that the answer is positive.

THEOREM 3.3.20. $\mathcal{C}^{\mathrm{int}}$ *is a modular tensor category with simple objects* V_λ ($\lambda \in C$),

$$s_{\lambda\mu} = |P/\varkappa Q^\vee|^{-1/2}\, \mathrm{i}^{|\Delta_+|} \sum_{w \in W} (-1)^{l(w)} q^{2\langle\langle w(\lambda+\rho), \mu+\rho \rangle\rangle}, \tag{3.3.7}$$

$$t_{\lambda\mu} = \delta_{\lambda\mu} q^{\langle\langle \lambda, \lambda+2\rho \rangle\rangle}, \tag{3.3.8}$$

and

$$D = \sqrt{|P/\varkappa Q^\vee|} \prod_{\alpha \in \Delta_+} \bigl(2\sin(\pi\langle \alpha, \rho\rangle/\varkappa)\bigr)^{-1}, \tag{3.3.9}$$

$$\zeta = e^{2\pi \mathrm{i} c/24}, \quad c = (\varkappa - h^\vee)\dim \mathfrak{g}/\varkappa. \tag{3.3.10}$$

PROOF. The calculations in the proof of Theorem 3.3.9 and Eq. (3.1.5) give

$$\tilde{s}_{\lambda\mu} = \chi_\mu(q^{2(\lambda+\rho)}) \dim_q V_\lambda = \frac{1}{\delta(q^{2\rho})} \sum_{w\in W} (-1)^{l(w)} q^{2\langle\langle w(\lambda+\rho),\mu+\rho\rangle\rangle}.$$

To show that $\det \tilde{s} \neq 0$, we will calculate the matrix $\tilde{s}^2$. First note that if we use the formula above to extend $\tilde{s}_{\lambda\mu}$ for $\lambda, \mu \in P$, this extended matrix will be antisymmetric with respect to the shifted action of the affine Weyl group W^a:

$$\tilde{s}_{w.\lambda,\mu} = (-1)^{l(w)} \tilde{s}_{\lambda,\mu}, \qquad w \in W^a. \tag{3.3.11}$$

In particular, $\tilde{s}_{\lambda\mu} = 0$ when λ or μ are on the walls of C.

Since $\sum_{\mu\in C} \tilde{s}_{\lambda\mu}\tilde{s}_{\mu\nu}$ is symmetric with respect to the shifted action of W^a on μ and C is the fundamental domain for the action of W^a on P, we can replace the range of summation with P/W^a. Since $W^a \simeq W \ltimes \varkappa Q^\vee$, this sum equals

$$\begin{aligned} \frac{1}{|W|} &\sum_{\mu\in P/\varkappa Q^\vee} \tilde{s}_{\lambda\mu}\tilde{s}_{\mu\nu} \\ &= \frac{1}{|W|} \sum_{w,w'\in W} \sum_{\mu\in P/\varkappa Q^\vee} \delta(q^{2\rho})^{-2} (-1)^{l(w)+l(w')} q^{2\langle\langle \mu+\rho, w(\lambda+\rho)+w'(\nu+\rho)\rangle\rangle}. \end{aligned}$$

Now we need an obvious lemma.

LEMMA 3.3.21. $\displaystyle\sum_{\mu\in P/\varkappa Q^\vee} q^{2\langle\langle\mu,a\rangle\rangle} = \begin{cases} 0 & \text{for } a \notin \varkappa Q^\vee, \\ |P/\varkappa Q^\vee| & \text{for } a \in \varkappa Q^\vee. \end{cases}$

Note that $w(\lambda+\rho) + w'(\nu+\rho) = w(\lambda+\rho) - w'w_0(\nu^*+\rho) \in \varkappa Q^\vee$ iff $\lambda+\rho \in w^{-1}w'w_0(\nu^*+\rho) + \varkappa Q^\vee$ where w_0 is the longest elment in W. But since both λ and ν^* are in C, which is a fundamental domain of W^a, this is only possible if $\lambda+\rho = \nu^*+\rho$, $w^{-1}w' = w_0$. Therefore

$$\sum_{\mu\in C} \tilde{s}_{\lambda\mu}\tilde{s}_{\mu\nu} = \frac{|P/\varkappa Q^\vee|}{\delta(q^{2\rho})^2} (-1)^{l(w_0)} \delta_{\lambda,\nu^*}.$$

This number is non-zero, hence $\det \tilde{s} \neq 0$.

This also gives D since $(\tilde{s}^2)_{\lambda\nu} = D^2\delta_{\lambda,\nu^*}$. Formula (3.3.8) for the twist follows directly from Example 2.2.6. The rest of the proof is straightforward and is left to the reader. $\square$

EXAMPLE 3.3.22. When $\mathfrak{g} = \mathfrak{sl}_2$, we have:

$$s_{\lambda\mu} = \sqrt{\frac{2}{\varkappa}} \sin\left(\pi \frac{(\lambda+1)(\mu+1)}{\varkappa}\right), \qquad 0 \le \lambda, \mu \le \varkappa - 2.$$

The arguments of Theorem 3.3.20 can be repeated for $q = e^{\pi \mathrm{i}/m\varkappa}$, $\varkappa \in \mathbb{Q}$, but in this case the matrix $\tilde{s}$ may be degenerate.

Note that the formulas for the matrices s, t coincide with the Kac–Peterson formula [**KP**] for the modular transformations of characters of the affine Lie algebra $\hat{\mathfrak{g}}$ when $q = e^{\pi \mathrm{i}/m\varkappa}$ (their matrix T corresponds to the matrix t/ζ in our notations). This fact will be explained later.

Finally, let us discuss the Verlinde algebra for $\mathcal{C}^{\mathrm{int}}$. Let $\mathcal{V} = K(\mathcal{R}ep_f(\mathfrak{g}))\otimes\mathbb{C}$ be the complexified Grothendieck ring of $\mathcal{R}ep_f(\mathfrak{g})$; similarly, denote $\mathcal{V}_k = K(\mathcal{C}^{\mathrm{int}})\otimes\mathbb{C}$ (where, as before, $\varkappa = k + h^\vee$).

PROPOSITION 3.3.23. *The Verlinde algebra $\mathcal{V}_k$ is the quotient of $\mathcal{V}$, namely, $\mathcal{V}_k = \mathcal{V}/\mathcal{I}_k$, where $\mathcal{I}_k \subset \mathcal{V}$ is the linear span of $\langle V_\lambda \rangle - (-1)^{l(w)}\langle V_{w.\lambda}\rangle$ for $\lambda \in P_+, w \in W^a, w.\lambda \in P_+$.*

PROOF. The construction given in Theorem 3.1.11 defines a surjective map $\mu\colon \mathcal{V} \to \mathcal{V}_k$. It follows from Weyl character formula that $\mathcal{I}_k \subset \ker \mu$. On the other hand, it follows from Theorem 3.3.6(iii) that $\dim \mathcal{V}/\mathcal{I}_k = |C| = \dim \mathcal{V}_k$. □

EXERCISE 3.3.24. (i) Show that for $\mathfrak{g} = A_n$, the ideal $\mathcal{I}_k$ is the linear span of $\langle V_\lambda \rangle$ for $\lambda \in P_+, (\lambda + \rho, \theta^\vee) = \varkappa$.

(ii) Show that for $\mathfrak{g} = E_8$ this is not so.

(iii) Show that the fusion rules for $U_q(\mathfrak{sl}_2)$ for $q = e^{\pi \mathrm{i}/(k+2)}$ are given by

$$\langle V_m \rangle \langle V_n \rangle = \sum_l N_{mn}^l \langle V_l \rangle,$$

where

$$N_{mn}^l = \begin{cases} 1 & \text{for } |m-n| \le l \le m+n,\ l \le 2k-(m+n),\ l+m+n \in 2\mathbb{Z}, \\ 0 & \text{otherwise} \end{cases}$$

(cf. Example 2.1.10).