LMU Munich
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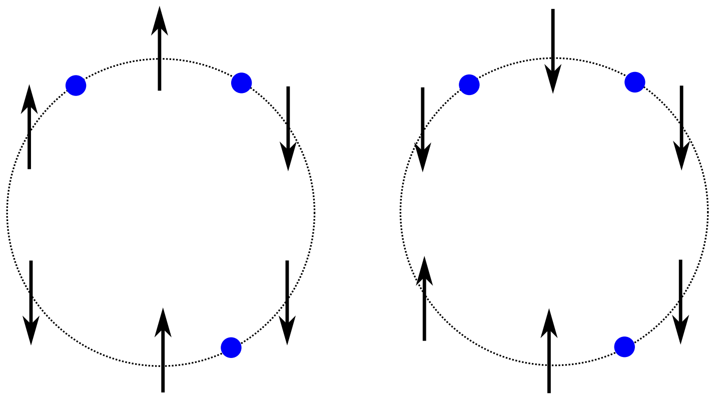
Exercises on Mathematical Statistical Physics
Math Sheet 3

Problem 1 (Kac ring). The Kac ring is a simple toy model which exemplifies how a Poincaré recurrent system can nevertheless show equilibration. Consider a ring with N sites. On each site, there is an arrow (call it spin if you like) either pointing up or down. On the N spaces between the arrows, F flippers are distributed (depicted by blue circles). The configuration of the flippers never changes. The dynamics of the system is as follows: during each time step, every arrow moves clockwise to the next site. If it passes a flipper, it reverses its orientation: Up becomes down, down becomes up. The picture shows one time step in the evolution of a Kac ring for $N = 6, F = 3$.



a) Formulate the dynamics in terms of a dynamical system $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ with transformation T, such that the measure $\mathbb{P}$ is preserved by T.

b) Convince yourself that the dynamics is reversible and that Poincaré recurrence holds (in an ever stronger sense than for continuous phase space). What is the recurrence time scale?

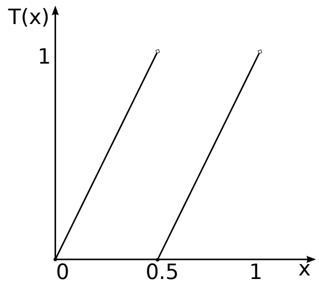
c) Now assume that the positions of the flippers and the configuration of the arrows are independent of each other for all times. Arrive at an equation for the time evolution of the average number of arrows pointing up, p_t, and solve it. What is the time scale of approach to equilibrium?

d) How do your results seemingly contradict each other? Why only seemingly?

Problem 2 (A dynamical system). We consider the dynamical system given by the probability space $(\Omega = [0,1), \mathcal{B}(\Omega), ?)$ and the transformation

$$T : \Omega \to \Omega,\ T(x) = 2x \text{ mod} 1,$$

whose graph is depicted below. Note that T is not invertible. As stressed in the lecture, finding the right probability measure is usually done after specifying the dynamics.



a) Prove that the Lebesgue measure on Ω is preserved by T.

b) Can you find other probability measures that are preserved by T? (*Hint: You can.*)

c) The map T is a paradigmatic example for a *chaotic transformation.* This concept may be mathematically defined in different ways. We prove two different version of chaotic behaviour. (*Hint: It might be helpful to express the numbers in Ω in binary representation.*)

i) Prove that T features sensitive dependence on initial conditions at each $x \in \Omega$, which is defined as follows: There is an $\varepsilon > 0$ such that

$$\forall \delta > 0\ \exists y \in \Omega, n \in \mathbb{N} : \|y - x\| < \delta\ \wedge\ \|T^n(y) - T^n(x)\| > \varepsilon.$$

ii) Prove that there is at least one point x_0 in Ω whose orbit, i.e. the set $\{T^n(x_0) \mid n \in \mathbb{N}_0\}$, is dense in Ω.

Problem 3 (Different convergence concepts in probability theory). We investigate two different concepts of convergence for random variables. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(X_n)_{n\in\mathbb{N}}$ a sequence of random variables. The sequence is said to converge to a random variable X

- **in probability** iff for all $\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(|X_n - X| \leq \varepsilon) = 1$.
- **almost surely** iff $\mathbb{P}(\{\omega \in \Omega | \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1$.

Now let Ω be the unit circle, $\mathcal{A}$ its Borel set and $\mathbb{P}$ the uniform distribution. We define a sequence of random variables $(X_n)_{n\in\mathbb{N}}$, each mapping $\Omega \to \{0,1\}$ by

$$X_j(\omega) = \begin{cases} 1 & if\ \omega \in \left[\sum_{k=1}^{j-1} \frac{1}{k}, \sum_{k=1}^{j} \frac{1}{k}\right] \\ 0 & else \end{cases} .$$

Here it is understood that $\omega + 2\pi \equiv \omega$.

a) Show that $(X_n)_{n\in\mathbb{N}}$ converges to $X = 0$ in probability.

b) Show that $(X_n)_{n\in\mathbb{N}}$ does not converge to X almost surely, but that even $\mathbb{P}(\{\omega \in \Omega | \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 0$.

The solutions to these exercises will be discussed on Friday, 13.05. and Wednesday, 18.05., 10-12 (replacement for the Monday group because of the holiday) in room C111.