CHAPTER 1

Braided Tensor Categories

In this chapter, we give basic definitions related to braided tensor categories, such as commutativity and associativity isomorphisms, and state various coherence theorems. We also give two basic examples of tensor categories: the category $\mathcal{C}(\mathfrak{g})$ of representations of a quantum group $U_q(\mathfrak{g})$ (for formal q, i.e., over the field of rational functions in q) and Drinfeld's category $\mathcal{D}(\mathfrak{g}, \varkappa), \varkappa \notin \mathbb{Q}$, which as abelian category coincides with the category of finite-dimensional complex representations of a simple Lie algebra $\mathfrak{g}$, but has commutativity and associativity isomorphisms defined in terms of asymptotics of the Knizhnik–Zamolodchikov equations.

1.1. Monoidal tensor categories

We will work over a field k of characteristic 0. Recall the following definition (for details, see, e.g., [**Mac**]).

Definition 1.1.1. A category $\mathcal{C}$ is an *additive category* over k if the following conditions are satisfied:

(i) All $\mathrm{Hom}_{\mathcal{C}}(U,V) \equiv \mathrm{Mor}_{\mathcal{C}}(U,V)$ are k-vector spaces and the compositions

$$\mathrm{Hom}_{\mathcal{C}}(V,W) \times \mathrm{Hom}_{\mathcal{C}}(U,V) \to \mathrm{Hom}_{\mathcal{C}}(U,W), \qquad (\varphi, \psi) \mapsto \varphi \circ \psi$$

are k-bilinear ($U, V, W \in \mathrm{Ob}\,\mathcal{C}$).

(ii) There exists a zero object $0 \in \mathrm{Ob}\,\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(0,V) = \mathrm{Hom}_{\mathcal{C}}(V,0) = 0$ for all $V \in \mathrm{Ob}\,\mathcal{C}$.

(iii) Finite direct sums exist in $\mathcal{C}$.

An additive category $\mathcal{C}$ is called *abelian* if it satisfies the following condition:

(iv) Every morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(U,V)$ has a kernel $\ker\varphi \in \mathrm{Mor}\,\mathcal{C}$ and a cokernel $\mathrm{coker}\,\varphi \in \mathrm{Mor}\,\mathcal{C}$. Every morphism is a composition of an epimorphism followed by a monomorphism. If $\ker\varphi = 0$, then $\varphi = \ker(\mathrm{coker}\,\varphi)$; if $\mathrm{coker}\,\varphi = 0$, then $\varphi = \mathrm{coker}(\ker\varphi)$.

Informally speaking, an abelian category is an additive category in which we can use the notions of a kernel and a cokernel of a morphism in the same way as, say, in the category of vector spaces over k.

Functors between additive categories will always be assumed to be k-linear on the spaces of morphisms.

Example 1.1.2. The following categories are abelian:

(i) The category of k-vector spaces $\mathcal{V}ec(k)$ and the category of finite dimensional k-vector spaces $\mathcal{V}ec_f(k)$.

(ii) The category $\mathcal{R}ep(A)$ of representations of a k-algebra A.

(iii) The category $\mathcal{R}ep(G)$ of representations of a group G over k.

DEFINITION 1.1.3. An object U in an abelian category $\mathcal{C}$ is called *simple* if any injection $V \hookrightarrow U$ is either 0 or isomorphism.

An abelian category $\mathcal{C}$ is called *semisimple* if any object V is isomorphic to a direct sum of simple ones:

$$V \simeq \bigoplus_{i\in I} N_i V_i$$

where V_i are simple objects, I is the set of isomorphism classes of non-zero simple objects in $\mathcal{C}$, $N_i \in \mathbb{Z}_+$ and only a finite number of N_i are non-zero.

Throughout the book we will consider only semisimple categories satisfying the following additional property:

$$\operatorname{End} V_i = k \quad \text{for all } i \in I. \tag{1.1.1}$$

This automatically holds if the base field k is algebraically closed (Schur's Lemma). It easily follows from the definition that

$$\operatorname{Hom}(V_i, V_j) = 0, \quad i \neq j. \tag{1.1.2}$$

Let $\mathcal{C}$ be an abelian category. We want to define a notion of tensor product $\otimes$ on $\mathcal{C}$ with natural associativity and commutativity properties. Let us first digress to a well-known example. Recall that for a k-vector space A, a bilinear map $\cdot\colon A\times A \to A$ is called *associative* if any two expressions

$$(a_1 \cdot a_2) \cdot \big((a_3 \cdot a_4) \cdots a_n\big), \tag{1.1.3}$$

obtained by placing brackets in the product $a_1 \cdot a_2 \cdots a_n$, are equal. This definition of associativity is equivalent to the usual one.

THEOREM 1.1.4. *A bilinear map* $\cdot\colon A \times A \to A$ *is associative if and only if*

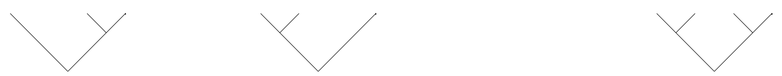
$$(a_1 \cdot a_2) \cdot a_3 = a_1 \cdot (a_2 \cdot a_3) \tag{1.1.4}$$

for all $a_1, a_2, a_3 \in A$.

Of course, this fact is well-known. Nevertheless, we will give a proof because its method will be useful later.

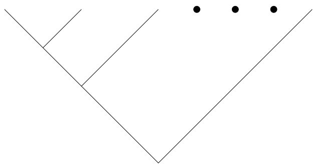
PROOF. Let A_n be the set of all ways of placing brackets (grammatically correctly) in the product $a_1 \cdot a_2 \cdots a_n$, i.e., the set of all expressions of the form (1.1.3). Let us connect two points in A_n if they can be obtained from one another by applying (once) the relation (1.1.4). Our goal is to show that this makes A_n a *connected* graph. It is easy to see that the elements of A_n are in 1-1 correspondence with binary trees with n leaves, e.g.

$a_1 \cdot (a_2 \cdot a_3)$ $\qquad$ $(a_1 \cdot a_2) \cdot a_3$ $\qquad$ $(a_1 \cdot a_2) \cdot (a_3 \cdot a_4)$

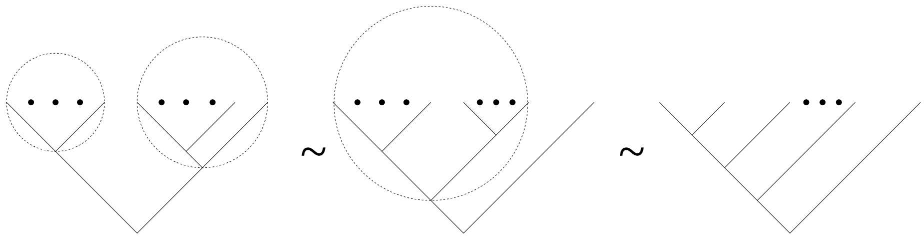


We will prove by induction that every tree can be reduced by applying (1.1.4) to the tree

$$(\cdots((a_1 \cdot a_2) \cdot a_3) \cdots) \cdot a_n$$



This is clear by the figure



The first equivalence $\sim$ is obtained by applying the inductive assumption to the two smaller subtrees and then the elementary relation (1.1.4); the second one—by the inductive assumption. $\square$

EXERCISE 1.1.5. Prove that $|A_n| = \frac{1}{n}\binom{2n-2}{n-1}$. This is known as the $(n-1)$st *Catalan number* c_{n-1}.
Hint: $|A_n| = \sum_{k=1}^{n-1} |A_k||A_{n-k}|$.

Now, what kind of associativity to require for a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ for an abelian category $\mathcal{C}$? We cannot assert that

$$(U \otimes V) \otimes W = U \otimes (V \otimes W)$$

for all $U, V, W \in \mathcal{C}$ because this is not true even in the category of vector spaces $\mathcal{V}ec(k)$. We may ask that instead

$$(U \otimes V) \otimes W \simeq U \otimes (V \otimes W)$$

but now this condition is too weak. For example, in $\mathcal{V}ec_f(k)$ every two vector spaces of equal dimension are isomorphic. What we need is the existence of a canonical isomorphism.

Recall that for two functors $F, G\colon \mathcal{C} \to \mathcal{C}'$ a *functorial morphism* $\varphi\colon F \to G$ is a collection of morphisms

$$\varphi_U\colon F(U) \to G(U), \quad U \in \operatorname{Ob}\mathcal{C},$$

such that for every $f \in \operatorname{Hom}_{\mathcal{C}}(U, V)$ the following diagram is commutative:

$$\begin{tikzcd} F(U) \arrow[r, "F(f)"] \arrow[d, "\varphi_U"'] & F(V) \arrow[d, "\varphi_V"] \\ G(U) \arrow[r, "G(f)"'] & G(V) \end{tikzcd} \tag{1.1.5}$$

.

Functorial morphisms are also called "natural transformations". Sometimes they are also referred to as "canonical morphisms"; however, we will use the latter term in a slightly different situation (see below).

EXAMPLE 1.1.6. (i) In $\mathcal{V}ec_f(k)$ there exists a functorial isomorphism between a vector space and its double dual, $V \simeq V^{**}$, but there is no functorial isomorphism between V and V^*.

(ii) Let G be a group, $g \in Z(G)$ (center of G), V be a G-module. Then the action of g is a functorial isomorphism $V \simeq V$ (of the identity functor in $\mathcal{R}ep(G)$).

DEFINITION 1.1.7. A *monoidal category* is a category $\mathcal{C}$ with the following additional data:

(i) a bifunctor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;

(ii) a functorial isomorphism

$$\alpha_{UVW} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W) \tag{1.1.6}$$

(*associativity isomorphism*) of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;

(iii) a *unit object* $\mathbf{1} \in \operatorname{Ob} \mathcal{C}$ and functorial isomorphisms

$$\lambda_V : \mathbf{1} \otimes V \xrightarrow{\sim} V, \tag{1.1.7}$$

$$\rho_V : V \otimes \mathbf{1} \xrightarrow{\sim} V \tag{1.1.8}$$

for $V \in \operatorname{Ob} \mathcal{C}$.

They have to satisfy the following

(iv) *Associativity axiom.* Suppose X_1 and X_2 are two expressions obtained from $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ by inserting $\mathbf{1}$'s and brackets, e.g.,

$$(V_1 \otimes \mathbf{1}) \otimes \big((V_2 \otimes V_3) \otimes \cdots \otimes V_n\big). \tag{1.1.9}$$

Then all isomorphisms $\varphi \colon X_1 \xrightarrow{\sim} X_2$, composed of α's, λ's, ρ's and their inverses, have to be equal.

Usually, we will consider additive monoidal categories. In this case, we additionally require that $\otimes$ is bilinear on the spaces of morphisms and that

(v) $\mathbf{1}$ is a simple object in $\mathcal{C}$ and $\operatorname{End}_{\mathcal{C}}(\mathbf{1}) = k$.

The associativity axiom 1.1.7(iv) implies that we have a uniquely defined functorial isomorphism $X_1 \xrightarrow{\sim} X_2$. We will call such an isomorphism "canonical". Note that a composition of two such canonical isomorphisms is again a canonical isomorphism.

EXAMPLE 1.1.8. The following categories are monoidal:

(i) $\mathcal{V}ec(k)$ and $\mathcal{V}ec_f(k)$;

(ii) The category $\mathcal{R}ep(\mathfrak{g})$ (respectively $\mathcal{R}ep_f(\mathfrak{g})$) of representations (respectively finite dimensional representations) of a Lie algebra $\mathfrak{g}$ over k.

(iii) Let A be a *bialgebra* over k, i.e., a k-algebra provided with algebra homomorphisms $\Delta \colon A \to A \otimes A$ (comultiplication) and $\varepsilon \colon A \to k$ (counit) satisfying

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta, \tag{1.1.10}$$

$$(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}\,. \tag{1.1.11}$$

Then the category $\mathcal{R}ep(A)$ of representations of A (as a k-algebra) is a monoidal category. ($\otimes$ is the tensor product of vector spaces and $\mathbf{1} = k$ with the following action of A: $x(v \otimes w) := \Delta(x)(v \otimes w)$, $xc := \varepsilon(x)c$ for $x \in A$, $v \in V$, $w \in W$, $c \in k$, $V, W \in \operatorname{Ob} \mathcal{R}ep(A)$.)

In fact, the axioms (1.1.10, 1.1.11) can be deduced from the requirement that the category $\mathcal{R}ep(A)$ be monoidal.

THEOREM 1.1.9 (MacLane Coherence Theorem). *Suppose we are given the data* $(\mathcal{C}, \otimes, \alpha, \lambda, \rho)$ *as above. Then* $\mathcal{C}$ *is a monoidal category if and only if the following properties are satisfied.*

(i) Pentagon axiom. *For any* $V_i \in \mathrm{Ob}\,\mathcal{C}$ $(i = 1, 2, 3, 4)$ *the following diagram is commutative*

$$\begin{tikzcd}[column sep=small]
& ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \arrow[dl, "\alpha_{1,2,3} \otimes \mathrm{id}_4"'] \arrow[dr, "\alpha_{12,3,4}"] & \\
(V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 \arrow[d, "\alpha_{1,23,4}"] & & (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \arrow[d, "\alpha_{1,2,34}"'] \\
V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) \arrow[rr, "\mathrm{id}_1 \otimes \alpha_{2,3,4}"'] & & V_1 \otimes (V_2 \otimes (V_3 \otimes V_4))
\end{tikzcd}$$

(ii) Triangle axiom. *For any* $V_1, V_2 \in \mathrm{Ob}\,\mathcal{C}$ *the diagram*

$$\begin{tikzcd}
(V_1 \otimes \mathbf{1}) \otimes V_2 \arrow[rr, "\alpha"] \arrow[dr, "\rho \otimes \mathrm{id}"'] & & V_1 \otimes (\mathbf{1} \otimes V_2) \arrow[dl, "\mathrm{id} \otimes \lambda"] \\
& V_1 \otimes V_2 &
\end{tikzcd}$$

is commutative.

We do not want to give the proof of this theorem here as it is rather technical and does not help in any way to understand the structure of a monoidal category; instead, we refer the reader to [**Mac**, Sect. VII.2]. Also, we want to stress that this theorem is in a sense just a technical tool, similar to describing a group by generators and relations. Of course, such a description may be useful, but a "global" description—such as a group of automorphisms of some object—is much more important, and in many cases, more useful. For monoidal categories, such a global description is given by the associativity axiom 1.1.7(iv).

REMARK 1.1.10. In analogy with Theorem 1.1.4, we can reformulate the statement of the above theorem as follows (forgetting about $\mathbf{1}$). Suppose that we are given $(\mathcal{C}, \otimes, \alpha)$ as above, satisfying the pentagon axiom 1.1.9(i). Recall the definition of the set A_n from the proof of Theorem 1.1.4; then Theorem 1.1.4 claims that if we add to A_n an edge for every elementary associativity equality (1.1.4) then A_n becomes a connected graph: $\pi_0(A_n) = 1$.

Let us construct a 2-dimensional complex by gluing to A_n pentagons corresponding to each commutative diagram of the form 1.1.9(i) and squares corresponding to the functoriality condition (1.1.5) for α. Then the Coherence Theorem 1.1.9 states that the resulting 2-complex is simply connected: $\pi_1(A_n) = 0$. It is easy to see that this implies that the category $\mathcal{C}$ is monoidal. In fact, Stasheff has shown that this 2-complex is a 2-skeleton of the sphere S^{n-3}.

Similarly, the Coherence Theorem in the form stated—with the unit object and isomorphisms λ, ρ—is equivalent to the statement that the 2-complex M_n whose vertices are labeled by the expressions of the form (1.1.9), is simply connected. Note that M_n has infinite number of vertices; however, this does not cause any problems.

In a monoidal category we can use the canonical isomorphisms to "identify" all expressions of the form (1.1.9) and write the tensor product without brackets, in the same way we do with tensor product of vector spaces. Here is an appropriate formalism (warning: this terminology is not standard).

DEFINITION 1.1.11. For a category $\mathcal{C}$, define its *universal cover* $\hat{\mathcal{C}}$ to be the category with:

Objects: An object of $\hat{\mathcal{C}}$ is a collection $(V_\alpha, \varphi_{\alpha\alpha'})_{\alpha,\alpha'\in A}$, where A is any set, V_α are objects of $\mathcal{C}$, and $\varphi_{\alpha\alpha'} \in \mathrm{Hom}_{\mathcal{C}}(V_{\alpha'}, V_\alpha)$ are isomorphisms satisfying $\varphi_{\alpha_1\alpha_2}\varphi_{\alpha_2\alpha_3} = \varphi_{\alpha_1\alpha_3}$.

Morphisms: For two objects $X = (V_\alpha, \varphi_{\alpha\alpha'})_{\alpha,\alpha'\in A}$ and $Y = (V_\beta, \varphi_{\beta\beta'})_{\beta,\beta'\in B}$ of $\hat{\mathcal{C}}$, the space of morphisms $\mathrm{Hom}_{\hat{\mathcal{C}}}(X, Y)$ is defined to be the set of all collections $(f_{\alpha\beta} \in \mathrm{Hom}_{\mathcal{C}}(V_\alpha, V_\beta))_{\alpha\in A,\beta\in B}$ satisfying $f_{\alpha\beta}\varphi_{\alpha\alpha'} = f_{\alpha'\beta}$ and $\varphi_{\beta'\beta}f_{\alpha\beta} = f_{\alpha\beta'}$.

In other words, objects of $\hat{\mathcal{C}}$ are collections of objects of $\mathcal{C}$ related by canonical isomorphisms, and morphisms are collections of morphisms compatible with these isomorphisms. In particular, taking collections containing only one object, we see that $\mathcal{C}$ itself is a full subcategory of $\hat{\mathcal{C}}$.

LEMMA 1.1.12. *The category $\hat{\mathcal{C}}$ is equivalent to $\mathcal{C}$.*

The proof of this lemma is left to the reader as an exercise.

Now we can say that in a monoidal category, for given $V_1, \dots, V_n$, the collection of all expressions of the form (1.1.9) with the canonical isomorphisms between them forms an object of the category $\hat{\mathcal{C}}$, which we will denote $V_1 \otimes \dots \otimes V_n$. Since the categories $\hat{\mathcal{C}}$ and $\mathcal{C}$ are equivalent, we can as well think of $V_1 \otimes \dots \otimes V_n$ as an object of $\mathcal{C}$ without bothering about brackets. In other words, in a monoidal category iterations of $\otimes$ give a functor $\mathcal{C}^{\times n} \to \mathcal{C}$, which we also denote by $\otimes$. From now on, we will frequently use this remark and omit parentheses and the associativity and unit isomorphisms in our formulas. This is no less rigorous than omitting the parentheses in tensor products of vector spaces, which usually is considered as too trivial to mention. Readers who feel uneasy about this can spell out all the formulas, writing all the associativity and unit isomorphisms explicitly.

More standard (but in our opinion, more artificial) way to deal with the same problem is to use the notion of a strict category.

DEFINITION 1.1.13. A monoidal category $\mathcal{C}$ is called *strict* if

$$V \otimes \mathbf{1} = V, \quad \mathbf{1} \otimes V = V, \quad (V_1 \otimes V_2) \otimes V_3 = V_1 \otimes (V_2 \otimes V_3) \tag{1.1.12}$$

for all objects $V, V_i \in \mathrm{Ob}\,\mathcal{C}$ and all α's, λ's, ρ's are the identity isomorphisms.

In a strict monoidal category one can write multiple tensor products $V_1 \otimes V_2 \otimes \dots \otimes V_n$ without bothering about brackets.

Note that the category of vector spaces $\mathcal{V}ec(k)$ is not strict. However, we have the following result.

THEOREM 1.1.14 (MacLane). *Every monoidal category is equivalent to a strict one.*

PROOF. See, e.g., [**Ka**, Sect. XI.5] or [**Mac**]. $\square$

Finally, let us note that if $\mathcal{C}$ is an additive monoidal category, then the functor $\otimes: \mathcal{C}^{\times n} \to \mathcal{C}$ is polylinear on the spaces of morphisms. For many applications, it is convenient to define the notion of "tensor product of additive categories" as below.

DEFINITION 1.1.15. Let $\mathcal{C}_1, \mathcal{C}_2$ be additive categories over k. Their tensor product $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ is the category with the following objects and morphisms:

$$\mathrm{Ob}(\mathcal{C}_1 \boxtimes \mathcal{C}_2) = \text{finite sums of the form } \bigoplus X_i \boxtimes Y_i, \quad X_i \in \mathrm{Ob}\,\mathcal{C}_1, Y_i \in \mathrm{Ob}\,\mathcal{C}_2,$$

$$\mathrm{Hom}_{\mathcal{C}_1 \boxtimes \mathcal{C}_2}(\bigoplus X_i \boxtimes Y_i, \bigoplus X'_j \boxtimes Y'_j) = \bigoplus_{i,j} \mathrm{Hom}(X_i, X'_j) \otimes \mathrm{Hom}(Y_i, Y'_j).$$

One easily sees that $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ is again an additive category, and that for any additive category $\mathcal{C}$ there is a natural bijection between additive functors $\mathcal{C}_1 \boxtimes \mathcal{C}_2 \to \mathcal{C}$ and *bilinear* functors $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}$. In a similar way, one can define $\mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n$. More generally, for every finite set A (not necessarily ordered) and a collection of additive categories $\mathcal{C}_a, a \in A$, we can define $\boxtimes_{a \in A} \mathcal{C}_a$. As usual, we will use the notation $\mathcal{C}^{\boxtimes n}$, $\mathcal{C}^{\boxtimes A}$ if all the $\mathcal{C}_i$ are equal to $\mathcal{C}$ (respectively, all the $\mathcal{C}_a$ are equal to $\mathcal{C}$). It is also convenient to define $\mathcal{C}^0 = \mathcal{C}^\emptyset = \mathcal{V}ec_f(k)$.

Using this definition, we see that a structure of a monoidal category on an additive category $\mathcal{C}$ gives rise to a collection of functors $\otimes: \mathcal{C}^{\boxtimes n} \to \mathcal{C}$, satisfying some natural compatibility conditions (see [**De2**]). We will discuss this in detail later.

1.2. Braided tensor categories

In the category of vector spaces $\mathcal{V}ec(k)$ we also have a commutativity isomorphism $\sigma_{VW}: V \otimes W \xrightarrow{\sim} W \otimes V$. It is naturally compatible with the isomorphisms α, λ, ρ, and $\sigma^2 = \mathrm{id}$. We would like to axiomatize this kind of structure; however, we want to allow that $\sigma^2 \neq \mathrm{id}$, since this is what happens in most interesting examples. It turns out that the simplest way to formulate the compatibility axioms is based on the so-called braid group.

DEFINITION 1.2.1. A *braid* in n strands is an isotopy class of a union of n non-intersecting segments of smooth curves (strands) in $\mathbb{R}^3$ with end points $\{1, \ldots, n\} \times \{0\} \times \{0, 1\}$, such that for each of these strands the third coordinate z is strictly increasing from 0 to 1 (so strands are considered as "going up").

An example of a braid is depicted in Figure 1.1 below. All braids form a group called the *braid group* in n strands B_n.

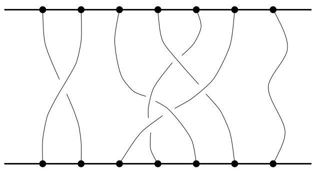


FIGURE 1.1. A braid in 7 strands.

We multiply two braids by putting one of them on the top of the other:

$$b'b'' = \begin{array}{|c|}\hline b' \\ \hline b'' \\ \hline\end{array} \quad . \tag{1.2.1}$$

The unit element is the braid shown in Figure 1.2.

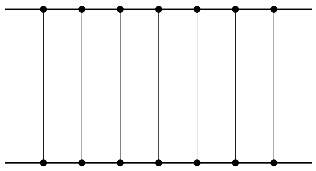


FIGURE 1.2. The unit of B_7.

THEOREM 1.2.2 (E. Artin). *The braid group B_n has generators b_i, $i = 1, \dots, n-1$ (see Figure 1.3) and relations* (braid relations):

$$b_i b_j = b_j b_i, \quad |i-j| > 1, \tag{1.2.2}$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}. \tag{1.2.3}$$

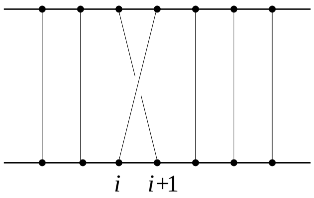
$b_i =$ 

FIGURE 1.3. Generators of the braid group.

PROOF. See, e.g., [**B1**, Theorem 1.8]. □

Let $\mathcal{C}$ be a monoidal category with functorial isomorphisms

$$\sigma_{VW} \colon V \otimes W \xrightarrow{\sim} W \otimes V \tag{1.2.4}$$

for all objects $V, W \in \operatorname{Ob} \mathcal{C}$. The functoriality of σ means that

$$\sigma_{V'W'}(f \otimes g) = (g \otimes f)\sigma_{VW} \tag{1.2.5}$$

for any two morphisms $f \colon V \to V'$ and $g \colon W \to W'$.

For given objects $V_1, \dots, V_n$ in $\mathcal{C}$, we consider all expressions of the form

$$\big((V_{i_1} \otimes V_{i_2}) \otimes (\mathbf{1} \otimes V_{i_3})\big) \otimes \cdots \otimes V_{i_n}, \tag{1.2.6}$$

obtained from $V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n}$ by inserting some $\mathbf{1}$'s and brackets, where $(i_1, \dots, i_n)$ is a permutation of $\{1, \dots, n\}$. To any composition of α's, λ's, ρ's, σ's and their inverses, acting on the element (1.2.6), we assign an element of the braid group B_n as follows. To α, λ and ρ we assign 1, to $\sigma_{V_{i_k} V_{i_{k+1}}}$ the generator b_k.

For example, both isomorphisms

$$(V_1 \otimes V_2) \otimes V_3 \xrightarrow{\sim} (V_2 \otimes V_1) \otimes V_3$$

and

$$(V_3 \otimes V_2) \otimes V_1 \xrightarrow{\sim} (V_2 \otimes V_3) \otimes V_1$$

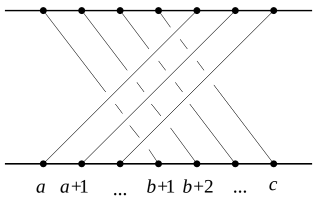
correspond to the element b_1.

More generally, to the isomorphism

$$\begin{aligned}(1.2.7)\quad \sigma_{AB}\colon \cdots\otimes(V_{i_a}\otimes\cdots\otimes V_{i_b})\otimes(V_{i_{b+1}}\otimes\cdots\otimes V_{i_c})\otimes\cdots\\ \xrightarrow{\sim}\cdots\otimes(V_{i_{b+1}}\otimes\cdots\otimes V_{i_c})\otimes(V_{i_a}\otimes\cdots\otimes V_{i_b})\otimes\cdots\end{aligned}$$

flipping two blocks A and B we assign the braid

$b_{AB} =$



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To a composition of morphisms we associate the product of the corresponding braids. In view of (1.2.1), one can say that "operators act from bottom to top".

DEFINITION 1.2.3. A *braided tensor category* (BTC) $\mathcal{C}$ is a category with $\otimes$, $\mathbf{1}$, α, λ, ρ, σ as above, such that for any two expressions X_1, X_2 of the form (1.2.6) and $\varphi\colon X_1 \xrightarrow{\sim} X_2$ obtained by composing α's, λ's, ρ's, σ's and their inverses, φ depends only on its image in the braid group B_n.

The functorial isomorphism σ is called the *commutativity isomorphism.*

Unless stated otherwise, we will always assume that $\mathcal{C}$ is abelian, in which case we also require that $\otimes$ is bilinear and that $\mathbf{1}$ is simple, with $\mathrm{End}(\mathbf{1}) = k$.

REMARK 1.2.4. Any braided tensor category is a monoidal category.

THEOREM 1.2.5 (Coherence Theorem). *The data* $(\mathcal{C},\otimes,\mathbf{1},\alpha,\lambda,\rho,\sigma)$ *constitute a braided tensor category iff they satisfy the following axioms*:

(i) Pentagon axiom 1.1.9(i).

(ii) Triangle axiom 1.1.9(ii).

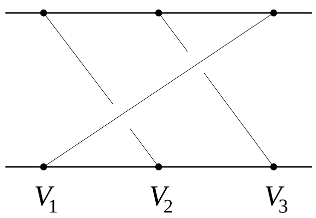
(iii) Hexagon axioms:

(a) *For any* $V_i \in \mathrm{Ob}\,\mathcal{C}$ $(i = 1,2,3)$ *the following diagram is commutative*

$$\begin{tikzcd}
 & V_1\otimes(V_2\otimes V_3) \arrow[r, "\sigma_{1,23}"] & (V_2\otimes V_3)\otimes V_1 \arrow[dr, "\alpha_{2,3,1}"] & \\
(V_1\otimes V_2)\otimes V_3 \arrow[ur, "\alpha_{1,2,3}"] \arrow[dr, "\sigma_{1,2}\otimes \mathrm{id}_3"'] & & & V_2\otimes(V_3\otimes V_1) \\
 & (V_2\otimes V_1)\otimes V_3 \arrow[r, "\alpha_{2,1,3}"'] & V_2\otimes(V_1\otimes V_3) \arrow[ur, "\mathrm{id}_2\otimes\sigma_{1,3}"'] &
\end{tikzcd}$$

(b) *The same as* (a) *but with* σ^{-1} *instead of* σ.

PROOF. It is easy to see that these axioms are necessary; for example, the hexagon axiom claims equality of two isomorphisms, both corresponding to the following element of the braid group:



We refer the reader to [**Mac**] for the proof that these axioms are sufficient. □

EXERCISE 1.2.6. Show that in a BTC, the braid relation (1.2.3) gives rise to identity of two morphisms $V_1 \otimes V_2 \otimes V_3 \xrightarrow{\sim} V_3 \otimes V_2 \otimes V_1$ and deduce this identity from the pentagon and hexagon axioms. This identity is called the *Yang–Baxter equation*.

DEFINITION 1.2.7. A braided tensor category $\mathcal{C}$ is called a *symmetric tensor category* (STC) if all isomorphisms σ satisfy $\sigma_{WV}\sigma_{VW} = \mathrm{id}_{V\otimes W}$.

EXAMPLE 1.2.8. (i) The categories of vector spaces $\mathcal{V}ec(k)$ and $\mathcal{V}ec_f(k)$ are symmetric tensor categories.

(ii) Let A be a *cocommutative* bialgebra (i.e., a bialgebra with $\Delta = \Delta^{\mathrm{op}} := P\Delta$, where $P(a \otimes b) = b \otimes a$). Then the category $\mathcal{R}ep(A)$ of A-modules is a symmetric tensor category, with the same σ as in $\mathcal{V}ec(k)$ (that is, $\sigma = P$).

(iii) Let A be a *quasi-triangular* bialgebra, i.e., a bialgebra possessing a *universal R-matrix* R: an invertible element $R \in A \otimes A$ satisfying

$$\Delta^{\mathrm{op}}(x) = R\Delta(x)R^{-1}, \qquad x \in A, \tag{1.2.8}$$

$$(\mathrm{id} \otimes \Delta)R = R_{13}R_{12}, \tag{1.2.9}$$

$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23}. \tag{1.2.10}$$

(Here we use the standard notation $R_{13} := \sum a_i \otimes 1 \otimes b_i \in A \otimes A \otimes A$, etc., for $R = \sum a_i \otimes b_i \in A \otimes A$.)

Then, taking $\sigma_{VW} = PR\colon V \otimes W \xrightarrow{\sim} W \otimes V$ (where $P(v \otimes w) = w \otimes v$), we see that $\mathcal{R}ep(A)$ is a braided tensor category. As in Example 1.1.8(iii), the axioms for R are in fact equivalent to the requirement that $\mathcal{R}ep(A)$ be a BTC (for details, see, e.g., [**Ka**]).

EXERCISE 1.2.9. Show that the universal R-matrix satisfies

$$(\varepsilon \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \varepsilon)(R) = 1, \tag{1.2.11}$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{1.2.12}$$

and interpret these identities in terms of the BTC structure of $\mathcal{R}ep(A)$. Eq. (1.2.12) is called the *quantum Yang–Baxter equation* (QYBE), cf. Exercise 1.2.6.

Finally, we will need a notion of functors between braided tensor categories which agree in a certain sense with the tensor product.

DEFINITION 1.2.10. Let $\mathcal{C}_1, \mathcal{C}_2$ be braided tensor categories. A *tensor functor* from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a pair (F, J), where F is a functor $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ and J is a functorial isomorphism

$$J_{U,V}\colon F(U \otimes V) \xrightarrow{\sim} F(U) \otimes F(V)$$

such that:

(i) $F(\alpha_1) = \alpha_2$, where α_1, α_2 are the associativity isomorphisms in $\mathcal{C}_1, \mathcal{C}_2$, respectively, and

$$F(\alpha_1)\colon F((V_1 \otimes V_2) \otimes V_3) \to F(V_1 \otimes (V_2 \otimes V_3))$$

is considered as an operator

$$(F(V_1) \otimes F(V_2)) \otimes F(V_3) \to F(V_1) \otimes (F(V_2) \otimes F(V_3))$$

using J.

(ii) $F(\sigma_1) = \sigma_2$, where σ_1, σ_2 are the commutativity isomorphisms in $\mathcal{C}_1, \mathcal{C}_2$.

The notions of *functorial isomorphisms* between two tensor functors and of *equivalence* of braided tensor categories are defined in an obvious way.

In the next two sections we will give two major examples of braided tensor categories: the category of representations of a quantum group and the Drinfeld category arising from the Knizhnik–Zamolodchikov equations.

1.3. Quantum groups

We assume that the reader is familiar with the basics of representation theory of simple Lie algebras and quantum groups, so our exposition is very brief. More detailed information can be found in [**Hum**] (classical theory) and in [**L2**], [**CP**], [**Jan**] (quantum groups). Let us first fix the notation.

$\mathfrak{g}$ will be a finite dimensional simple Lie algebra over $\mathbb{C}$,

$\mathfrak{h}$—its Cartan subalgebra,

$\Delta \subset \mathfrak{h}^*$—the root system,

$\Pi = \{\alpha_1, \dots, \alpha_r\} \subset \Delta$—the set of simple roots,

$h_i = \alpha_i^\vee \in \mathfrak{h}$—the dual roots (coroots),

$A = (a_{ij})_{1 \le i,j \le r}$—the Cartan matrix, $a_{ij} = (\alpha_i^\vee, \alpha_j)$,

$P \subset \mathfrak{h}^*$—the weight lattice,

$P_+ \subset P$—the cone of dominant integer weights,

$Q \subset \mathfrak{h}^*$—the root lattice,

$Q^\vee \subset \mathfrak{h}$—the dual root lattice (coroot lattice).

Let $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ be an invariant bilinear form on $\mathfrak{g}$ normalized by $\langle\!\langle \alpha, \alpha \rangle\!\rangle = 2$ for *short* roots α. Then $d_i := \langle\!\langle \alpha_i, \alpha_i \rangle\!\rangle / 2 \in \mathbb{Z}_+$ for all $i = 1, \dots, r$.

Finally, $\mathbb{C}_q$ will be the field $\mathbb{C}(q^{1/|P/Q|})$ where q is a formal variable.

Definition 1.3.1. The *quantum group* $U_q(\mathfrak{g})$ is the associative algebra over $\mathbb{C}_q$ with generators e_i, f_i $(i = 1, \dots, r)$, q^h $(h \in Q^\vee)$ and relations

$$q^{h'} q^{h''} = q^{h'+h''}, \quad q^0 = 1, \quad h', h'' \in Q^\vee, \tag{1.3.1}$$

$$q^h e_i q^{-h} = q^{(h,\alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h,\alpha_i)} f_i, \tag{1.3.2}$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q^{d_i} - q^{-d_i}}, \tag{1.3.3}$$

and the Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i e_i^{1-a_{ij}-k} e_j e_i^k = 0, \qquad i \neq j, \tag{1.3.4}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i f_i^{1-a_{ij}-k} f_j f_i^k = 0, \qquad i \neq j, \tag{1.3.5}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_i := \frac{[n]_i!}{[k]_i!\,[n-k]_i!}, \quad [n]_i! := [1]_i[2]_i \cdots [n]_i, \quad [n]_i := \frac{q^{d_i n} - q^{-d_i n}}{q^{d_i} - q^{-d_i}}. \tag{1.3.6}$$

$U_q(\mathfrak{g})$ has the following *Hopf algebra* structure:

comultiplication

$$\Delta(q^h) = q^h \otimes q^h, \tag{1.3.7}$$

$$\Delta(e_i) = e_i \otimes q^{d_i h_i} + 1 \otimes e_i, \tag{1.3.8}$$

$$\Delta(f_i) = f_i \otimes 1 + q^{-d_i h_i} \otimes f_i, \tag{1.3.9}$$

counit

$$\varepsilon(q^h) = 1, \;\; \varepsilon(e_i) = \varepsilon(f_i) = 0, \tag{1.3.10}$$

and antipode

$$\gamma(q^h) = q^{-h}, \;\; \gamma(e_i) = -e_i q^{-d_i h_i}, \;\; \gamma(f_i) = -q^{d_i h_i} f_i. \tag{1.3.11}$$

It also has a *quasi-triangular* structure: an invertible element R in a certain completed tensor product $U_q(\mathfrak{g}) \widehat{\otimes} U_q(\mathfrak{g})$ satisfying (1.2.8–1.2.10). R has the form

$$R = q^{\sum x_i \otimes x^i}(1 + \cdots), \tag{1.3.12}$$

where $\{x_i\}$ and $\{x^i\}$ are dual bases in $\mathfrak{h}$ with respect to $\langle\langle \cdot, \cdot \rangle\rangle$ and the terms in the brackets belong to $U_q(\mathfrak{g})^+ \otimes U_q(\mathfrak{g})^-$. Here $U_q(\mathfrak{g})^+$ (respectively $U_q(\mathfrak{g})^-$) is the subalgebra of $U_q(\mathfrak{g})$ generated by the elements q^h, e_i (respectively q^h, f_i).

Let $\mathcal{C}(\mathfrak{g})$ be the category of finite dimensional representations of $U_q(\mathfrak{g})$ over $\mathbb{C}_q$ which have a weight decomposition:

$$V = \bigoplus_{\lambda \in P} V^\lambda, \qquad q^h|_{V^\lambda} = q^{(h,\lambda)} \operatorname{id}_{V^\lambda}. \tag{1.3.13}$$

Then $\mathcal{C}(\mathfrak{g})$ is a braided tensor category with the usual $\otimes$, $\mathbf{1} = \mathbb{C}_q$, α, λ, ρ — the same as in the category of vector spaces, and

$$\sigma_{VW} = PR_{VW} \colon V \otimes W \xrightarrow{\sim} W \otimes V. \tag{1.3.14}$$

This is a well-defined operator even though R lies in a completion of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, i.e., the sum in (1.3.12) becomes finite when applied to any vector from $V \otimes W$.

The category $\mathcal{C}(\mathfrak{g})$ is semi'simple with the same simple objects as $\mathcal{R}ep_f(\mathfrak{g})$, i.e., V_λ ($\lambda \in P_+$), and moreover,

$$V_\lambda \otimes V_\mu = \sum_{\nu \in P_+} N_{\lambda\mu}^\nu V_\nu \tag{1.3.15}$$

with the same multiplicities $N_{\lambda\mu}^{\nu}$ as in $\mathcal{R}ep_f(\mathfrak{g})$ [**L1**]. However, $\mathcal{C}(\mathfrak{g})$ is not equivalent to $\mathcal{R}ep_f(\mathfrak{g})$ as a BTC (or even as a monoidal category—see Remark 1.3.3 below).

There is a version of $\mathcal{C}(\mathfrak{g})$ (due to Lusztig [**L2**]) in which we allow q to be a complex number instead of a formal variable. Let $\mathcal{A} = \mathbb{Z}[q^{\pm 1/|P/Q|}]$ and $U_q(\mathfrak{g})_{\mathbb{Z}}$ be the $\mathcal{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by the elements

$$(1.3.16)\qquad e_i^{(n)} = \frac{e_i^n}{[n]_i!},\ \ f_i^{(n)} = \frac{f_i^n}{[n]_i!},\ \ q^h, \qquad n = 1,2,\dots,\ i = 1,\dots,r,\ h \in Q^\vee.$$

Fix $\varkappa \in \mathbb{C}^\times$ and consider $\mathbb{C}$ as an $\mathcal{A}$-module via the homomorphism

$$\mathcal{A} \to \mathbb{C}, \quad q^a \mapsto e^{a\pi \mathrm{i}/m\varkappa}$$

where

$$(1.3.17)\qquad m := \max d_i = \frac{\langle\!\langle \alpha, \alpha \rangle\!\rangle}{2} \quad \text{for a long root } \alpha.$$

Note that $m = 1$ when the Lie algebra $\mathfrak{g}$ is simply-laced.

Now we define

$$(1.3.18)\qquad U_q(\mathfrak{g})|_{q=e^{\pi \mathrm{i}/m\varkappa}} := U_q(\mathfrak{g})_{\mathbb{Z}} \otimes_{\mathcal{A}} \mathbb{C}.$$

The quasi-triangular Hopf algebra structure of $U_q(\mathfrak{g})$ can be defined for $U_q(\mathfrak{g})_{\mathbb{Z}}$ and hence for all algebras $U_q(\mathfrak{g})|_{q=e^{\pi \mathrm{i}/m\varkappa}}$ [**L2**]. Thus, we can define a braided tensor category $\mathcal{C}(\mathfrak{g}, \varkappa)$—the category of finite dimensional representations of $U_q(\mathfrak{g})|_{q=e^{\pi \mathrm{i}/m\varkappa}}$ over $\mathbb{C}$ possessing a weight decomposition. Note that there is a subtlety in defining the notion of weight decomposition for $\varkappa \in \mathbb{Q}$; we will discuss it in detail in Section 3.3.

THEOREM 1.3.2 (Lusztig [**L2**]). *If $\varkappa \notin \mathbb{Q}$ then the category $\mathcal{C}(\mathfrak{g}, \varkappa)$ is semisimple with the same simple objects and multiplicities $N_{\lambda\mu}^{\nu}$ (1.3.15) as $\mathcal{R}ep_f(\mathfrak{g})$.*

REMARK 1.3.3. $\mathcal{C}(\mathfrak{g}, \varkappa)$ is not equivalent to $\mathcal{R}ep_f(\mathfrak{g})$ as a monoidal category. In fact, for $\mathfrak{g} = \mathfrak{sl}_n$, $\varkappa \notin \mathbb{Q}$, the categories $\mathcal{C}(\mathfrak{g}, \varkappa)$ and $\mathcal{C}(\mathfrak{g}, \varkappa')$ are equivalent as monoidal categories iff $\varkappa = \pm\varkappa'$. (See [**KW**].)

1.4. Drinfeld category

In this section we use the same notation as in the previous one. Let $\langle \cdot, \cdot \rangle$ be an invariant bilinear form on $\mathfrak{g}$ normalized by $\langle \alpha, \alpha \rangle = 2$ for *long* roots α. Define the element

$$(1.4.1)\qquad \Omega := \sum a_i \otimes a^i \in U(\mathfrak{g}) \otimes U(\mathfrak{g}),$$

where $\{a_i\}$ and $\{a^i\}$ are dual bases in $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle$. Recall that the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ has a comultiplication determined by

$$(1.4.2)\qquad \Delta(x) := x \otimes 1 + 1 \otimes x \in U(\mathfrak{g}) \otimes U(\mathfrak{g}), \quad x \in \mathfrak{g}.$$

LEMMA 1.4.1. (i) $[\Delta(x), \Omega] = 0$ *for all* $x \in \mathfrak{g}$.

(ii) $(1 \otimes \Delta)\Omega = \Omega_{12} + \Omega_{13}$ *in* $U(\mathfrak{g})^{\otimes 3}$, *where* $\Omega_{12} = \sum a_i \otimes a^i \otimes 1$, *etc., as before.*

(iii) *For any* $V, W \in \mathcal{R}ep_f(\mathfrak{g})$, Ω *is diagonalizable on* $V \otimes W$. *If* $V_\nu \subset V_\lambda \otimes V_\mu$ $(\lambda, \mu, \nu \in P_+)$, *then* Ω *acts on* V_ν *as the constant*

$$(1.4.3)\qquad \frac{1}{2}\big(\langle \nu, \nu + 2\rho \rangle - \langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle\big).$$

PROOF. Part (i) is a standard exercise; (ii) is obvious; (iii) follows from the identity $\Omega = \frac{1}{2}(\Delta(D) - D \otimes 1 - 1 \otimes D)$, where

$$D = \sum a_i a^i \tag{1.4.4}$$

is the Casimir element. Details can be found, for example, in [**Hum**]. $\square$

Consider the *Knizhnik–Zamolodchikov equation*

$$\varkappa \frac{d}{dt} f(t) = \left(\frac{\Omega_{12}}{t} + \frac{\Omega_{23}}{t-1} \right) f(t) \tag{KZ$'_3$}$$

for a function $f\colon (0,1) \to V_1 \otimes V_2 \otimes V_3$, $V_i \in \mathcal{R}ep_f(\mathfrak{g})$. The equation (KZ$'_3$) has two singular points: $t = 0$ and $t = 1$, both of them regular.

DEFINITION 1.4.2. Let $v \in V_1 \otimes V_2 \otimes V_3$ be an eigenvector of Ω_{12}, i.e., $\Omega_{12} v = \lambda v$, $\lambda \in \mathbb{C}$. We say that a solution $f(t)$ of (KZ$'_3$) is an *asymptotic solution* around 0 corresponding to v, and write $f(t) \sim t^{\lambda/\varkappa} v$, if

$$f(t) = t^{\lambda/\varkappa}(v + r(t)) \tag{1.4.5}$$

for some vector valued function $r(t)$ analytic in a neighborhood of 0 and vanishing at $t = 0$.

LEMMA 1.4.3. *If $\varkappa \notin \mathbb{Q}$ then for every eigenvector $v \in V_1 \otimes V_2 \otimes V_3$ of Ω_{12}, there exists a unique asymptotic solution of* (KZ$'_3$) *around* 0 *corresponding to v. Extended by linearity, this correspondence gives an isomorphism*

$$\phi_0\colon \Gamma(V_1, V_2, V_3) \to V_1 \otimes V_2 \otimes V_3, \tag{1.4.6}$$

where $\Gamma(V_1, V_2, V_3)$ is the space of solutions of (KZ$'_3$) *on the interval* $(0,1)$.

PROOF. Follows from the standard results on asymptotics of solutions of ordinary differential equations (see, e.g., [**CL**, Chapter 4]) and the fact that Ω_{12} is diagonalizable and eigenvalues of $\Omega_{12}/\varkappa$ can not differ by a non-zero integer. The latter statement follows from Lemma 1.4.1(iii) and the irrationality of $\varkappa$. $\square$

REMARK 1.4.4. The assumption $\varkappa \notin \mathbb{Q}$ is essential, the map ϕ_0 may have poles for rational values of $\varkappa$.

Similarly, we can define the notion of an asymptotic solution of (KZ$'_3$) around 1 and get an isomorphism

$$\phi_1 : \Gamma(V_1, V_2, V_3) \to V_1 \otimes V_2 \otimes V_3. \tag{1.4.7}$$

Note that it easily follows from Lemma 1.4.1(i) that $\Gamma(V_1, V_2, V_3)$ is a $\mathfrak{g}$-module, and that ϕ_0, ϕ_1 are morphisms of $\mathfrak{g}$-modules.

THEOREM 1.4.5 (Drinfeld [**Dr1**]). *Let $\mathcal{D}(\mathfrak{g}, \varkappa)$ for $\varkappa \notin \mathbb{Q}$ be the category of all finite dimensional representations of $\mathfrak{g}$ over $\mathbb{C}$ with the usual $\otimes$, $\mathbf{1} = \mathbb{C}$, λ and ρ, but with*

$$\alpha_{V_1 V_2 V_3} = \phi_1 \phi_0^{-1}\colon (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3) \tag{1.4.8}$$

and

$$\sigma_{V_1 V_2} = P e^{\pi \mathrm{i} \Omega/\varkappa}\colon V_1 \otimes V_2 \to V_2 \otimes V_1. \tag{1.4.9}$$

Then $\mathcal{D}(\mathfrak{g}, \varkappa)$ is a braided tensor category.

The proof of this theorem will be sketched below. The associativity morphism α (1.4.8) is called the *Drinfeld associator*. The following theorem relates the Drinfeld category $\mathcal{D}(\mathfrak{g}, \varkappa)$ with the category of representations of quantum group, considered in the previous section.

THEOREM 1.4.6 (Drinfeld, Kazhdan–Lusztig). *For $\varkappa \notin \mathbb{Q}$, the categories $\mathcal{D}(\mathfrak{g}, \varkappa)$ and $\mathcal{C}(\mathfrak{g}, \varkappa)$ are equivalnet as braided tensor categories.*

The proof of this theorem is extremely difficult. It was proved by Drinfeld ([**Dr1**], [**Dr4**]) over the ring of formal power series in $1/\varkappa$ (which corresponds to the infinitesimal neighborhood of the point $q = 1$). Kazhdan and Lusztig in their series of papers [**KL**] proved that for simply-laced Lie algebras this theorem holds for numeric values of $\varkappa$ provided that $\varkappa \notin \mathbb{Q}_+$. Note that for $\varkappa \in \mathbb{Q}_-$, our definition of the category $\mathcal{D}(\mathfrak{g}, \varkappa)$ does not work; Kazhdan and Lusztig use a different definition, based on representation theory of affine Lie algebras, which works for all $\varkappa \neq 0$, and which coincides with our definition for $\varkappa \notin \mathbb{Q}$. Finally, the case of non-simply laced Lie algebras was treated by Lusztig in [**L3**].

Let us explain why Drinfeld's category $\mathcal{D}(\mathfrak{g}, \varkappa)$ is indeed a BTC, and why the equation (KZ_3') is so special. To do so, we will need to introduce more general *Knizhnik–Zamolodchikov equations*

$$\varkappa \frac{\partial}{\partial z_i} f = \Big(\sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j}\Big) f, \qquad 1 \leq i \leq n, \tag{KZ_n}$$

for a function $f(z_1, \ldots, z_n)$ with values in $V_1 \otimes \cdots \otimes V_n$, $V_i \in \mathcal{R}ep_f(\mathfrak{g})$.

LEMMA 1.4.7. *The KZ equations are compatible. In other words, this system of equations defines a flat connection: locally, every $v \in V_1 \otimes \cdots \otimes V_n$ can be extended uniquely to a solution.*

PROOF. We have to check that:

$$\Big[\varkappa \frac{\partial}{\partial z_i} - \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j}, \varkappa \frac{\partial}{\partial z_k} - \sum_{j \neq k} \frac{\Omega_{kj}}{z_k - z_j}\Big] = 0.$$

This is verified by an explicit calculation, based on the identity $[\Omega_{12} + \Omega_{13}, \Omega_{23}] = 0$, which easily follows from Lemma 1.4.1 (i), (ii). □

Knizhnik–Zamolodchikov equations were introduced in [**KZ**] in the study of Wess–Zumino–Witten model of Conformal Field Theory. These equations also play important role in representation theory of affine Lie algebras and have been studied in many papers (see [**EFK**] and references therein). We will return to these equations later when we discuss the relations between tensor categories and modular functors.

Here is another argument which explains why these equations are so important. Suppose we want to write a system of differential equations of the form

$$\frac{\partial}{\partial z_i} f = \Big(\sum_{j \neq i} r_{ij}(z_i, z_j)\Big) f, \tag{1.4.10}$$

where f takes values in the tensor product $V_1 \otimes \cdots \otimes V_n$ and $r_{ij}(z_i, z_j)$ acts on $V_i \otimes V_j$. Moreover, let us assume that V_i are representations of $\mathfrak{g}$ and that we have some function $r(z, w) \in \mathfrak{g} \otimes \mathfrak{g}$ such that $r_{ij}(z_i, z_j)$ is just the action of $r(z_i, z_j)$ in $V_i \otimes V_j$. All such $r(z, w)$ giving a flat connection were determined by Belavin and

Drinfeld [**BD**]. In particular, they showed that up to certain equivalences, all non-degenerate $r(z,w)$ which give a compatible system of differential equations (1.4.10) are of the form

$$r(z,w) = \frac{\Omega}{z-w} + \text{regular}.$$

Therefore, in a sense, (KZ_n) is the simplest compatible system of the form (1.4.10).

REMARK 1.4.8. Let f be a solution of (KZ_n). Then it is easy to see that:

(i) $(\sum_{i=1}^n \partial/\partial z_i)f = 0$, hence f is translation invariant: $f(z_1+c,\dots,z_n+c) = f(z_1,\dots,z_n)$ for all $c \in \mathbb{C}$.

(ii) $\varkappa(\sum_{i=1}^n z_i(\partial/\partial z_i))f = (\sum_{i<j}\Omega_{ij})f$ and the operator $\sum_{i<j}\Omega_{ij}$ commutes with all Ω_{ij}. Therefore, $f(cz_1,\dots,cz_n) = c^{\varkappa^{-1}\sum\Omega_{ij}} f(z_1,\dots,z_n)$.

This remark shows that (KZ_n) can be reduced to an equation in $n-2$ variables. For example, when $n=3$, setting $z_1=0$, $z_2=t$, $z_3=1$, we see that (KZ_3) reduces to (KZ_3').

Now, let us return to explaining why the category $\mathcal{D}(\mathfrak{g},\varkappa)$ is a BTC. We will try to convey the main idea, referring the reader to the original papers of Drinfeld for the formal proof (which is not difficult).

Let $V_1,\dots,V_n$ be finite-dimensional representations of $\mathfrak{g}$. Define

$$\begin{aligned}\Gamma(V_1,V_2,\dots,V_n) &= \{\text{space of solutions of the equations } (\mathrm{KZ}_n) \text{ on } \Delta^n\},\\ \Delta^n &= \{(z_1,\cdots,z_n)\in\mathbb{R}^n \mid z_1<\cdots<z_n\}.\end{aligned} \tag{1.4.11}$$

As before, we note that $\Gamma(V_1,V_2,\dots,V_n)$ is a $\mathfrak{g}$-module, and that for every point $z^0 = (z_1^0,\dots,z_n^0)\in\Delta^n$, the map $f\mapsto f(z^0)$ gives an isomorphism $\Gamma(V_1,V_2,\dots,V_n)\to V_1\otimes\cdots\otimes V_n$.

Our next goal is to define an analogue of the morphisms ϕ_0,ϕ_1, corresponding to taking asymptotics of a solution. Unlike the case $n=3$, when our system can be reduced to an ODE with two regular singularities, for general n we have more choices.

Let X be any bracket arrangement in the tensor product $V_1\otimes\cdots\otimes V_n$, e.g., $X = (V_1\otimes V_2)\otimes(V_3\otimes V_4)$. As was discussed in the proof of Theorem 1.1.4, such expressions are in bijection with binary trees. For each such X, we define an isomorphism

$$\phi_X\colon \Gamma(V_1,V_2,\dots,V_n)\to V_1\otimes\cdots\otimes V_n \tag{1.4.12}$$

as follows. First, we choose a curve

$$\gamma\colon (0,1)\to\{z=(z_1,\dots,z_n)\in\mathbb{R}^n | z_1<\cdots<z_n\}$$

such that as $t\to 0$, we have $z_i(t)-z_j(t)\sim t^{d-d_{ij}}$, where

$$\begin{aligned} d_{ij} &= \text{depth of the minimal subtree in } X \text{ containing both } V_i \text{ and } V_j,\\ d &= \max d_{ij} = \text{depth of } X.\end{aligned}$$

In other words, the closer are V_i and V_j in the tree, the faster $z_i - z_j$ approaches zero. For example, for $X = (V_1\otimes V_2)\otimes(V_3\otimes V_4)$, we can take the curve $\gamma(t) = (0,t,1,1+t)$. For $X = ((V_1\otimes V_2)\otimes V_3)\otimes V_4$, we can take $\gamma(t) = (0,t^2,t,1)$.

Now, let us restrict the system of equations (KZ_n) to this curve, i.e., rewrite them in terms of the variable t. We claim that this gives an ODE with a regular singularity at $t=0$; thus, we can define an isomorphism ϕ_X as the operation of taking asymptotics as $t\to 0$, similarly to Lemma 1.4.3. Moreover, this isomorphism does not depend on the choice of the curve as long as it satisfies the conditions formulated above. Details can be found in [**EFK**, Lecture 8].

EXERCISE 1.4.9. Prove that for $n = 3, X = (V_1 \otimes V_2) \otimes V_3$, the morphism ϕ_X coincides with the morphism ϕ_0 defined in Lemma 1.4.3, and for $X = V_1 \otimes (V_2 \otimes V_3)$, ϕ_X coincides with ϕ_1.

Now we are ready to prove the pentagon axiom 1.1.9(i). Let us consider the following diagram:

(1.4.13)

$$\begin{tikzcd}
 & ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \arrow[dl] \arrow[dr] & \\
(V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 \arrow[d] & \Gamma(V_1, V_2, V_3, V_4) \arrow[l] \arrow[r] \arrow[u] \arrow[dl] \arrow[dr] & (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \arrow[d] \\
V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) \arrow[rr] & & V_1 \otimes (V_2 \otimes (V_3 \otimes V_4))
\end{tikzcd}$$

where the arrows forming the outer sides of the pentagon are the same as in the pentagon axiom, with the associativity given by the Drinfeld associator (1.4.8), and the arrows originating at $\Gamma(V_1, V_2, V_3, V_4)$ are the morphisms ϕ_X defined above.

LEMMA 1.4.10. *Each of the five triangles forming the diagram above is commutative.*

Obviously, this lemma immediately implies that the Drinfeld associator satisfies the pentagon axiom. Let us prove, for example, that the following triangle is commutative:

(1.4.14)

$$\begin{tikzcd}
 & \Gamma(V_1, V_2, V_3, V_4) \arrow[dl] \arrow[dr] & \\
((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \arrow[rr] & & (V_1 \otimes V_2) \otimes (V_3 \otimes V_4)
\end{tikzcd} .$$

Commutativity of this triangle is equivalent to saying that the map

$$\phi_Y \phi_X^{-1} : X = ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \to Y = (V_1 \otimes V_2) \otimes (V_3 \otimes V_4),$$

obtained from comparing asymptotics of the KZ equations in 4 variables, coincides with the map

$$\phi_{Y'} \phi_{X'}^{-1} : X' = (V_1' \otimes V_3) \otimes V_4 \to Y' = V_1' \otimes (V_3 \otimes V_4),$$

obtained from the KZ equations in 3 variables, with $V_1' = V_1 \otimes V_2$.

Let us make two observations. First, the maps

$$\begin{aligned} \phi_X &: \Gamma(V_1, V_2, V_3, V_4) \to ((V_1 \otimes V_2) \otimes V_3) \otimes V_4, \\ \phi_Y &: \Gamma(V_1, V_2, V_3, V_4) \to (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \end{aligned}$$

are determined only by the behavior of the solutions for $|z_1 - z_2| \ll |z_1 - z_3|$, $|z_1 - z_2| \ll |z_1 - z_4|$. More formally, let us define

$$D_\varepsilon = \{(z_1, \ldots, z_4) \in \mathbb{R}^4 \, | z_1 < \cdots < z_4, |z_1 - z_2| < \varepsilon |z_1 - z_3|, \ |z_1 - z_2| < \varepsilon |z_1 - z_4|\}.$$

Then for every $\varepsilon > 0$, the curves $\gamma(t)$ used in the definition of the operators ϕ_X, ϕ_Y satisfy $\gamma(t) \in D_\varepsilon$ for t small enough.

Second, let us consider the following system of equations:

$$(\mathrm{KZ}'_4)\qquad \begin{aligned}
\varkappa\frac{\partial}{\partial z_1}f &= \Big(\frac{\Omega_{12}}{z_1-z_2}+\frac{\Omega_{13}}{w_2-z_3}+\frac{\Omega_{14}}{w_2-z_4}\Big)f,\\
\varkappa\frac{\partial}{\partial z_2}f &= \Big(\frac{\Omega_{12}}{z_2-z_1}+\frac{\Omega_{23}}{w_2-z_3}+\frac{\Omega_{24}}{w_2-z_4}\Big)f,\\
\varkappa\frac{\partial}{\partial z_3}f &= \Big(\frac{\Omega_{13}}{z_3-w_2}+\frac{\Omega_{23}}{z_3-w_2}+\frac{\Omega_{34}}{z_3-z_4}\Big)f,\\
\varkappa\frac{\partial}{\partial z_4}f &= \Big(\frac{\Omega_{14}}{z_4-w_2}+\frac{\Omega_{24}}{z_4-w_2}+\frac{\Omega_{34}}{z_4-z_3}\Big)f,
\end{aligned}$$

where $w_2 := (z_1 + z_2)/2$. The only difference between these equations and the equations (KZ_4) is that in the right hand side we have replaced everywhere z_1 and z_2 with w_2, except in the terms $\Omega_{12}/(z_1-z_2)$ and $\Omega_{12}/(z_2-z_1)$. One can show that this is still a compatible system of equations. Moreover, since—as was mentioned before—asymptotics only depend on the behavior of solutions for $|z_1 - z_2| \ll |z_1 - z_3|$, $|z_1 - z_2| \ll |z_1 - z_4|$, this replacement does not change the asymptotics: the operator $\phi_Y\phi_X^{-1}$ defined from the equations (KZ'_4) coincides with that obtained from (KZ_4). (Of course, this requires a rigorous proof which is not too difficult.)

Let $w_1 = (z_1 - z_2)/2$, $w_2 = (z_1 + z_2)/2$, then $\frac{\partial}{\partial w_1} = \frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2}$ and $\frac{\partial}{\partial w_2} = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}$. The system (KZ'_4) is equivalent to the following system of equations:

$$\varkappa\frac{\partial}{\partial w_1}f = 2\frac{\Omega_{12}}{w_1}f + \Big(\frac{\Omega_{13}-\Omega_{23}}{w_2-z_3}+\frac{\Omega_{14}-\Omega_{24}}{w_2-z_4}\Big)f,$$

and

$$\begin{aligned}
\varkappa\frac{\partial}{\partial w_2}f &= \Big(\frac{\Omega_{13}+\Omega_{23}}{w_2-z_3}+\frac{\Omega_{14}+\Omega_{24}}{w_2-z_4}\Big)f,\\
\varkappa\frac{\partial}{\partial z_3}f &= \Big(\frac{\Omega_{13}+\Omega_{23}}{z_3-w_2}+\frac{\Omega_{34}}{z_3-z_4}\Big)f,\\
\varkappa\frac{\partial}{\partial z_4}f &= \Big(\frac{\Omega_{14}+\Omega_{24}}{z_4-w_2}+\frac{\Omega_{34}}{z_4-z_3}\Big)f.
\end{aligned}$$

Let us call (KZ''_4) the system obtained from this one by discarding the last term in the first equation. Again, this does not change the asymptotics for $|z_1 - z_2| \ll |z_1 - z_3|$, $|z_1 - z_2| \ll |z_1 - z_4|$. Then the first equation becomes

$$\varkappa\frac{\partial}{\partial w_1}f = 2\frac{\Omega_{12}}{w_1}f$$

which is equivalent to the KZ system in 2 variables. Noting that $\Omega_{13} + \Omega_{23} = \Omega_{V_1\otimes V_2, V_3}$, we see that this system can be written as a sum of two systems of equations:

$$(1.4.15)\qquad \mathrm{KZ}''_4(V_1, V_2, V_3, V_4) = \mathrm{KZ}_2(V_1, V_2) \otimes \mathrm{id} + \mathrm{KZ}_3(V_1 \otimes V_2, V_3, V_4).$$

Moreover, by virtue of Lemma 1.4.1(i), the right-hand sides of these two systems of equations commute. It is not difficult to show from this that the operators $\phi_Y\phi_X^{-1}$ defined from the equations $\mathrm{KZ}''_4(V_1, V_2, V_3, V_4)$ coincide with those defined from the equations $\mathrm{KZ}_3(V_1 \otimes V_2, V_3, V_4)$. This proves that the triangle (1.4.14) is commutative. In a similar way, one proves commutativity of all the other triangles in (1.4.13) and thus, proves the lemma.

It is clear that the crucial step in the above arguments was that the KZ equations have certain nice "factorization" properties, like (1.4.15). The seeming informality and awkwardness of these arguments can be avoided if we use the appropriate language—namely, of local systems with regular singularities on the appropriate moduli spaces—which we will do later (see Section 6.5).

Finally, we leave it to the reader to prove the unit and hexagon axioms. The proofs are easy once you notice that the Drinfeld's associator is trivial if one of the representations V_1, V_2, V_3 is equal to $\mathbb{C}$, and that the commutativity constraint (1.4.9) relates the asymptotics of a solution of the KZ equations in 2 variables in the zone $z_1 < z_2$ and its analytic continuation to the zone $z_2 < z_1$ via the upper half-plane.