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Summer term 2016 | L. Nickel, C. Velasco

Exercises on Mathematical Statistical Physics
Math Sheet 6

Problem 1 (Bosons and fermions in the microcanonical ensemble). We consider a gas of free quantum particles and want to derive the occupation numbers for energy levels.

a) Calculate the number of possibilites to distribute n bosons to g different states

b) Calculate the number of possibilites to distribute n fermions to g different states.

c) Consider a gas of N free fermions at total energy E. There are energy levels $e_i, i \in \mathbb{N}$, which have a respective degeneracy of g_i. A macrostate is given by the set of n_i, the occupation numbers of the energy states e_i (so not counting the different degenerate states on the energy shell). Calculate the Boltzmann entropy for a given macrostate and show that it is (using Stirling's approximation) extremal under the given constraints if

$$n_i = \frac{g_i}{e^{\beta(e_i - \mu)} + 1}.$$

d) Doing the same for bosons gives

$$n_i = \frac{g_i}{e^{\beta(e_i - \mu)} - 1}.$$

Derive this equation in another way, by calculating the expectation value of n_i in the grand canonical ensemble.

Problem 2 (Bose-Einstein condensation). Consider a gas of N free bosons in a d-dimensional box of volume L^d. We assume the Hamiltonian has a discrete spectrum with eigenstates given by $|k\rangle = |k_1, ..., k_d\rangle, k_j \in \mathbb{Z}$, energies

$$e(k) = \pi \frac{k_1^2 + ... + k_d^2}{L^2},$$

and occupation numbers

$$n(k_1, ..., k_d) = \frac{1}{z^{-1} e^{\beta e(k)} - 1},$$

where z is the fugacity, which for bosons satisfies $0 < z \leq 1$. $n_0 = n(0, ..., 0)$ is the occupation number of the ground state. We say that Bose-Einstein-condensation is present iff the ground state is macroscopically occupied, i.e.

$$\lim_{N\to\infty} \frac{n_0}{N} > 0, \quad \text{limit to be taken at constant density, i.e. also } L \to \infty.$$

Your task is to show that this happens at some positive temperature T iff $d \geq 3$. To achieve this, write $N = n_0 + \sum_k n(k)$, approximate the sum by an integral in momentum space (justify this!) and investigate whether the integral is finite.

Problem 3 (Bosons in the canonical ensemble). For quantum gases, one usually uses the grand canonical ensemble because it simplifies calculations. In this exercise, we consider the canonical ensemble, which gives different results as long as the thermodynamic limit is not taken. We look at N free identical bosons in a harmonic potential in one dimension. Therefore, the one-particle energy levels are given by

$$e_j = \Delta(\frac{1}{2} + j), \quad j \in \mathbb{N}_0,$$

where $\Delta = \hbar\omega$.

a) Calculate the canonical partition function

$$Z_N = \sum_{n_0=0}^{N} \sum_{n_1=0}^{N} \sum_{n_2=0}^{N} \cdots \; e^{-\beta \sum_{j=0}^{\infty} e_j n_j} \; \delta_{N, \sum_{j=0}^{\infty} n_j}$$

by deriving for $\tilde{Z}_N = e^{\beta N \frac{\Delta}{2}} Z_N$ a recursion relation of the form

$$\tilde{Z}_N = \sum_{\nu=0}^{N} e^{-\beta\nu\Delta} \tilde{Z}_\nu.$$

b) Calculate the average energy E.

The solutions to these exercises will be discussed on Friday, July 1, and Monday, July 4.