CHAPTER 2

Ribbon Categories

In this chapter, we continue the study of the theory of tensor categories. We define the notion of ribbon category (in other terminology, rigid balanced braided tensor category) as a category in which every object has a dual satisfying some natural properties and there are functorial isomorphisms $V^{**} \simeq V$ compatible with the tensor product. We develop the "graphic calculus", allowing one to present morphisms in a ribbon category by ribbon (framed) tangles. In particular, this shows that every ribbon category gives rise to invariants of links (Reshetikhin–Turaev invariants). We also show that both examples of Chapter 1 — that is, the categories $\mathcal{C}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g}, \varkappa)$ — are ribbon.

2.1. Rigid monoidal categories

Now we will discuss the notion of duality in a tensor category. To motivate the definitions, let us consider again the category $\mathcal{V}ec_f(k)$ of finite dimensional vector spaces over a field k. For each $V \in \operatorname{Ob} \mathcal{V}ec_f(k)$, there is a dual vector space $V^* \in \operatorname{Ob} \mathcal{V}ec_f(k)$ and natural morphisms

$$e_V : V^* \otimes V \to k, \tag{2.1.1}$$

$$i_V : k \to V \otimes V^*. \tag{2.1.2}$$

Here e_V is the evaluation map and $i_V(1) := \sum v_i \otimes v^i$ where $\{v_i\}$ and $\{v^i\}$ are dual bases in V and V^*, i.e., $i_V(1)$ corresponds to id_V via the isomorphism $V \otimes V^* \simeq \operatorname{End}_k(V)$.

DEFINITION 2.1.1. Let $\mathcal{C}$ be a monoidal category and V be an object in $\mathcal{C}$. A *right dual* to V is an object V^* with two morphisms

$$e_V : V^* \otimes V \to \mathbf{1}, \tag{2.1.3}$$

$$i_V : \mathbf{1} \to V \otimes V^*, \tag{2.1.4}$$

such that the composition

$$V \xrightarrow{i_V \otimes \mathrm{id}_V} V \otimes V^* \otimes V \xrightarrow{\mathrm{id}_V \otimes e_V} V \tag{2.1.5}$$

is equal to id_V, and the composition

$$V^* \xrightarrow{\mathrm{id}_{V^*} \otimes i_V} V^* \otimes V \otimes V^* \xrightarrow{e_V \otimes \mathrm{id}_{V^*}} V^* \tag{2.1.6}$$

is equal to id_{V^*}.

The properties (2.1.5) and (2.1.6) are called the *rigidity axioms*.

As mentioned before, we skip the canonical associativity and unit isomorphisms in our formulas. Otherwise, we would have to write the first map in (2.1.5) as follows:

$$V \xrightarrow{\lambda_V^{-1}} \mathbf{1} \otimes V \xrightarrow{i_V \otimes \mathrm{id}_V} (V \otimes V^*) \otimes V \xrightarrow{\alpha} V \otimes (V^* \otimes V).$$

Similarly to 2.1.1, we can define a *left dual* of an object V to be an object ${}^*V$ with morphisms

$$e'_V : V \otimes {}^*V \to \mathbf{1}, \tag{2.1.7}$$

$$i'_V : \mathbf{1} \to {}^*V \otimes V \tag{2.1.8}$$

and similar axioms.

DEFINITION 2.1.2. A monoidal category $\mathcal{C}$ is called *rigid* if every object in $\mathcal{C}$ has right and left duals.

EXAMPLE 2.1.3. As we already discussed, the category of finite dimensional vector spaces $\mathcal{V}ec_f(k)$ has duals satisfying the rigidity axioms. For example, (2.1.5) is equivalent to the well-known identity

$$\sum_i v_i(v^i, v) = v, \qquad v \in V. \tag{2.1.9}$$

EXAMPLE 2.1.4. Let A be a *Hopf algebra* over a field k, i.e., a bialgebra A with an algebra anti-isomorphism $\gamma\colon A \to A$, called the antipode, satisfying

$$\mu(\mathrm{id} \otimes \gamma)\Delta = \varepsilon = \mu(\gamma \otimes \mathrm{id})\Delta, \tag{2.1.10}$$

where $\mu\colon A \otimes A \to A$ is the multiplication.

Let $\mathcal{C} = \mathcal{R}ep_f(A)$ be the category of finite dimensional representations of A. It is a monoidal category (see Exercise 1.1.8(iii)). For an object V we define its dual V^* to be the dual vector space with the following action of A:

$$(av^*, v) := (v^*, \gamma(a)v) \tag{2.1.11}$$

for $a \in A$, $v^* \in V^*$, $v \in V$. Then the canonical maps of vector spaces $k \to V \otimes V^*$ and $V^* \otimes V \to k$ are morphisms of A-modules, and thus, $\mathcal{C}$ is a rigid monoidal category.

LEMMA 2.1.5. *If it exists, the right dual is unique up to a unique isomorphism compatible with e and i, i.e., for any two duals $(V^*_{(1)}, e_{(1)}, i_{(1)})$ and $(V^*_{(2)}, e_{(2)}, i_{(2)})$ of an object V, there is a unique isomorphism $\varphi\colon V^*_{(1)} \xrightarrow{\sim} V^*_{(2)}$ such that the diagrams*

$$\begin{tikzcd} V^*_{(1)} \otimes V \arrow[rr, "\varphi \otimes \mathrm{id}_V"] \arrow[dr, "e_{(1)}"'] & & V^*_{(2)} \otimes V \arrow[dl, "e_{(2)}"] \\ & \mathbf{1} & \end{tikzcd} \qquad \begin{tikzcd} V \otimes V^*_{(1)} \arrow[rr, "\mathrm{id}_V \otimes \varphi"] & & V \otimes V^*_{(2)} \\ & \mathbf{1} \arrow[ul, "i_{(1)}"] \arrow[ur, "i_{(2)}"'] & \end{tikzcd}$$

are commutative.

PROOF. Take φ to be the composition

$$V^*_{(1)} \xrightarrow{\mathrm{id} \otimes i_{(2)}} V^*_{(1)} \otimes V \otimes V^*_{(2)} \xrightarrow{e_{(1)} \otimes \mathrm{id}} V^*_{(2)}.$$

The rest of the proof is left as an exercise. □

Note that if V^* and ${}^*V$ exist, then there are canonical isomorphisms

$$V = {}^*(V^*) = ({}^*V)^*. \tag{2.1.12}$$

LEMMA 2.1.6. *Suppose that V has a dual V^*. Then there exist canonical isomorphisms*

$$\operatorname{Hom}(U \otimes V, W) = \operatorname{Hom}(U, W \otimes V^*), \tag{2.1.13}$$

$$\operatorname{Hom}(U, V \otimes W) = \operatorname{Hom}(V^* \otimes U, W). \tag{2.1.14}$$

PROOF. To $\psi \in \operatorname{Hom}(U \otimes V, W)$ we associate the composition

$$U \xrightarrow{\operatorname{id} \otimes i_V} U \otimes V \otimes V^* \xrightarrow{\psi \otimes \operatorname{id}} W \otimes V^*$$

which is an element of $\operatorname{Hom}(U, W \otimes V^*)$.

Similarly, to $\varphi \in \operatorname{Hom}(U, W \otimes V^*)$ we assign

$$U \otimes V \xrightarrow{\varphi \otimes \operatorname{id}} W \otimes V^* \otimes V \xrightarrow{\operatorname{id} \otimes e_V} W.$$

One can easily check that these two maps are inverse to each other, establishing (2.1.13). The proof of (2.1.14) is similar. □

In particular, if both U and V have duals, then by Lemma 2.1.6

$$\operatorname{Hom}(U, V) = \operatorname{Hom}(V^*, U^*) = \operatorname{Hom}(\mathbf{1}, V \otimes U^*). \tag{2.1.15}$$

(In the language of abstract nonsense, this means that the category $\mathcal{C}$ has internal Hom's when it has duals.) For $f \in \operatorname{Hom}(U, V)$ its image in $\operatorname{Hom}(V^*, U^*)$ via the isomorphism (2.1.15) will be denoted by f^*.

If the right dual $*$ exists for all objects in $\mathcal{C}$, then by (2.1.15) it is a contravariant functor, i.e., a functor $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ where $\mathcal{C}^{\mathrm{op}}$ is the opposite (or dual) category to $\mathcal{C}$. (Recall that $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$ but with all arrows reversed.)

EXERCISE 2.1.7. Show that, in a rigid category, $*$ is an equivalence of categories $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$.

Rigidity is a very restrictive requirement. As an illustration, let us prove the following proposition.

PROPOSITION 2.1.8. *In an abelian rigid monoidal category the tensor product functor $\otimes$ is exact, i.e., for any short exact sequence $0 \to U \to V \to W \to 0$ and an object X, the sequences*

$$0 \to U \otimes X \to V \otimes X \to W \otimes X \to 0$$

and

$$0 \to X \otimes U \to X \otimes V \to X \otimes W \to 0$$

are exact.

PROOF. The sequence

$$0 \to U \otimes X \to V \otimes X \to W \otimes X$$

is exact iff

$$0 \to \operatorname{Hom}(Y, U \otimes X) \to \operatorname{Hom}(Y, V \otimes X) \to \operatorname{Hom}(Y, W \otimes X)$$

is exact for any object Y. But by (2.1.12, 2.1.13), $\operatorname{Hom}(Y, U \otimes X) = \operatorname{Hom}(Y \otimes {}^*X, U)$. Since the functor $\operatorname{Hom}(Y, -)$ is left exact, it follows that $- \otimes X$ is left exact. Using Exercise 2.1.7 (or repeating the same argument with duals), we see that it is also right exact. □

Definition 2.1.9. For an abelian category $\mathcal{C}$, its *Grothendieck group* $K(\mathcal{C})$ is the quotient of the free abelian group on the set of all isomorphism classes of objects in $\mathcal{C}$ modulo the relations $\langle V\rangle = \langle U\rangle + \langle W\rangle$ for any short exact sequence $0 \to U \to V \to W \to 0$. Here $\langle U\rangle$ denotes the isomorphism class of U. When the category $\mathcal{C}$ is rigid monoidal we can make $K(\mathcal{C})$ a ring—the *Grothendieck ring* of $\mathcal{C}$—by defining $\langle U\rangle\langle V\rangle = \langle U \otimes V\rangle$. Note that we need the exactness of $\otimes$ in order that this be well-defined.

Obviously, $K(\mathcal{C})$ is an associative ring with unit; if in addition $\mathcal{C}$ is braided, then $K(\mathcal{C})$ is commutative.

Example 2.1.10. Let $\mathcal{C} = \mathcal{R}ep_f(\mathfrak{sl}_2)$ be the category of finite dimensional representations of $\mathfrak{sl}_2$ over $\mathbb{C}$. It is well-known that every object in $\mathcal{C}$ is a direct sum of simple ones and the simple objects are classified by their dimension: for any $n \in \mathbb{Z}_+$ there is a unique up to isomorphism irreducible $\mathfrak{sl}_2$-module V_n of dimension $n+1$. Therefore the Grothendieck group of $\mathcal{C}$ is

$$K(\mathcal{C}) = \bigoplus_{n=0}^{\infty} \mathbb{Z}\,\langle V_n\rangle. \tag{2.1.16}$$

The ring structure is also well-known:

$$\langle V_m\rangle\langle V_n\rangle = \sum_k N_{mn}^k \langle V_k\rangle, \tag{2.1.17}$$

where

$$N_{mn}^k = \begin{cases} 1 & \text{for } |m-n| \le k \le m+n,\ k+m+n \in 2\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.18}$$

Here is a different description of $K(\mathcal{C})$. Define the map

$$K(\mathcal{C}) \to \mathbb{Z}[x, x^{-1}]^{S_2} = \{f(x) \in \mathbb{Z}[x, x^{-1}] \mid f(x) = f(x^{-1})\} \tag{2.1.19}$$

by

$$V \mapsto \operatorname{tr}_V x^h \qquad \text{for } V \in \operatorname{Ob}\mathcal{C}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2. \tag{2.1.20}$$

Then it is easy to see that this map is an isomorphism.

As we discussed before, the category $\mathcal{C}(\mathfrak{sl}_2, \varkappa)$ of representations of the quantum group $U_q(\mathfrak{sl}_2)$ with $q = e^{\pi \mathrm{i}/\varkappa}$, $\varkappa \notin \mathbb{Q}$, has the same Grothendieck ring.

Let again $\mathcal{C}$ be a rigid monoidal category. One may ask whether the duality is compatible with the monoidal structure $(\otimes, \mathbf{1}, \alpha, \lambda, \rho)$. It turns out that this is true without imposing further restrictions.

Lemma 2.1.11. *Let $\mathcal{C}$ be a rigid monoidal category. Then:*

(i) $\mathbf{1}^* = \mathbf{1} = {}^*\mathbf{1}$.

(ii) $(V \otimes W)^* = W^* \otimes V^*$.

(iii) $(\alpha_{V_1 V_2 V_3})^* = \alpha_{V_3^* V_2^* V_1^*}$.

If $\mathcal{C}$ is a BTC then, in addition, we have:

(iv) $(\sigma_{VW})^* = \sigma_{V^* W^*}$.

(v) $e_{V \otimes W} = (e_V \otimes e_W)(\sigma_{W^*, V^* \otimes V} \otimes \mathrm{id})$, *and the same with σ^{-1} instead of σ.*

(vi) $i_{V \otimes W} = (\mathrm{id} \otimes \sigma_{V^*, W \otimes W^*})(i_V \otimes i_W)$, *and the same with σ^{-1} instead of σ.*

(In the statements (i), (ii) equality means existence of a unique isomorphism; these isomorphisms are used in (iii)–(vi).)

PROOF. (i) and (ii) are obvious since the dual is unique up to a unique isomorphism. The statements (iii)–(vi) are also easy, but we find it more instructive to give their proof in Section 2.3 using the pictorial technique developed there. □

2.2. Ribbon categories

In our basic example—the category of finite dimensional k-vector spaces—we also have functorial isomorphisms

$$\delta_V : V \xrightarrow{\sim} V^{**} \tag{2.2.1}$$

with the properties:

$$\delta_{V\otimes W} = \delta_V \otimes \delta_W, \tag{2.2.2}$$

$$\delta_{\mathbf{1}} = \mathrm{id}, \tag{2.2.3}$$

$$\delta_{V^*} = (\delta_V^*)^{-1}, \tag{2.2.4}$$

where for $f \in \mathrm{Hom}(U,V)$, $f^* \in \mathrm{Hom}(V^*,U^*)$ is defined by (2.1.15).

The existence of such isomorphisms δ does not follow from the other axioms of a rigid BTC. Let us introduce the last formal definition of this chapter.

DEFINITION 2.2.1. A *ribbon category* is a rigid braided tensor category with functorial isomorphisms (2.2.1) satisfying (2.2.2—2.2.4) for all objects V and W.

As before, unless otherwise specified we assume that $\mathcal{C}$ is abelian, $\otimes$ is bilinear, $\mathbf{1}$ is simple and $\mathrm{End}(\mathbf{1}) = k$.

The term "ribbon", introduced by Reshetikhin and Turaev [**RT1**], will be explained later. Ribbon BTCs are also called "tortile categories" [**JS**] or "balanced rigid braided tensor categories". The word "balancing" will be explained below.

Note that in any rigid BTC $\mathcal{C}$ one can construct functorial isomorphisms

$$\psi_V : V^{**} \xrightarrow{\sim} V \tag{2.2.5}$$

as the composition

$$V^{**} \xrightarrow{i\otimes\mathrm{id}} V \otimes V^* \otimes V^{**} \xrightarrow{\mathrm{id}\otimes\sigma^{-1}} V \otimes V^{**} \otimes V^* \xrightarrow{\mathrm{id}\otimes e} V. \tag{2.2.6}$$

However, ψ does not satisfy the property (2.2.2) unless $\mathcal{C}$ is symmetric.

LEMMA 2.2.2. *In any rigid braided tensor category $\mathcal{C}$, we have*:

$$\psi_{V\otimes W} = \sigma_{WV}\sigma_{VW}(\psi_V \otimes \psi_W),$$
$$\psi_{\mathbf{1}} = \mathrm{id}\,.$$

If, in addition, $\mathcal{C}$ is ribbon, then

$$\psi_{V^*} = \delta_V^* \psi_V^* \delta_V^*.$$

PROOF. Left as an exercise, which is best done using the pictorial technique of Section 2.3. □

Let $\mathcal{C}$ be a ribbon category. We define functorial isomorphisms

$$\theta_V = \psi_V \delta_V : V \xrightarrow{\sim} V, \qquad V \in \mathrm{Ob}\,\mathcal{C}. \tag{2.2.7}$$

The isomorphisms θ are usually called *balancing isomorphisms*, or *twists*. It follows from Lemma 2.2.2 that they satisfy the *balancing axiom*:

$$\theta_{V\otimes W} = \sigma_{WV}\sigma_{VW}(\theta_V \otimes \theta_W), \tag{2.2.8}$$
$$\theta_{\mathbf{1}} = \mathrm{id}, \tag{2.2.9}$$
$$\theta_{V^*} = (\theta_V)^*. \tag{2.2.10}$$

Conversely, a system of twists satisfying the balancing axioms (2.2.8–2.2.10) uniquely defines δ_V satisfying (2.2.2–2.2.4).

COROLLARY 2.2.3. *If $\mathcal{C}$ is a symmetric tensor category, then $\delta_V = \psi_V^{-1}$ defines a structure of a ribbon category on $\mathcal{C}$.*

A non-trivial example of a ribbon category is provided by the category $\mathcal{R}ep_f U_q(\mathfrak{g})$ of finite dimensional representations of the quantum group $U_q(\mathfrak{g})$ (see Section 1.3 and Example 2.1.4). Here q will be a formal variable. Recalling (2.1.11), we see that $V^{**} \simeq V$ as a vector space, but has a different action of $U_q(\mathfrak{g})$, namely

$$\pi_{V^{**}}(a) = \pi_V(\gamma^2(a)), \qquad a \in U_q(\mathfrak{g}). \tag{2.2.11}$$

These two representations are isomorphic:

PROPOSITION 2.2.4. *The category $\mathcal{R}ep_f U_q(\mathfrak{g})$ is a ribbon category with balancing $\delta_V = q^{2\rho}\colon V \xrightarrow{\sim} V^{**}$, where ρ is half of the sum of positive roots in $\mathfrak{g}$. (On the weight subspace $V^\lambda \subset V$, $\lambda \in \mathfrak{h}^*$, $q^{2\rho}$ acts as a multiplication by $q^{\langle\langle 2\rho,\lambda\rangle\rangle}$ and $\langle\langle \rho, \alpha_i\rangle\rangle := \langle\langle \alpha_i, \alpha_i\rangle\rangle/2 = d_i$.)*

PROOF. Quite trivial and based on the identity

$$\gamma^2(a) = q^{2\rho} a q^{-2\rho}, \qquad a \in U_q(\mathfrak{g}),$$

which can be easily checked on the generators of $U_q(\mathfrak{g})$ (see (1.3.11)). $\square$

Let us calculate ψ and θ in the category $\mathcal{C} = \mathcal{R}ep_f U_q(\mathfrak{g})$. Let R be the universal R-matrix of $U_q(\mathfrak{g})$ (see Example 1.2.8(iii) and (1.3.12)). Write

$$R_{21}^{-1} = \sum a_i \otimes b_i, \quad R_{21} = P(R), \tag{2.2.12}$$

where $P(a\otimes b) := b \otimes a$. Recall that $\sigma = PR = R_{21}P$, so $\sigma^{-1} = PR_{21}^{-1}$. Let $\{v_i\}$, $\{v^i\}$ be dual bases in V, V^*, and let $x \in V^{**}$. Then we compute $\psi(x)$ using (2.1.9), (2.2.6):

$$x \mapsto \sum_i v_i \otimes v^i \otimes x \mapsto \sum_{i,j} v_i \otimes b_j(x) \otimes a_j(v^i) \mapsto$$
$$\mapsto \sum_{i,j} v_i\,(b_j(x), a_j(v^i)) = \sum_{i,j} v_i\,(\gamma^{-1}(a_j)b_j(x), v^i) = \sum_j \gamma^{-1}(a_j)b_j(x).$$

Thus we obtained:

LEMMA 2.2.5. *$\theta = u^{-1}q^{2\rho}$, where $u^{-1} := \sum_j \gamma^{-1}(a_j)b_j$ lies in a certain completion of $U_q(\mathfrak{g})$.*

EXERCISE 2.2.6. Deduce from the previous lemma that θ acts as a multiplication by $q^{\langle\langle \lambda,\lambda+2\rho\rangle\rangle}$ on the irreducible $U_q(\mathfrak{g})$-module V_λ with highest weight λ.

The central element $\theta = u^{-1}q^{2\rho}$ is called a (universal) *Casimir operator* for $U_q(\mathfrak{g})$. The element u was introduced and studied by Drinfeld [**Dr2**].

Our next example of a ribbon category will be the Drinfeld category $\mathcal{D}(\mathfrak{g}, \varkappa)$ (see Section 1.4).

THEOREM 2.2.7 (Drinfeld [**Dr2**]). (i) *The category $\mathcal{D}(\mathfrak{g}, \varkappa)$ is a ribbon category, with V^*, e, i, δ the same as in the category of vector spaces.*

(ii) *In this category, the universal twist is given by $\theta = e^{\pi \mathrm{i} D/\varkappa}$ where D is the Casimir operator of $\mathfrak{g}$ defined by* (1.4.4).

Note that this theorem is not immediately obvious since both the rigidity axioms and the definition of θ involve the Drinfeld associator α (1.4.8), i.e., the asymptotics of solutions of KZ equations. However, it is not too difficult.

The formula for the universal twist given by this theorem agrees with $q^{\langle\langle \lambda, \lambda+2\rho \rangle\rangle}$ if $q = e^{\pi \mathrm{i}/m\varkappa}$ (where m is given by (1.3.17)) and confirms the equivalence of $\mathcal{D}(\mathfrak{g}, \varkappa)$ and $\mathcal{C}(\mathfrak{g}, \varkappa)$ as ribbon categories (cf. Theorem 1.4.6).

Finally, we note that a given rigid BTC may be endowed with different structures of a ribbon category, i.e., the maps δ are not uniquely defined by the associativity and commutativity isomorphisms. Here is an example.

EXAMPLE 2.2.8. Let $\mathcal{C} = \mathcal{R}ep_f(\mathfrak{sl}_2)$ be the category of finite dimensional representations of $\mathfrak{sl}_2$ over $\mathbb{C}$ (see Example 2.1.10). This category is symmetric, and thus, has a structure of ribbon category given by Corollary 2.2.3. This definition of δ coincides with the canonical isomorphism of vector spaces $V \xrightarrow{\sim} V^{**}$.

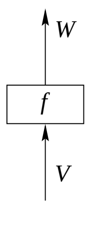
Let $Z = (-1)^h$. It is obvious that for every $\mathfrak{sl}_2$-module V, the map $Z\colon V \to V$ commutes with the action of $\mathfrak{sl}_2$; thus, Z is a functorial isomorphism (of the identity functor). On the simple module V_n it acts as 1 for even n and as -1 for odd n. Define $\tilde{\delta} = Z\delta$. Then $\tilde{\delta}$ also satisfies axioms (2.2.2–2.2.4), which follows from $\Delta(Z) = Z \otimes Z, Z|_{\mathbb{C}} = 1, \gamma(Z) = Z = Z^{-1}$. Thus, $\tilde{\delta}$ defines a new structure of a ribbon category on $\mathcal{C}$.

EXERCISE 2.2.9. Show that in any ribbon symmetric tensor category $\theta^2 = \mathrm{id}$. This exercise is done much easier using the results of the next section.
Hint: By Lemma 2.2.2, $(\theta_V^2)^* = \psi_{V^*}\psi_V^*$.

2.3. Graphical calculus for morphisms

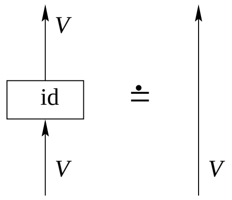
Let $\mathcal{C}$ be a ribbon category. We will introduce a pictorial technique for representing morphisms in $\mathcal{C}$ (cf. [**RT1**], [**T**]).

A morphism $f\colon V \to W$ in $\mathcal{C}$ will be represented by the figure

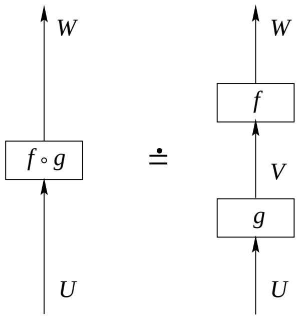


Note that this diagram should be read from bottom to top, as indicated by the arrows. (Some authors use other conventions.)

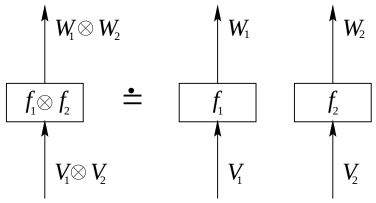
When f is the identity morphism the box will be omitted:



Here and below we relate by $\doteq$ diagrams which give equal morphisms in $\mathcal{C}$. The composition $f \circ g$ of two morphisms is obtained by placing the diagram of f on top of that of g:

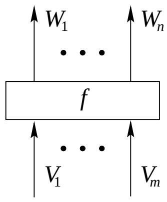


The tensor product of two morphisms f_1 and f_2 will be depicted by placing the diagram of f_1 to the left of the diagram of f_2:



A morphism $f\colon V_1 \otimes \cdots \otimes V_m \to W_1 \otimes \cdots \otimes W_n$ will be depicted as

(2.3.1)

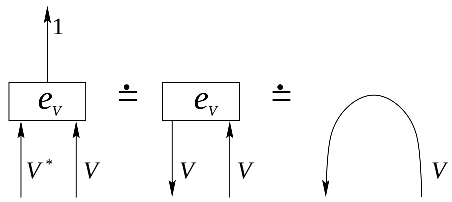


Since $\mathbf{1} \otimes V \simeq V \otimes \mathbf{1} \simeq V$ for any object V, we can add arrows labeled with $\mathbf{1}$'s to any picture without changing the morphism it represents. Hence, the empty picture represents the identity endomorphism of $\mathbf{1}$.

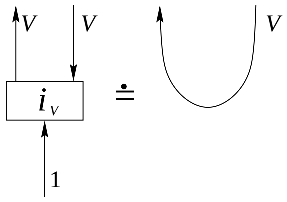
Note that there is ambiguity in defining $V_1 \otimes \cdots \otimes V_m$ for $m/ge3$ when the category $\mathcal{C}$ is not strict. In practice, however, we will use pictures of the form (2.3.1) only for $m, n \leq 2$. An accurate formulation of this formalism, that works for non-strict categories, is provided by Theorem 2.3.9 below.

We will depict duals by simply reversing the arrows and will skip the arrows labeled by $\mathbf{1}$. Also we identify V^{**} with V via δ_V. Then, for example, the morphism

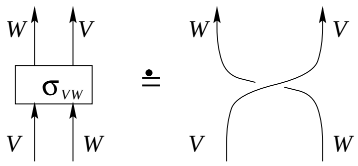
e_V (2.1.3) corresponds to

(2.3.2)


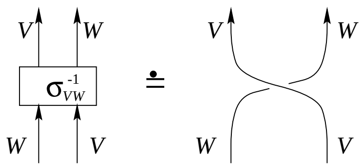
Similarly, i_V (2.1.4) corresponds to

(2.3.3)


The braiding σ_{VW} (1.2.4) will be depicted as

(2.3.4)


and its inverse σ_{VW}^{-1} as

(2.3.5)


Using these pictures as building blocks one can represent arbitrary functorial morphism in $\mathcal{C}$ which is a composition of $\alpha, \lambda, \rho, \sigma, e, i, \delta$ and their inverses.

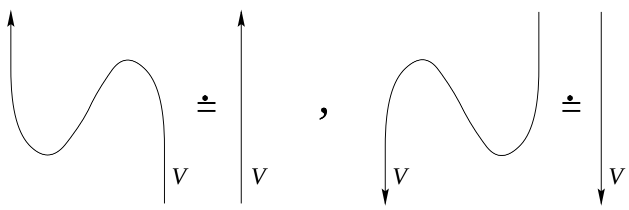
REMARK 2.3.1. The associativity α and the balancing δ are lost in this formalism. A more refined version which keeps track of α was proposed by Bar-Natan [**BN**].

The rigidity axioms (2.1.5, 2.1.6) in the case of a strict category read

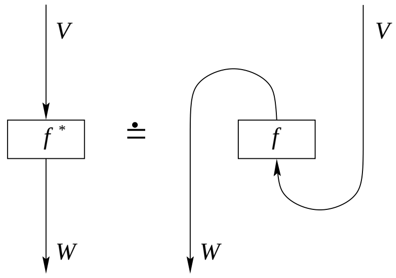
$$(\mathrm{id}_V \otimes e_V)(i_V \otimes \mathrm{id}_V) = \mathrm{id}_V, \tag{2.3.6}$$

$$(e_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes i_V) = \mathrm{id}_{V^*}, \tag{2.3.7}$$

and can be represented graphically as

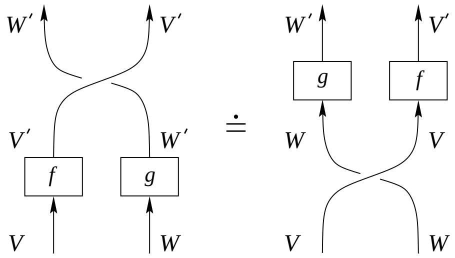


For a morphism $f\colon V \to W$ in $\mathcal{C}$, its dual $f^*\colon W^* \to V^*$ is given by the following picture (see the proof of Lemma 2.1.6):



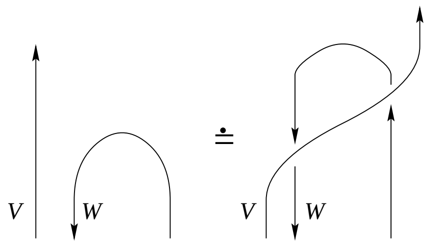
The functoriality of σ (1.2.5) is represented by:

(2.3.8)



Another example is the following lemma.

LEMMA 2.3.2.

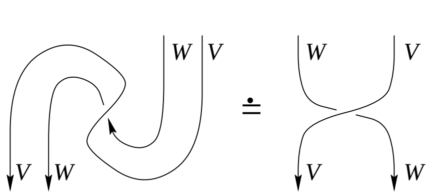


PROOF. This is equivalent to the commutative diagram

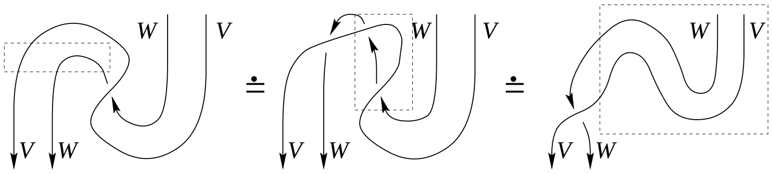
$$
(2.3.9)\qquad
\begin{tikzcd}
V \otimes W^* \otimes W \arrow[r, "\sigma_{V,W^*\otimes W}"] \arrow[d, "\mathrm{id}_V \otimes e_W"'] & W^* \otimes W \otimes V \arrow[d, "e_W \otimes \mathrm{id}_V"] \\
V \otimes \mathbf{1} \arrow[r, "\sigma_{V,\mathbf{1}}"'] & \mathbf{1} \otimes V
\end{tikzcd}
$$

which follows from the functoriality of $\sigma_{V,-}$. □

As an immediate corollary we obtain the identity from Lemma 2.1.11(v). Next, let us prove the identity $(\sigma_{VW})^* = \sigma_{V^*W^*}$ from Lemma 2.1.11(iv). Its pictorial representation is



We manipulate the left hand side as follows using Lemma 2.3.2:



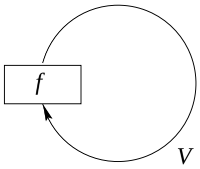
Then applying rigidity, we obtain the right hand side.

It is not difficult to write this proof formally in terms of axioms—it is not long but looks completely mysterious. (We leave it to the suspicious reader.) Representing morphisms with pictures is an intuitive way to visualize the axioms.

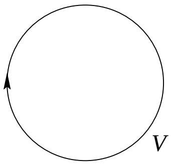
DEFINITION 2.3.3. Let V be an object in a ribbon category $\mathcal{C}$ and f be an endomorphism of V. We define the *trace* of f—$\operatorname{tr} f \in \operatorname{End}_k(\mathbf{1}) \simeq k$—to be the composition

$$\mathbf{1} \xrightarrow{i_V} V \otimes V^* \xrightarrow{f \otimes \mathrm{id}} V \otimes V^* \xrightarrow{\delta_V \otimes \mathrm{id}} V^{**} \otimes V^* \xrightarrow{e_{V^*}} \mathbf{1}. \tag{2.3.10}$$

Its picture is

(2.3.11) $\operatorname{tr} f =$ 

In particular, for $f = \mathrm{id}_V$, we define the *dimension* of V to be $\dim V := \operatorname{tr} \mathrm{id}_V$

(2.3.12) $\dim V =$ 

EXERCISE 2.3.4. (i) Show that in the category $\mathcal{R}ep_f U_q(\mathfrak{g})$ we have:

$$\operatorname{tr}_q f = \operatorname{tr}_V q^{2\rho} f, \quad \dim_q V = \operatorname{tr}_V q^{2\rho}. \tag{2.3.13}$$

Here we denote by $\dim_q$ and tr_q the above defined dimension and trace to distinguish them from the ordinary dimension and trace of V considered as a vector space.

(ii) Using their pictorial presentation, deduce from the axioms that

$$\operatorname{tr}(f \otimes g) = \operatorname{tr} f \operatorname{tr} g, \quad \operatorname{tr}(f^*) = \operatorname{tr} f, \quad \operatorname{tr}(fg) = \operatorname{tr}(gf). \tag{2.3.14}$$

In particular,

$$\dim(V \otimes W) = \dim V \dim W, \quad \dim V^* = \dim V. \tag{2.3.15}$$

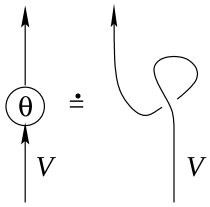
(A solution to this exercise can be found in [**Ka**, Theorem XIV.4.2].)

After the examples given above, it is natural to ask if it is true that any two morphisms giving rise to isotopic pictures are equal. To answer this question, we must finally formalize what we mean by "pictures". As it was with braids, the most natural approach is to consider these pictures as plane projections of certain 3-dimensional objects.

DEFINITION 2.3.5. A *tangle* is an isotopy class of a collection of non-intersecting smooth curves in $\mathbb{R}^2 \times [0,1]$, allowed to have ends only on the lines $\mathbb{R}\times\{0\}\times\{0\}$ and $\mathbb{R}\times\{0\}\times\{1\}$. If all the curves are closed, the tangle is called a *link*; in particular, a link which consists of just one curve is called a *knot*.

In particular, this includes as a special case the notion of a braid (Definition 1.2.1).

?! Now we can return to the question: is it true that any two morphisms corresponding to isotopic tangles are equal? The answer is "no". For example, the twist $\theta_V : V \to V$ defined by (2.2.7) is represented by the picture

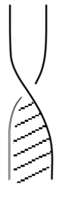


which is obviously isotopic to the picture of id_V, but $\theta \neq \mathrm{id}$ unless the category $\mathcal{C}$ is symmetric. To count the twists, one has to consider so-called framed, or ribbon, tangles.

We will call a *ribbon* a homeomorphic image of a rectangle in $\mathbb{R}^3$. We will always assume that a ribbon has distinguished "bases", i.e., a distinguished pair of opposite edges (in the pictures, they will be usually shown as "short sides"), and a distinguished side—"face side"—which will be shown in white, as opposed to the "back side", which will be shown in gray. We will also allow homeomorphic images of annuli, which, again, must have a distinguished side.

DEFINITION 2.3.6. A *ribbon tangle* (or a *framed tangle*) of n strands is an isotopy class of a union of n non-intersecting ribbons in $\mathbb{R}^2 \times [0,1]$, such that the bases of all ribbons lie on the lines $\mathbb{R}\times\{0\}\times\{0,1\}$, and near these lines, the ribbons are turned with their face side upward. A ribbon tangle composed only of annuli is called a *ribbon* (or *framed*) *link*, and a ribbon link consisting of a single annulus is called a *framed knot*.

Examples of ribbon tangles are shown on the next several pages. Note, however, that by definition, every ribbon must have an even number of twists; in particular, ribbons like this:



are not allowed.

We will call a *coupon* a rectangle in $\mathbb{R}^3$ lying in a plane parallel to $\mathbb{R}\times\{0\}\times\mathbb{R}$ and having edges parallel to $\mathbb{R}\times\{0\}\times\{0\}$ and $\{0\}\times\{0\}\times\mathbb{R}$. Each coupon is provided with a labeling of its sides: "face" and "back", and its edges: "bottom", "left", "top", and "right". The difference between coupons and ribbons is that we do not allow coupons to be twisted.

DEFINITION 2.3.7. A *generalized ribbon tangle* is an isotopy class of a union of several non-intersecting ribbons and several coupons in $\mathbb{R}^2 \times [0,1]$, such that the bases of all ribbons lie on the lines $\mathbb{R} \times \{0\} \times \{0,1\}$ or on edges of coupons parallel to them, and near them ribbons are turned with their face side upward.

So, a generalized ribbon tangle is almost the same as a ribbon tangle, but we allow coupons and allow the ribbons to end on them.

DEFINITION 2.3.8. If $\mathcal{C}$ is a category, a *$\mathcal{C}$-colored ribbon tangle* is a generalized ribbon tangle with the following additional structure:

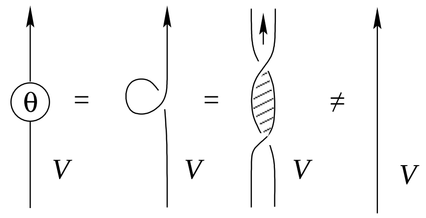
(i) Each ribbon strand is directed.

(ii) Each ribbon strand is labeled (or colored) by an object of $\mathcal{C}$.

(iii) Each coupon is labeled by a morphism of $\mathcal{C}$, so that the following condition is satisfied. For any fixed coupon, let $V_1, \dots, V_m$ be the labels of the ribbons ending on the bottom edge of the coupon, and let $\varepsilon_i = +$ if the direction of the ith ribbon is "incoming" (i.e., pointing towards the coupon), and $\varepsilon_i = -$ otherwise. To this we associate the object $X = V_1^{\varepsilon_1} \otimes \dots \otimes V_m^{\varepsilon_m}$, where $V^+ := V$ and $V^- := V^*$ ($X := \mathbf{1}$ if there are no ribbons ending on the bottom edge of the coupon). We do a similar thing for the top edge of the coupon, where now we put $\varepsilon = +$ for the "outgoing" ribbons, and get an object Y. Then we require that the coupon is labeled by a morphism $f\colon X \to Y$.

Note that any plane projection of a usual tangle can be considered as a ribbon tangle which is "lying flat", i.e., always with the face side up, as in Example 2.3.10 (this is called the "blackboard framing"). For technical reasons, we will often draw lines instead of ribbons, always assuming the blackboard framing. Also, we will often omit the arrows pointing up when there is no ambiguity.

Note, however, that different projections of the same (non-framed) tangle now give rise to different framed tangles. For example, the ribbon tangle corresponding to the twist θ_V is no longer isotopic to the trivial tangle. We will insert a circle containing the letter θ (or θ^{-1}) to represent twists:



It turns out that this was the only problem: now it is true that the isomorphisms corresponding to isotopic ribbon tangles are equal. This was proved by Reshetikhin and Turaev in [**RT1**]; see also expositions in [**T, Ka**]. We give here their result in a slightly modified form (cf. [**T**, Theorem 2.5]).

Let $\mathcal{C}$ be a ribbon category. Fix objects $V_1, \dots, V_n$ in $\mathcal{C}$ and consider all possible expressions of the form

$$X = ((V_1 \otimes V_2) \otimes V_4) \otimes ((V_1^{***} \otimes \mathbf{1}) \otimes {}^*V_2) \otimes \cdots \tag{2.3.16}$$

where we take the tensor product of $V_1, \dots, V_n$ in arbitrary order and allow repetitions, arbitrary number of left and right stars and $\mathbf{1}$'s.

To each expression X as above we assign a sequence $F(X)$ of arrows and labels by the following rule: to an object ${}^{*\cdots*}V^{*\cdots*}$ we assign $\downarrow_V$ if the total number of stars is odd and $\uparrow_V$ if it is even. All $\mathbf{1}$'s are skipped. For example, to the element (2.3.16) we assign the sequence

$$\uparrow V_1 \quad \uparrow V_2 \quad \uparrow V_4 \quad \downarrow V_1 \quad \downarrow V_2$$

For two such expressions X_1 and X_2 consider all morphisms $\varphi\colon X_1 \to X_2$ which can be obtained as a composition of the elementary morphisms $\alpha^{\pm 1}$, $\lambda^{\pm 1}$, $\rho^{\pm 1}$, $\sigma^{\pm 1}$, e, i, $\delta^{\pm 1}$, as well as a number of other morphisms of $\mathcal{C}$. To each such composition of morphisms $\varphi\colon X_1 \to X_2$, we assign a $\mathcal{C}$-colored ribbon tangle $T = F(\varphi)$ with

$$\mathrm{bottom}(T) = F(X_1), \quad \mathrm{top}(T) = F(X_2)$$

by the following rules.

A morphism $f\colon X_1 \to X_2$ corresponds to a coupon labeled with f, so that the ribbons ending on its bottom edge are labeled by $F(X_1)$, and those ending on its top edge by $F(X_2)$ (cf. Definition 2.3.8(iii)). The morphisms $\alpha, \lambda, \rho, \delta$ and their inverses correspond to the trivial tangles, $F(e)$, $F(i)$, $F(\sigma)$, and $F(\sigma^{-1})$ are given by the right hand sides of (2.3.2), (2.3.3), (2.3.4), and (2.3.5), respectively, and

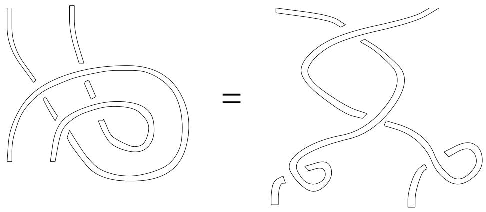
$$F(\varphi_1 \otimes \varphi_2) = \boxed{F(\varphi_1) \mid F(\varphi_2)}, \qquad F(\varphi_1\varphi_2) = \boxed{\begin{array}{c} F(\varphi_1) \\ \hline F(\varphi_2) \end{array}}.$$

In other words, we just apply the rules of the graphical calculus introduced earlier, but we use ribbons instead of lines so that we keep track of the twists. Then we have the following crucial result.

Theorem 2.3.9 (Reshetikhin–Turaev [**RT1**]). *The morphism φ depends only on the isotopy class of the tangle $F(\varphi)$, i.e., if $F(\varphi_1)$ and $F(\varphi_2)$ are isotopic as ribbon tangles then $\varphi_1 = \varphi_2$.*

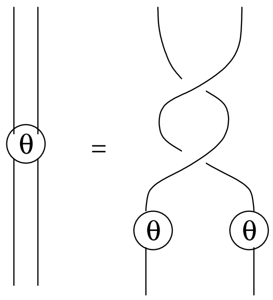
The proof of this theorem will be sketched later, when we give another reformulation.

Example 2.3.10. The identity $\theta_{V\otimes W} = \sigma_{WV}\sigma_{VW}(\theta_V \otimes \theta_W)$ (2.2.8) corresponds to the following isotopic ribbon graphs:



Another way to present this identity, using the conventions formulated before (i.e., drawing lines instead of ribbons and omitting upward arrows), is shown below.

(2.3.17)



An important corollary of Theorem 2.3.9 is that any two isomorphisms $\varphi\colon X_1 \xrightarrow{\sim} X_2$, composed of $\alpha, \lambda, \rho, \delta$ and their inverses, are equal and therefore, if $F(X_1) = F(X_2)$ then there exists a *canonical* isomorphism $X_1 \xrightarrow{\sim} X_2$.

Typically, the Reshetikhin–Turaev Theorem is used in the opposite direction. We will reformulate the theorem in another form which appeared in their original paper [**RT1**].

THEOREM 2.3.11 (Reshetikhin–Turaev [**RT1**]). *Let $\mathcal{C}$ be a ribbon category. Then for every $\mathcal{C}$-colored ribbon tangle T we can define a morphism $F^{-1}(T)\colon X_1 \to X_2$ in $\mathcal{C}$ where*

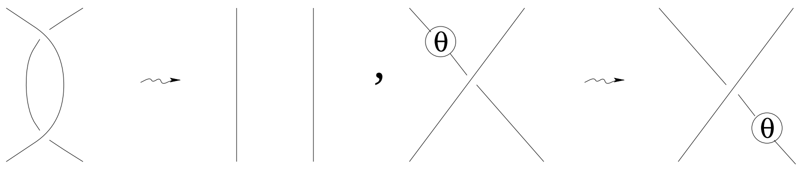
$$X_1 = F^{-1}(\mathrm{bottom}(T)), \quad X_2 = F^{-1}(\mathrm{top}(T)).$$

The objects X_1 and X_2 are defined up to a canonical isomorphism and the morphism $F^{-1}(T)$ depends only on the isotopy type of the tangle T.

OUTLINE OF PROOF. First, one shows that two ribbon tangles are isotopic if and only if the corresponding diagrams in the plane can be obtained one from another by applying a sequence of the following elementary operations:

a) isotopy of $\mathbb{R}^2$ and

b) one of a finite number of "simple moves", such as



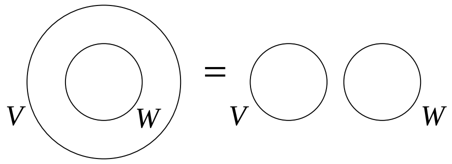
the braid relation and some more—see a complete list in [**RT1**] or [**T, Ka**]. These simple moves generalize the so-called *Reidemeister moves* which play the same role for unframed tangles.

Now, it suffices to check that $F^{-1}(T)$ is unchanged under any of these moves, which is straightforward: for example, the braid relation and the first of these moves follow from the definition of a braided category, and the second relation follows from the functoriality of the commutativity isomorphism. We refer the reader to the original papers for details. □

COROLLARY 2.3.12. *For every $\mathcal{C}$-colored framed link T, $F^{-1}(T) \in k$ is a number which depends only on the isotopy type of T.*

Therefore, every ribbon category gives a number of invariants of links. These invariants are usually called *Reshetikhin–Turaev invariants*. In particular, if we take $\mathcal{C}$ to be the category $\mathcal{C}(\mathfrak{g}, \varkappa)$ of representations of a quantum group then these invariants are rational functions in $q^{1/n}$. If they are rewritten as formal power series in $1/\varkappa$ then every coefficient of such a series is again an invariant; moreover, every such coefficient is an invariant of a very special type—*Vassiliev invariant* (see, for example, [**PS**]). However, it was recently proved that not all Vassiliev invariants can be obtained in this way [**Vo**].

Theorem 2.3.9 is the most effective way of proving identities in ribbon categories—just draw the corresponding pictures and manipulate with them. For example, the identity $\dim(V \otimes W) = \dim V \dim W$ (cf. Exercise 2.3.4) is now obvious from the following isotopy of ribbon tangles:



We also have the following somewhat technical but useful fact.

PROPOSITION 2.3.13. *Let T be a $\mathcal{C}$-colored ribbon tangle. Let T' be obtained from T by reversing the direction of one of the strands and replacing the corresponding to it label V by V^*. Then we get the same morphism $F^{-1}(T) = F^{-1}(T')$.*

The proof is similar to that of Theorem 2.3.9. It is immediate to deduce from Proposition 2.3.13 that $\dim V = \dim V^*$ (cf. Exercise 2.3.4).

2.4. Semisimple categories

In this section we study some properties of semisimple abelian tensor categories and, in particular, semisimple ribbon categories.

Let $\mathcal{C}$ be a semisimple ribbon category. Let I be the set of equivalence classes of non-zero simple objects in $\mathcal{C}$, and let $\{V_i\}_{i\in I}$ be representatives of those classes. Since we assumed that $\mathbf{1}$ is a simple object, it also can be written as V_i for some $i \in I$. Traditionally, this index is denoted by 0: $V_0 = \mathbf{1}$.

Semisimplicity immediately implies a number of properties:

1. For $i \in I$, V_i^* is also simple, hence $V_i^* \simeq V_{i^*}$ for some $i^* \in I$. The map $*: I \to I$ is an involution and $0^* = 0$. (Note, however, that in general there is no canonical way to define the isomorphism $V_i^* \simeq V_{i^*}$, see Remark 2.4.2 below.)

2. We can define the multiplicity coefficients $N_{ij}^k \in \mathbb{Z}_+$ by

$$V_i \otimes V_j \simeq \bigoplus_k N_{ij}^k V_k. \tag{2.4.1}$$

In the physics literature, this formula is called a *fusion rule*, and the coefficients N_{ij}^k are called "fusion coefficients". They satisfy:

$$N_{ij}^k = \dim \operatorname{Hom}(V_k, V_i \otimes V_j) = \dim \operatorname{Hom}(\mathbf{1}, V_i \otimes V_j \otimes V_k^*), \tag{2.4.2}$$

$$N_{ij}^k = N_{ji}^k = N_{ik^*}^{j^*} = N_{i^*j^*}^{k^*}, \quad N_{ij}^0 = \delta_{ij^*}. \tag{2.4.3}$$

These numbers are nothing but the structure coefficients of the Grothendieck ring $K(\mathcal{C})$, which has the basis $x_i = \langle V_i \rangle$ with the multiplication rule $x_i x_j = \sum_k N_{ij}^k x_k$.

3. Since $\operatorname{End} V_i = k$, we have

$$\theta_{V_i} = \theta_i \operatorname{id}_{V_i}, \qquad \dim V_i = d_i \tag{2.4.4}$$

for some $\theta_i, d_i \in k$. The number d_i is called the *quantum dimension* of V_i, cf. Exercise 2.3.4. We have

$$\theta_0 = 1, \;\; \theta_{i^*} = \theta_i, \tag{2.4.5}$$

$$d_0 = 1, \;\; d_{i^*} = d_i, \;\; d_i d_j = \textstyle\sum_k N_{ij}^k d_k. \tag{2.4.6}$$

LEMMA 2.4.1. *In a semisimple ribbon category, all dimensions $d_i = \dim V_i$ of simple objects are non-zero.*

PROOF. Since $N^0_{ii^*} = N^i_{i0} = 1$, we can write $V_i \otimes V_i^* = X \oplus X'$, where $X \simeq \mathbf{1}$ (not canonically), and $\mathrm{Hom}(\mathbf{1}, X') = 0$. Thus, the maps $i\colon \mathbf{1} \to V_i \otimes V_i^*$ and $e\colon V_i \otimes V_i^* \to \mathbf{1}$ can be considered as maps $\mathbf{1} \to X$, $X \to \mathbf{1}$. Since both of these maps are non-zero (this follows from rigidity), their composition is also non-zero. $\square$

REMARK 2.4.2. It is tempting (and many authors do so) to choose some identification $V_i^* \xrightarrow{\sim} V_{i^*}$ to simplify the formulas. However, this should be avoided. Not only this cannot be done canonically, but in many cases it cannot be done at all! Indeed, in order for this to be useful we need to choose these isomorphisms $\phi_i\colon V_i^* \to V_{i^*}$ in such a way that the composition

$$V_i \xrightarrow{\phi_{i^*}^{-1}} V_{i^*}^* \xrightarrow{\phi_i^*} V_i^{**}$$

coincides with δ_{V_i}. We leave it as an exercise to the reader that for the category $\mathcal{R}ep_f(\mathfrak{sl}_2)$ (with the standard balancing), considered in Example 2.2.8, this is impossible: regardless of the choice of ϕ_i, the composition above will give $\tilde{\delta}$ rather than δ.

When $\mathcal{C} = \mathcal{R}ep_f G$, G—a finite group, an important role in the representation theory of G is played by the regular representation $R = Fun(G)$. This is a G-bimodule (i.e., an element of $\mathcal{C}^{\boxtimes 2}$); as a bimodule, it is isomorphic to $\bigoplus V_i \boxtimes V_i^*$. More generally, any rigid semisimple abelian category automatically gives rise to an object $R \in \mathrm{ind}{-}\mathcal{C}^{\boxtimes 2}$ defined by

$$R = \bigoplus_{i \in I} V_i \boxtimes V_i^*, \tag{2.4.7}$$

where we denote by $\mathrm{ind}{-}\mathcal{C}^{\boxtimes 2}$ the category whose objects are infinite sums of the form $\sum A_i \boxtimes B_i$ with $A_i, B_i \in \mathcal{C}$. The object R does not depend on the choice of representatives V_i of the isomorphism classes: if we choose another representative $\tilde{V}_i$, then one has a canonical isomorphism $R \xrightarrow{\sim} \tilde{R}$. In particular, if $\mathcal{C}$ is balanced, then R is symmetric in the following sense: we say that an object $R \in \mathrm{ind}{-}\mathcal{C}^{\boxtimes 2}$ is *symmetric* if we are given an isomorphism $s\colon R^{\mathrm{op}} \xrightarrow{\sim} R$ such that $ss^{\mathrm{op}} = \mathrm{id}$. Here the functor op is defined by $(A \boxtimes B)^{\mathrm{op}} = B \boxtimes A$.

For the object R defined by (2.4.7), s can be written explicitly as follows: for every i, choose an isomorphism $\varphi_i\colon V_i^* \xrightarrow{\sim} V_{i^*}$, and define $\psi_i\colon V_i^* \boxtimes V_i \to V_{i^*} \boxtimes V_{i^*}^*$ by $\psi_i = \varphi_i \boxtimes \delta_{V_i}(\varphi_i^*)^{-1}$. This obviously does not depend on the choice of φ_i (since $\mathrm{End}\, V_i = k$). Now, define

$$s = \bigoplus \psi_i \colon R \xrightarrow{\sim} R^{\mathrm{op}}. \tag{2.4.8}$$

We will also frequently use the following object of $\mathrm{ind}{-}\mathcal{C}$ associated with R :

$$H = \otimes(R) = \bigoplus V_i \otimes V_i^*. \tag{2.4.9}$$

The previous arguments show that H is also canonically isomorphic to $\bigoplus V_i^* \otimes V_i$. Also, H is canonically isomorphic to $H^* = \bigoplus V_i^{**} \otimes V_i^*$. Both these properties will be used many times in the sequel.

One of the main goals of the next lectures will be to answer the following question: given a semisimple abelian category $\mathcal{C}$ and a symmetric object $R \in \mathrm{ind}{-}\mathcal{C}^{\boxtimes 2}$, what extra data are needed to reconstruct the structure of a ribbon category on $\mathcal{C}$? We will give an answer to this question in Chapter 5.

Finally, when working with an object $R \in \mathrm{ind}-\mathcal{C}^{\boxtimes 2}$ (not necessarily the one given by (2.4.7)), it is convenient to use the following notation: if $R = \bigoplus A_i \boxtimes B_i$, then we write $\mathrm{Hom}(X, R^{(1)} \otimes Y \otimes R^{(2)})$ instead of $\bigoplus_i \mathrm{Hom}(X, A_i \otimes Y \otimes B_i)$, etc. If R is symmetric, then the superscripts "(1)", "(2)" can be omitted, and we will just write $\mathrm{Hom}(X, R \otimes Y \otimes R)$. Similarly, we can use several copies $R_1, \dots, R_m$ of R and write, for example,

$$\begin{aligned} R_1 \otimes X \otimes R_2 \otimes R_2 \otimes R_1 &= R_1^{(1)} \otimes X \otimes R_2^{(1)} \otimes R_2^{(2)} \otimes R_1^{(2)} \\ &= \bigoplus_{i,j} A_i \otimes X \otimes A_j \otimes B_j \otimes B_i. \end{aligned}$$