LMU Munich
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Exercises on Mathematical Statistical Physics
Math Sheet 7

Problem 1 (Operate but leave no trace). In this exercise, we compare the trace-norm $\|A\|_{\text{tr}}$ and the operator norm $\|A\|_{\text{op}}$ of self-adjoint operators on the Hilbert space $L^2(\mathbb{R}^3)$.

a) Show that the norms are not equivalent, i.e.

$$\|A_n\|_{\text{tr}} \overset{n\to\infty}{\longrightarrow} 0 \ \not\Leftrightarrow \|A_n\|_{\text{op}} \overset{n\to\infty}{\longrightarrow} 0.$$

Which one is stronger?

b) Show that for the convergence of a one-particle reduced density matrix μ^{ψ_n} to a projector $|\varphi\rangle\langle\varphi|$, the equivalence holds:

$$\left\|\mu^{\psi_n} - |\varphi\rangle\langle\varphi|\right\|_{\text{tr}} \overset{n\to\infty}{\longrightarrow} 0 \ \Leftrightarrow \left\|\mu^{\psi_n} - |\varphi\rangle\langle\varphi|\right\|_{\text{op}} \overset{n\to\infty}{\longrightarrow} 0.$$

Hint: For the difficult direction, recall the properties of density matrices and projectors and think about the possible eigenvalues.

Problem 2 (The α method). Recall the situation where we derived the Hartree equation

$$i\partial_t\varphi_t = h_t\varphi_t = \left(-\triangle + V * |\varphi_t|^2\right)\varphi_t$$

as mean-field approximation to the full Schrödinger equation of N interacting bosons with Hamiltonian

$$H = \sum_{j=1}^{N} -\triangle_j + \frac{1}{N-2}\sum_{j\neq k} V(x_j - x_k),$$

$\|V\|_\infty < \infty$ (denominator changed to $N-2$ for convenience). We define an object similar to the α from the lecture, which is

$$\beta(t) = \left\langle \psi_t, q_1^t q_2^t \psi_t \right\rangle.$$

For the situation of initial closeness to a product state, which implies that $\beta(0)$ vanishes like $\frac{1}{N^2}$ for $N\to\infty$, show that also

$$\beta(t) \sim \frac{1}{N^2}.$$

Hint: Bring the time derivative of β to the form

$$\dot{\beta} = i \langle \psi, [H - h_1 - h_2, q_1 q_2]\, \psi \rangle$$

and then reduce it to something like

$$\big\langle q_2\psi, (p_1 + q_1)(p_3 + q_3)(V(x_1 - x_3) - V * |\varphi|^2(x_1)) q_1 (p_3 + q_3) q_2 \psi \big\rangle + \dots$$

Problem 3 (Tracer particle in a Fermi sea). We consider a gas of N free identical fermions (coordinates $\mathbf{x}_k$) in a d-dimensional box of volume L^d (with periodic boundary conditions).

a) What is the ground state ψ_G of this system? The respective ground state energy will be called E_G in the following.

b) We now specialize to the case of $d = 1$. Draw a picture of the ground state in k-space and compute the energy gap between the ground state and the lowest excited state.

c) We now add one particle of a different sort called *tracer particle*, whose coordinate is called $\mathbf{y}$, and consider first the free case, i.e. the total Hamiltonian

$$\mathcal{H} = \sum_{k=1}^{N} -\triangle_{\mathbf{x}_k} - \triangle_{\mathbf{y}}.$$

Assume the total wave function $\psi(\mathbf{x}_1, ...\mathbf{x}_N, \mathbf{y})$ is an eigenstate of the total momentum operator, i.e. $\hat{P}\psi = P\psi$. Using the results of b), prove the following statement: If N large enough, then

$$\forall \psi : \quad \langle \psi, \mathcal{H}\psi \rangle \geq P^2 + E_G$$

and equality holds if and only if $\psi(\mathbf{x}_1, ...\mathbf{x}_N, \mathbf{y}) = L^{-0.5}\ \psi_G(\mathbf{x}_1, ...\mathbf{x}_N) \otimes e^{iPy}$.

d) Prove that the statement of c) does not hold in $d = 2$ (or higher dimensions).

e) Back to one dimension ($d = 1$). We now switch on an interaction between the fermions and the tracer particle:

$$\mathcal{H}' = \sum_{k=1}^{N} -\triangle_{\mathbf{x}_k} - \triangle_{\mathbf{y}} + \sum_{k=1}^{N} V(\mathbf{x}_k - \mathbf{y}).$$

We assume that V is very small and that in the beginning, our tracer particle is flying through the homogeneous Fermi sea, so we start with a state $\psi(t_0) = L^{-0.5}\ \psi_G \otimes e^{iPy}$. Give a heuristic argument using the spectrum of $\mathcal{H}'$ why the tracer particle effectively propagates freely.

This is the last exercise sheet of the math part. The solutions will be discussed on Friday, July 8, and tba.