LMU Munich
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Exercises on Mathematical Statistical Physics
Math Sheet 4

Problem 1 (Equivalent definitions of ergodicity). Let $(\Omega, \mathcal{A}, T, \mathbb{P})$ be a dynamical system, with $\mathbb{P}$ being a preserved measure. Prove that the system is ergodic if and only if...

a) $T^{-1}(A) = A$ up to a set of measure zero implies $\mathbb{P}(A) \in \{0, 1\}$ (*this was sketched in the lecture*).

b) every $\varphi \in L^1(\Omega, \mu)$ which is T-invariant is constant almost everywhere.

c) every $\varphi \in L^1(\Omega, \mu)$ which is T-invariant almost everywhere is constant almost everywhere.

Problem 2 (Physical examples). For physical models, the property of ergodicity is rather difficult to prove. We therefore only discuss it heuristically here. Give arguments (but no proofs) why the Lebesgue measure on the following sets is ergodic or is not ergodic. Your answers may depend on the chosen initial conditions.

a) Ideal gas of N particles in a cubic box of volume 1, $\Omega = [0, 1]^{3N} \times \mathbb{R}^{3N}$.

b) Ideal gas of N particles in a spherical container of volume 1, $\Omega =$ energy shell.

c) Gas of N balls with radius $a > 0$ that elastically collide with each other and move on the two-dimensional torus T^2, $\Omega =$ energy shell.

d) One particle moving freely and colliding elastically with the walls of a quadratic table, $\Omega =$ energy shell.

e) Gas of N gravitating point particles in a cubic box, $\Omega =$ energy shell.

Problem 3 (Properties of ergodic measures). Recall or get to know the following definitions:
A set S is **convex** iff $\forall x, y \in S, \lambda \in [0,1] : \lambda x + (1-\lambda)y \in S$. A convex set (e.g. a ball, or a line) has **extremal points** (e.g. the points on the surface, or the ends of the line) which are defined to be the points p that satisfy: If for $x, y \in S$ and $\lambda \in (0,1)$, we have $p = \lambda x + (1-\lambda)y$, then $x = p = y$.
Two probability measures μ, ν on $(\Omega, \mathcal{B}(\Omega))$ are called **singular** iff there exists a subset $X \subset \Omega$ with $\mu(X) = 1$ and $\nu(X) = 0$.

Now let $(\Omega, \mathcal{B}(\Omega))$ a set endowed with its Borel algebra and $T : \Omega \to \Omega$ a (continuous) transformation. The aim of this exercise is some kind of uniqueness statement of ergodic measures.

a) Show that the set $\mathcal{I}_T$ of all probability measures that are preserved by T is a convex set.

b) Show that the ergodic measures are exactly the extremal points of $\mathcal{I}_T$. (*Hint: Use the ergodic theorem in some smart way.*)

c) Conclude that two different ergodic measures $\mu \neq \nu$ on Ω are singular with respect to each other.

The solutions to these exercises will be discussed on Friday, June 3 and on Monday, June 6.