MSP Math Sheet 7

Problem 1:

(a) Convergence in trace-norm is stronger, since

$\|\cdot\|_{op} \leq \|\cdot\|_{tr}$ always holds (as $\sup_{n\in\mathbb{N}} |\lambda_n| \leq \sum_{n\in\mathbb{N}} |\lambda_n|$),

but the other way around does not hold:

Consider (in some basis in L^2) the sequence $(A_n)_{n\in\mathbb{N}} : L^2 \to L^2$,

$$A_n = \begin{pmatrix} \frac{1}{n} & & & & & \\ & \frac{1}{n} & & & & \\ & & \ddots & & & \\ & & & \frac{1}{n} & & \\ & 0 & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \\ & & & & & & & \ddots \end{pmatrix} \quad \left.\right\} n \text{ times}$$

So $\|A_n\|_{tr} = n \cdot \frac{1}{n} = 1$

$\|A_n\|_{op} = \frac{1}{n} \xrightarrow{n\to\infty} 0.$

(b) Claim: $\|\mu^{\Psi_n} - |\varphi\rangle\langle\varphi|\|_{tr} \xrightarrow{n\to\infty} 0 \iff \|\mu^{\Psi_n} - |\varphi\rangle\langle\varphi|\|_{op} \xrightarrow{n\to\infty} 0$

Proof: "$\Rightarrow$" trivial (see a))

"$\Leftarrow$"

$A := \mu^{\Psi_n} - |\varphi\rangle\langle\varphi|$ Call its eigenvalues $\lambda_n, n\in\mathbb{N}$

μ^{Ψ_n} density matrix $\Rightarrow \operatorname{Tr}\mu^{\Psi_n} = 1$ and $0 \leq \mu^{\Psi_n} \leq 1$.

$|\varphi\rangle\langle\varphi|$ projector of rank 1 $\Rightarrow$ eigenvalues 0 and 1 with multiplicity 1

$\Rightarrow$ A can have at most one negative eigenvalue, (* more on this see next page) say λ_1.

$\operatorname{Tr}(A) = 0 \Rightarrow \sum_{n\in\mathbb{N}} \lambda_n = 0$

$0 = -|\lambda_1| + \sum_{n\geq 2} |\lambda_n|$ So λ_1 is the largest eigenvalue and

$\Rightarrow \sum_{n\in\mathbb{N}} |\lambda_n| = 2|\lambda_1| \Rightarrow \|A\|_{op} = |\lambda_1| = \frac{1}{2}\|A\|_{tr}$ $\square$

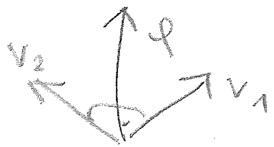
⇒ Why has $\hat{O} := \mu^{\varphi} - |\varphi\rangle\langle\varphi|$ only one negative eigenvalue?

Assume there are 2 normalized eigenvectors s.t.

$$\hat{O}\,|v^1\rangle = \lambda_1 |v^1\rangle$$

$$\hat{O}\,|v^2\rangle = \lambda_2 |v^2\rangle \qquad \lambda_1, \lambda_2 < 0$$

by s.-a. of $\hat{O}$, we know $\langle v^1, v^2\rangle = 0$



There is a linear combination $\alpha v^1 + \beta v^2$ perpendicular to φ!

But this implies

1.) $$\langle \alpha v^1 + \beta v^2 |\, \mu^{\varphi} - |\varphi\rangle\langle\varphi|(\alpha v^1 + \beta v^2)\rangle = \langle \alpha v^1 + \beta v^2 |\, \mu^{\varphi}(\alpha v^1 + \beta v^2)\rangle \geq 0$$

2.) $$\langle \alpha v^1 + \beta v^2 |\, \hat{O}(\alpha v^1 + \beta v^2)\rangle$$

$$= \langle \alpha v^1 + \beta v^2 \,|\, \alpha \lambda_1 v^1 + \beta \lambda_2 v^2\rangle$$

$$= |\alpha|^2 \lambda_1 + |\beta|^2 \lambda_2 < 0$$ ↯

Problem 2:

(omit index t here) $\qquad \alpha = \langle \psi, q_1 \psi \rangle \qquad \beta = \langle \psi, q_1 q_2 \psi \rangle$

$$\partial_t \psi = -i \mathcal{H} \psi \qquad \partial_t q_j = -i [h_j, q_j]$$

$$\partial_t \beta = \langle i\mathcal{H}\psi, q_1 q_2 \psi \rangle + \langle \psi, q_1 q_2 (-i\mathcal{H}\psi) \rangle$$
$$+ \langle \psi, -i [h_1, q_1] q_2 \psi \rangle + \langle \psi, q_1 (-i) [h_2, q_2 \psi \rangle$$
$$= i \langle \psi, \underbrace{[\mathcal{H} - h_1 - h_2, q_1 q_2]}_{(*)} \psi \rangle$$

remaining from $(*)$: $\left[\frac{1}{N} \left(\sum_{k=3}^{N} V(x_1 - x_k) + V(x_2 - x_k) + \frac{1}{N} V(x_1 - x_2) \right. \right.$
$$\left. - V * |\varphi|^2(x_1) - V * |\varphi|^2(x_2), q_1 q_2 \right]$$

$$= \underbrace{\frac{2i}{N} \langle \psi, [V_{12}, q_1 q_2] \psi \rangle}_{\text{in norm} \leq \frac{2}{N} |\langle \psi, V q_1 q_2 \psi \rangle| \leq \frac{2}{N} \|V\|_\infty \underbrace{\|q_1 q_2 \psi\|}_{= \sqrt{\beta}}} + 2i \langle \psi, [V_{13} - V * |\varphi|^2(x_1), q_1 q_2] \psi \rangle$$

$$\leq C \frac{1}{N} \sqrt{\beta} \leq C \left(\beta + \frac{1}{N^2} \right)$$

<u>using</u>: $ab \leq \frac{a^2 + b^2}{2}$

Second term: Insert identities $1 = p_j + q_j$

$$= 2i \left[\langle q_2 \psi, (p_1 + q_1)(p_3 + q_3)(\Delta V) q_1 (p_3 + q_3) q_2 \psi \rangle - \text{c.c.} \right]$$

where $\Delta V := V_{13} - V * |\varphi|^2(x_1)$

of the 8 terms, 4 vanish because they are equal to their c.c. using self-adjointness and symmetry

Three types of terms remain:

Type I: $p_1 p_3 \, \Delta V \, q_1 p_3$

II: $p_1 p_3 \, \Delta V \, q_1 q_3$

III: $q_1 p_3 \, \Delta V \, q_1 q_3$ (and similar)

We will estimate those in norm now.

$$|I| \leq |\langle q_2\psi, p_1 p_3 \Delta V \, q_1 p_3 \, q_2\psi\rangle| = 0$$

because, like in the lecture,

$$p_3 V(x_1 - x_3) p_3 = p_3 V * |\varphi|^2(x_1) p_3 .$$

$$|III| \leq |\langle q_2\psi, q_1 p_3 \Delta V \, q_1 q_3 \, q_2\psi\rangle|$$

$$= |\langle q_1 q_2\psi, \underbrace{p_3 \Delta V \, q_3}_{\text{bounded}} \, q_1 q_2 \psi\rangle| \leq C \cdot \|q_1 q_2 \psi\|^2 = C \cdot \beta$$

$$|II| \leq |\langle q_2\psi, p_1 p_3 V_{13} \, q_1 q_3 q_2 \psi\rangle|$$

$$= \left| \frac{1}{N-2} \langle q_2\psi, \Big(\sum_{j=3}^{N} p_1 p_j V_{1j} q_j\Big) q_1 q_2 \psi\rangle \right| \quad \text{using symmetry}$$

$$\overset{C.S.}{\leq} \frac{1}{N-2} \|\chi\| \cdot \underbrace{\|q_1 q_2 \psi\|}_{\sqrt{\beta}}$$

where $\chi = \sum_{j=3}^{N} q_j V_{1j} p_1 p_j q_2 \psi$

$$\|\chi\|^2 = \sum_{j,k=3}^{N} \langle q_2\psi, p_1 p_j V_{1j} q_j \, q_k V_{1k} p_1 p_k q_2 \psi\rangle$$

diagonal: (j=k) (N-2 terms):

$$\sum_{j=3}^{N} \langle q_2\psi, \underbrace{p_1 p_j V_{1j} q_j V_{1j} p_1 p_j}_{\text{bounded}} q_2 \psi\rangle \leq C \cdot N \cdot \|q_2\psi\|^2 \leq C \cdot N \cdot \alpha \leq C \frac{N}{N} \leq C$$

rest:

$$\Big|\sum_{j \neq k} \langle q_k q_2 \psi, p_1 p_j V_{1j} V_{1k} p_1 p_k \, q_j q_2 \psi\rangle$$

$$\leq N^2 \|V\|_\infty^2 \|q_k q_2 \psi\| \|q_2 q_j \psi\| \leq C N^2 \beta$$

$$\Rightarrow |II| \leq \frac{1}{N-2} (C + \tilde{C} N^2 \beta)^{1/2} \sqrt{\beta}$$

$$\leq \frac{\sqrt{\beta}}{N} C \left(\tilde{C} + \tilde{\tilde{C}} N \sqrt{\beta}\right) \quad \text{using } \sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$$

$$\leq C\left(\beta + \frac{1}{N}\sqrt{\beta}\right) \leq C\left(\beta + \frac{1}{N^2}\right) \quad \text{using } ab \leq \frac{a^2+b^2}{2}$$

All in all, $\dot{\beta}(t) \leq C\left(\beta(t) + \frac{1}{N^2}\right)$, so by Grönwall,

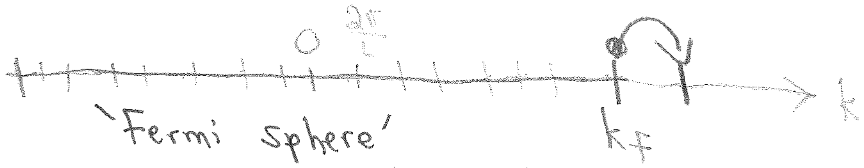
$\beta(t) \sim \frac{1}{N^2}$ if $\beta(0)$ does.

Problem 3:

a) possible k-values: $k=(k_1,\dots,k_d)$, $k_j=\frac{2\pi\cdot n}{L}$, $n\in\mathbb{N}_0$

$$\Rightarrow \psi_G = \bigwedge_{|k|\le k_F} e^{ikx} \cdot \text{Normalization}$$

b) $d=1$:



Lowest excited state: excite • $\quad E\propto k^2$!

$$\Delta E = \left(k_F+\frac{2\pi}{L}\right)^2 - k_F^2 = \frac{4\pi}{L}k_F + \left(\frac{2\pi}{L}\right)^2 \sim k_F \sim N$$

c) <u>Claim:</u> $\psi(x_1,\dots,x_N,y) = \frac{1}{\sqrt{L}}\psi_G(x_1,\dots,x_N)e^{iPy}$
is ground state of $\mathcal{H}=\sum_{k=1}^{N}-\Delta_{x_k}-\Delta_y$ with energy P^2+E_G.

<u>Proof:</u>
(i) $\langle \frac{1}{\sqrt{L}}\psi_G e^{iPy}, \mathcal{H}\frac{1}{\sqrt{L}}\psi_G e^{iPy}\rangle$

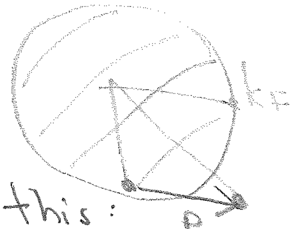
$= E_G\langle \frac{1}{\sqrt{L}}e^{iPy}, \frac{1}{\sqrt{L}}e^{iPy}\rangle + \langle\psi_G,\psi_G\rangle\frac{1}{L}\langle e^{iPy}, -\Delta_y e^{iPy}\rangle$

$= E_G + P^2$

(ii) We claim that $\forall\psi: \langle\psi,\mathcal{H}\psi\rangle \ge P^2+E_G$ if N large enough (specified later)

We know $\hat{P}\psi = P\psi$. By linearity, it suffices to consider product states.
If the tracer particle has the full momentum P, the ~~fermions~~ statement is true.
~~So assume $p_y<P$ (w.l.o.g.)~~.
If $p_y\ne P$, then $\sum_k p_{x_k}\ne 0$, so the fermions are not in ψ_G.
Each other state has at least energy $E_G+\frac{4\pi}{L}k_F+\left(\frac{2\pi}{L}\right)^2 \ge E_G+\frac{4\pi}{L}k_F$
For N large enough, k_F gets arbitrarily large $\Rightarrow \langle\psi,\mathcal{H}\psi\rangle\ge P^2+E_G$ □

d) In $d=2$,



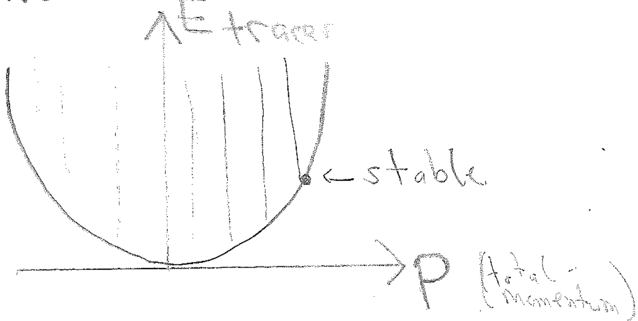
change ψ_G to this: and let y have no momentum.

$$\Delta E = \sqrt{k_F^2+p^2} - k_F^2 = k_F\left(\sqrt{1+\left(\frac{p}{k_F}\right)^2}-1\right)$$

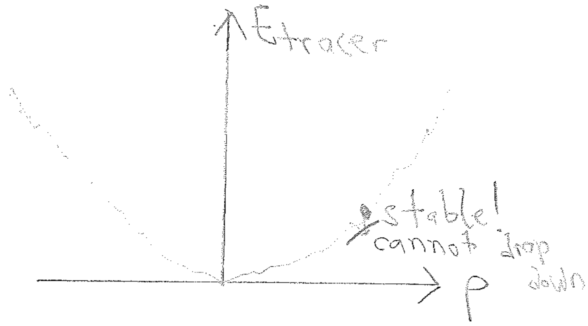
For large N, the energy difference goes to zero, so

$$\langle \psi, \mathcal{H}\psi \rangle = E_G + \Delta E < E_G + p^2 .$$

e) a very small V does not alter the spectrum too much. So the spectrum of $\mathcal{H}$ w.r.to the tracer particle is as follows:

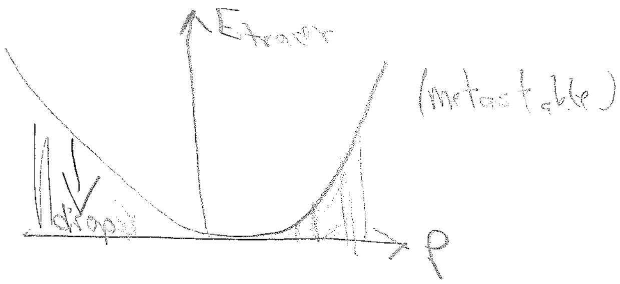


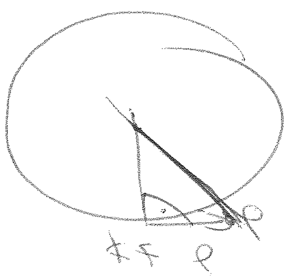
$V=0$



V small

in 2d hower:





$$\Delta E = \underbrace{\cancel{(k_F+p)^2}}_{k_F^2+p^2} - k_F^2 = p^2$$